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Supersymmetries and Essential Observables

by **J. M. Jauch*)** and **B. Misra**)**

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Summary. A new type of symmetrytransformation called *supersymmetry* is introduced and studied. It is shown that such symmetries always occur in a physical system with superselection rules. They are described by unitary transformations which commute with all the observables of the system. It is further shown that systems which admit a complete set of commuting observables and a supersymmetry must contain a certain class of observables, called essential observables, which must be represented in every complete system. Furthermore in that case the supersymmetries form an abelian group.

The applications of these results to C-number gauge transformations in quantum electrodynamics leads to the conclusion that the total charge operator is always a superselection operator. This explains why it is impossible to prepare a state which is a superposition of states corresponding to different eigenvalues of the charge. A corollary of this result is the fact that the 1- and 2-components of the isotopic spin cannot be observables.

1. Introduction

In quantum mechanics a symmetry transformation is a unitary or anti-unitary transformation of the space of statevectors which leaves the time evolution of the system invariant. The study of symmetry transformations has played an important rôle, not only in the practical application for the determination of term structures of stationary states, but also in the understanding of the fundamental aspects of the theory.

It occurs frequently that symmetry transformations leave, in addition to the Hamiltonian, other observables invariant. In this paper we direct our attention to a class of symmetry transformations which leave *all* the observables invariant. Obviously such transformations are only possible if there exist non-trivial transformations which commute with all observables, that is, the system must have superselection rules. For this reason we refer to such symmetries as *supersymmetries*.

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An example of such a transformation is for instance the iteration of the time reversal transformation. There is no reason to doubt that this transformation commutes with all the observables, corresponding to the situation in classical mechanics. However, in the presence of a spinorfield, this transformation is not trivial (that is, not just a multiplum of the identity), because it is possible to show that it must anticommute with the spinorfield[1]). It is precisely this property which was used for instance by WICK, WIGHTMAN and WIGNER in their proof that a spinorfield cannot be an observable[2]). Thus supersymmetries are expected to occur whenever there are superselection rules. This justifies the choice of terminology.

We shall show in this paper that supersymmetries are always connected with another phenomenon, which we have called the occurrence of *essential observables*. Such observables can be described in the following way: every complete set of commuting observables defines a certain algebra of bounded operators. Since such complete sets can be chosen in many different ways there exist many different such operator algebras. The intersection of all these algebras contains at least the multiples of the unit operator. If it contains any other operators, we shall say that there exist *essential observables*. Loosely speaking then, an essential observable is one which must be represented in every complete set of commuting observables.

The essential observables generate themselves an algebra which we have called the *core*. We shall demonstrate that the core is identical with the center of the algebra of *all* observables.

This result is a corrolary of the theorem that the commutator algebra of a von Neumann algebra $\mathfrak{N}$ (that is the algebra of all the bounded operators which commute with all operators in $\mathfrak{N}$) is abelian if and only if there exists a maximal abelian sub-algebra in $\mathfrak{N}$. We shall prove this theorem in section 3 (Theorem 3).

This theorem has another application which we mention here too although it is not the main subject of this paper. In the axiomatic formulation of quantum field theory it is customary to assume that the Hilbert space of statevectors can be represented as a direct sum of Hilbert spaces each consisting of physically realizable state vectors[3]). It is easy to see that this is equivalent to the assumption that the commutator algebra $\mathfrak{N}'$ of all observables is abelian[4]). Now, by the above theorem, this is equivalent to the assumption that there exist at least one maximal abelian subalgebra of $\mathfrak{N}$. But, as it was shown in a previous publication[4]), this means simply that the system permits a complete system of commuting observables. In this form the assumption that $\mathfrak{N}'$ be abelian has a direct physical interpretation and becomes therefore quite plausible.

2. Definition of supersymmetries

We shall assume that the pure states of the quantum mechanical system are described by normalized elements ψ from a separable Hilbert space $\mathfrak{H}$ with positive definite scalar product. The observables are represented by linear self adjoint operators operating on elements $\psi \in \mathfrak{H}$. Let $\mathfrak{S}$ be the system of all the observables and denote by $\mathfrak{N}$ the von Neumann algebra of all bounded operators generated by the observables[4]). This algebra is defined as the set of all bounded operators which commute with all bounded operators which commute with $\mathfrak{S}$

$$\mathfrak{N} = \{\mathfrak{S}\}'' . \tag{1}$$

Observables, as one knows from many examples, need not be bounded operators and if they are not, they would not be themselves contained in the algebra $\mathfrak{N}$. Nevertheless the algebra $\mathfrak{N}$ is a mathematically convenient and physically entirely adequate characterization of the kinematical structure of the set of all observables. For the unbounded observables we can introduce the weaker concept of *affiliation*. With this we mean the following property: every observable (bounded or not) is represented by a self adjoint linear operator A and such an operator defines a unique family of spectral projections. If every one of these spectral projections is contained in $\mathfrak{N}$ we say A is *affiliated* with $\mathfrak{N}$.

It follows from the definition (1) that every self adjoint operator representing an observable is affiliated with $\mathfrak{N}$ and that $\mathfrak{N}$ is a smallest von Neumann algebra with this property.

We introduce the concept of supersymmetry with the following *definition:*

A unitary operator $\mathfrak{U}$ is a supersymmetry if it differs from a multiplum of the identity and commutes with the set $\mathfrak{S}$ of all observables.

Two remarks which motivate this definition may be useful.

First, it should be noted that this definition makes only sense if every supersymmetry is also an ordinary symmetry. This is true if the Hamiltonian of the system (defined as the generator of the time-displacement operator) is also an observable. The Hamiltonian is interpreted as the total energy of the system and its observability is usually taken for granted. This amounts to the assertion that it is in principle possible to prepare states, by suitable physical arrangements, with an arbitrary small spread in energy.

Secondly we have defined a supersymmetry as a unitary transformation. There are of course also antiunitary symmetry transformations possible. Such transformations can never be supersymmetries. That is, an antiunitary transformation cannot commute with all the observables. The proof is based on the fact that the time reversal transformation T is always antiunitary[5]). Let X be an antiunitary supersymmetry transformation, then TX is also a time reversal transformation. But as a product of two antiunitary transformations it is itself unitary. This is not possible.

The following theorem gives a criterion for the existence of supersymmetries:

Theorem I

A quantum mechanical system has a supersymmetry if and only if the algebra $\mathfrak{R}$ of bounded operators generated by the set of all observables is reducible.

Proof: We say a von Neumann algebra $\mathfrak{R}$ is irreducible if its commutator algebra $\mathfrak{R}'$ is trivial (that is consists only of the multiples of the identity); otherwise it is called reducible. For the proof of the necessity of the condition we assume $\mathfrak{U}$ is unitary and it commutes with all observables $\mathfrak{S}$. That is

$$\mathfrak{U} \in \{\mathfrak{S}\}' .$$

We wish to show that

$$\mathfrak{U} \in \mathfrak{R}' . \tag{6}$$

In order to verify this, we need the following properties of von Neumann algebras:

(1) Every von Neumann algebra $\mathfrak{M}$ is identical with its double commutant:

$$\mathfrak{M}'' = \mathfrak{M} .$$

(2) The commutant of a set $\mathfrak{S}$ of self-adjoint operators is a von Neumann algebra.

Let us define then

$$\{\mathfrak{S}\}' = \mathfrak{M} .$$

It follows with properties (1) and (2)

$$\{\mathfrak{S}\}''' = \mathfrak{M}'' = \mathfrak{M} = \mathfrak{R}'$$

and therefore

$$\mathfrak{U} \in \mathfrak{R}' . \tag{7}$$

Since $\mathfrak{U}$ is different from the identity the commutant $\mathfrak{R}'$ is not trivial, hence $\mathfrak{R}$ is reducible. This proves the necessity of the condition.

To prove sufficiency, we assume $\mathfrak{R}'$ to be non trivial. We must show $\mathfrak{R}'$ contains at least one unitary operator. In order to show this, we use another property of von Neumann algebras: every von Neumann algebra can be generated by its unitary operators[6]). Since $\mathfrak{R}'$ is non trivial, there must exist at least one unitary operator $\mathfrak{U} \in \mathfrak{N}'$ which differs from the identity. This completes the proof.

We have previously shown[4]) that the reducibility of $\mathfrak{N}$ is characteristic for the existence of a superselection rule. Hence, we have the corollary:

A supersymmetry exists if and only if there exists a superselection rule.

3. Essential observables

We shall now assume that there exists at least one complete set of commuting observables A_i. In the above mentioned reference[4]), it was shown that such a set generates a maximal abelian von Neumann algebra

$$\mathfrak{A} = \{A_i\}'' , \tag{8}$$

$$\mathfrak{A} = \mathfrak{A}' . \tag{9}$$

Furthermore the algebra $\mathfrak{N}$ generated by all observables satisfies[4])

$$\mathfrak{N}' \subset \mathfrak{N} . \tag{10}$$

Let $\mathfrak{A}_i$ be the set of all maximal abelian algebras contained in $\mathfrak{N}$. The intersection of them is again a von Neumann algebra which we shall call the *core* $\mathfrak{C}$ of $\mathfrak{N}$

$$\mathfrak{C} \equiv \bigcap_i \mathfrak{A}_i . \tag{11}$$

From $\mathfrak{A}_i \subset \mathfrak{N}$ follows

$$\mathfrak{N}' \subset \mathfrak{A}_i' = \mathfrak{A}_i \subset \mathfrak{N} \tag{12}$$

and by taking the intersections

$$\mathfrak{N}' \subseteq \mathfrak{C} \subset \mathfrak{N} . \tag{13}$$

Thus the core always contains the center, which, in this case, is identical with $\mathfrak{N}'$. But for algebras generated by observables we can say even more.

We shall, in fact, show that for algebras which satisfy (10), the following theorem is true:

Theorem 2

For a von Neumann algebra with abelian commutator algebra the core is identical with the center.

Or

$$\mathfrak{N}' \subset \mathfrak{N} \quad \text{implies} \quad \mathfrak{Z} = \mathfrak{C} .$$

Proof: We note that $\mathfrak{Z} = \mathfrak{N}'$ and the inclusion

$$\mathfrak{Z} \subseteq \mathfrak{C}$$

is already proved (equation (13)). Thus, we proceed to establish

$$\mathfrak{C} \subseteq \mathfrak{Z}.$$

In order to do this, we need the following three lemmata.

Lemma 1: Let $\mathfrak{A} \subset \mathfrak{R}$ be an abelian subalgebra of a von Neumann algebra $\mathfrak{R}$ with abelian commutator algebra $\mathfrak{R}' \subset \mathfrak{R}$, then $\mathfrak{A} = \mathfrak{A}' \cap \mathfrak{R}$ implies $\mathfrak{A} = \mathfrak{A}'$. In other words, an algebra which is maximal abelian in $\mathfrak{R}$ does not have any abelian extension.

Proof of Lemma 1: Assume $\mathfrak{R}' \subset \mathfrak{R}$, $\mathfrak{A} \subset \mathfrak{R}$, and $\mathfrak{A} = \mathfrak{A}' \cap \mathfrak{R}$. We show first that $\mathfrak{Z} \subseteq \mathfrak{A}$. If this were not so, there would exist an operator $Z \in \mathfrak{Z}$ and $Z \notin \mathfrak{A}$. But this operator is both in $\mathfrak{R}$ (since it is in $\mathfrak{Z} \subseteq \mathfrak{R}$) and in $\mathfrak{A}'$ (since $\mathfrak{Z} = \mathfrak{R}' \subset \mathfrak{A}'$). Hence $\mathfrak{A} \subset \mathfrak{A}' \cap \mathfrak{R}$ and $\mathfrak{A} \neq \mathfrak{A}' \cap \mathfrak{R}$. This contradicts the assumption. Thus $\mathfrak{Z} \subseteq \mathfrak{A}$ and since $\mathfrak{Z} = \mathfrak{R}'$ also $\mathfrak{R}' \subseteq \mathfrak{A}$. Consequently $\mathfrak{A}' \subseteq \mathfrak{R}'' = \mathfrak{R}$ and so $\mathfrak{A} = \mathfrak{A}' \cap \mathfrak{R} = \mathfrak{A}'$. This proves the lemma.

Having established that maximal abelian algebras in $\mathfrak{R}$ have no abelian extensions, we show next that there exist maximal abelian algebras in $\mathfrak{R}$.

Lemma 2: Every abelian algebra in a von Neumann algebra $\mathfrak{R}$ can be extended to a maximal abelian algebra contained in $\mathfrak{R}$.

Proof of Lemma 2: Let $\mathfrak{A} \subset \mathfrak{R}$ be an abelian algebra in $\mathfrak{R}$. Let ϕ be the class of all abelian algebras in $\mathfrak{R}$, containing $\mathfrak{A}$. It is partially ordered by inclusion. If ϕ_0 is a linearly ordered subclass of ϕ, then the union of all elements in ϕ_0 is again an abelian algebra contained in ϕ and it is an upper bound for ϕ_0. By Zorn's lemma ϕ contains a maximal element $\mathfrak{A}_0$ for which $\mathfrak{A}_0 = \mathfrak{A}_0' \cap \mathfrak{R}$. By lemma 1 it follows $\mathfrak{A}_0' = \mathfrak{A}_0$, and this proves lemma 2.

A simple corollary of this is

Lemma 3: Every operator $T \in \mathfrak{R}$ is member of at least one maximal abelian algebra in $\mathfrak{R}$.

Proof of lemma 3: T generates the abelian algebra $\{T\}''$ and according to lemma 2 this algebra can be extended to a maximal abelian algebra in $\mathfrak{R}$. This proves lemma 3.

The proof of the theorem is now easily completed as follows. Let $S \in \mathfrak{C}$. If S were not in $\mathfrak{Z}$, then there exists an operator $T \in \mathfrak{R}$ which does not commute with S. But $S \in \mathfrak{A}_i$ for all $i \in I$. Thus T cannot be in any $\mathfrak{A}_i$. This contradicts the lemma 3. Thus $S \in \mathfrak{Z}$ and

$$\mathfrak{C} \subseteq \mathfrak{Z}.$$

This proves the theorem.

Observables in the core are called *essential observables*. A complete set of essential observables is a set of observables affiliated with the core with the property that the algebra generated by them is identical with the core. Every complete set of commuting observables contains a complete set of essential observables.

We mention now a few corollaries which follow from the preceding results. By combining theorem 1 and 2, we see that a physical system has essential observables if and only if it has superselection rules and therefore supersymmetries. The essential observables generate an abelian algebra which is identical with the center of the algebra generated by all observables. This establishes the connection between essential observables and supersymmetries mentioned in the introduction.

Another interesting consequence is obtained from lemma 3: either every observable is a member of a complete set of commuting observables or there exists no such complete set.

A further consequence of these results is the following: if $\mathfrak{N}'$ is abelian, then there exists a maximal abelian subalgebra of $\mathfrak{N}$. On the other hand, it is trivial[7]) to show that if there exists a maximal abelian algebra in $\mathfrak{N}$, then $\mathfrak{N}'$ is abelian. Thus, we have the

Theorem 3

The necessary and sufficient condition for the existence of a complete set of commuting observables is that the commutant $\mathfrak{N}'$ of the algebra $\mathfrak{N}$ generated by all observables is abelian.

This is the theorem which we have announced and briefly discussed in the introduction.

A simple corollary is: A physical system with no superselection rules always admits a complete set of commuting observables, a statement that would have been obvious in the first place.

4. Application to gauge transformations

One of the main problems in the investigation of the kinematical structure of physical systems is to find superselection rules. According to our theorem 1, this problem is identical with the finding of supersymmetries. In either case these properties amount to a physical assumption. Yet the formulation with the help of the concept of the supersymmetry has a certain heuristic advantage. One may, for instance, assume that transformations which do not affect the classical limit of observables are supersymmetries in their quantum mechanical interpretation.

Based on this assumption one would look for Abelian groups of symmetry transformations which leave the classical limit of the observable quantities invariant. Such transformations are for instance the gauge-transformations for a system of charged particles interacting with an electromagnetic field.

In the following, we shall make the assumption that C-number gauge-transformations are supersymmetries and we shall find the corresponding superselection rules. Since there does not yet exist a mathematically satisfactory formulation of a realistic field theory we shall use the conventional rules hoping that, in spite of the heuristic character of this theory, the symmetry transformations here considered, have a significance also in a future improved formulation of field theory.

Let $\Psi(x)$ represent a spinorfield interacting with an electromagnetic field represented by a vectorpotential $A_\lambda(x)$. The field operators satisfy the usual commutation rules and field equations[9]. A C-number gauge transformation is given by

$$\begin{aligned} A_\alpha(x) &\to A'_\lambda(x) = A_\alpha(x) + \partial_\lambda \Lambda(x) , \\ \Psi(x) &\to \Psi'(x) = \Psi(x)\, e^{i e \Lambda(x)} \end{aligned} \tag{14}$$

where $\Lambda(x)$ is a scalar function satisfying

$$\partial^\mu \partial_\mu \Lambda(x) = 0 . \tag{15}$$

These transformations are generated by the formally unitary transformations

$$\mathfrak{U} = e^{iF} \tag{16}$$

with

$$F = \int_\sigma d\sigma^\mu F_\mu , \tag{17}$$

$$\left.\begin{aligned} F_\mu &= \chi\, \partial_\mu \Lambda - \Lambda\, \partial_\mu \chi , \\ \chi &\equiv \partial^\lambda A_\lambda . \end{aligned}\right\} \tag{18}$$

The integration for F is extended over a space-like hyperplane σ which, because of

$$\partial^\mu \partial_\mu \chi = 0 \tag{19}$$

can be chosen arbitrary.

The operator F satisfies the commutation rules

$$\begin{aligned} i\,[F, A_\lambda(x)] &= \partial_\lambda \Lambda(x) , \\ i\,[F, \Psi(x)] &= i\, e\, \Psi(x)\, \Lambda(x) \end{aligned} \tag{20}$$

which are formally equivalent with

$$A'_\lambda(x) = \mathfrak{U}\, A_\lambda(x)\, \mathfrak{U}^{-1},$$
$$\Psi'(x) = \mathfrak{U}\, \Psi(x)\, \mathfrak{U}^{-1}. \tag{21}$$

Because of (20) and (19), the group of unitary operators $\mathfrak{U}\{\Lambda\}$ associated with all the different functions $\Lambda(x)$ is Abelian and satisfies

$$\mathfrak{U}\{\Lambda_1\}\, \mathfrak{U}\{\Lambda_2\} = \mathfrak{U}\{\Lambda_1 + \Lambda_2\}. \tag{22}$$

If we assume that the group $\mathfrak{U}\{\Lambda\}$ represents a supersymmetry, then only operators which commute with $\mathfrak{U}\{\Lambda\}$ can be observables.

We shall now show that this assumption leads to the conclusion that the total charge of any system is a superselection operator, that is an operator which commutes with all observables.

To this end, we use the identity

$$F_\mu = \chi\, \partial_\mu \Lambda - \Lambda\, (\partial_\nu F_\mu^{\ \nu} - J_\mu) \tag{23}$$

which follows from the field equation

$$\partial^\mu\, \partial_\mu A_\lambda = -\, J_\lambda \tag{24}$$

and the expression (18) for F_μ. Let us now consider as a special choice for Λ a constant, such that $\partial_\mu \Lambda = 0$. We obtain then for

$$F = \Lambda\, (F_0 - Q) \tag{25}$$

with

$$F_0 = - \int d\sigma_\mu\, \partial_\nu\, F^{\mu\nu}, \tag{26}$$

$$Q = - \int d\sigma_\mu\, J^\mu. \tag{27}$$

Because of $\partial_\mu\, \partial_\nu\, F^{\mu\nu} = 0$, the integral for F_0 can always be written as an integral over 3-space with only space-like derivatives

$$F_0 = \int_\sigma d^3 x\, \partial_i\, F^{0i}$$

where the space like plane σ can be placed anywhere. By an application of Gauss's theorem, the integral can be expressed as a surface integral at infinity. If one were dealing with numerical functions of one could introduce here the assumption that the fields vanish at infinity such that $F_0 = 0$. However, this often used argument is not applicable since we are dealing with field *operators* which vanish nowhere.

However, it is easy to see that the operator F_0 commutes with all the field variables at finite space-time points and therefore the operator $-\Lambda Q$ has the same commutation rules with all dynamical variables as the operator F. In particular, one verifies directly the relations

$$i\,[Q, A_\lambda(x)] = 0\,,$$

$$i\,[Q, \Psi(x)] = -\,i\,e\,\Psi(x)$$

which are obtained from (20) and (25) by specialization of $\Lambda(x)$ to a constant. These particular canonical transformations are therefore represented by the unitary operators

$$\mathfrak{U} = e^{-\Lambda Q}\,,$$

$$Q = -\int d\sigma_\mu\, J^\mu\,.$$

According to our assumption, these transformations represent supersymmetries and therefore any operator which does not commute with them, is not an observable. This leads to the conclusion that the total charge operator Q is a superselection operator. This corresponds to the well known fact that it is not possible to prepare a physical state which is a superposition of different charge states.

In order to emphasize the physical content of this conclusion, it should be placed in opposition to the situations which are obtained from other conserved quantities which are not superselection operators. Such operators are, for instance, the total momentum or the spin. Even though the total momentum is a conserved quantity in systems with displacement invariant interactions, it is perfectly possible to prepare states which are superpositions of the total momentum eigenstates. The same is of course true for the spincomponent in any fixed direction of space.

On the other hand, one must not confuse a statistical mixture with a superposition. It is quite possible to have a statistical mixture, represented by a density operator, which mixes states with different eigenvalues of the charge. What is excluded by the supersymmetry is the possibility of preparing a pure state which is a superposition of states with different charge eigenvalues.

An immediate consequence of this result is that the 1- and 2-components of isotopic spin cannot be observables. This is so because in the known elementary particles the 3-component of the isotopic spin differs from the total charge operator by an additive constant. Thus, the 1- and 2-components of isotopic spin can never commute with the total charge operator and according to the preceding result cannot be observables.

This result, which was known before, is derived here from the assumption that C-number gauge transformations are supersymmetries.

A further conclusion which follows from theorem 2 is that the total charge is an essential observable. Every complete system of observables must contain as one of them the total charge.

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[7]) See ref. [4]).

[8]) The non-trivial half of this theorem was communicated to us by G. TIXAIRE who has proved it with the theory of the direct integral. The proof given here uses only global methods and thereby avoids some measure theoretic complications.

[9]) Ref. [1]), page 69ff.