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Quantum Theory in Real Hilbert Space II (Addenda and Errats)

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(15. II. 1961)

Abstract: A more concise demonstration of the necessity of the thermodynamic signature of the metric $g^{\mu\nu}$ and of the pseudo-chronous character of $\breve{J}$ (corresponding to $i = \sqrt{-1}$) is given (§ 4 of I[1])). Furthermore, the anti-unitary operator (Annex 3 of I) is wrong. A more concise demonstration in terms of observables is given, showing the perfect correspondence between RHS (Real Hilbert Space) and CHS (Complex Hilbert Space).

A more conoise demonstration for the *thermodynamic signature* and the *pseudo-chronous character of* $\breve{J}$ is:

§ 4bis. The Thermodynamic Signature of $g^{\mu\nu}$ and the Pseudo-Chronous Character of $J = \breve{J}$

For a *local scalar observable,* we have the identity ($\alpha\,\beta\ldots = 12\ldots n$):

$$'F('x) = F(x) = F(L^{-1}\,'x) = O^{-1}\,F('x)\,O\,,$$

$$L \leftarrow O\colon\ 'x = L\,x = \{'x'^{\alpha} = L'^{\alpha}{}_{\alpha}(x^{\alpha} + L^{\alpha})\}\,,$$

$$g'^{\mu\,'\nu} = L'^{\mu}{}_{\mu}\,L'^{\nu}{}_{\nu}\,g^{\mu\nu}\,. \tag{4 bis. 1}$$

We form the *bilocal observable*

$$\breve{G}(x\,y) = \breve{J}\,[F(x),\,F(y)] \tag{4 bis. 2}$$

appearing in the UP:

$$\langle \Delta F(x)^2\rangle_{\Psi}\,\langle \Delta F(y)^2\rangle_{\Psi} \geqslant \frac{1}{4}\,\langle \breve{G}(x\,y)\rangle^2_{\Psi}\,. \tag{4 bis. 3}$$

*) Supported by the Swiss National Research Fund.

The hypothesis, that an *L-invariant vacuum state* Ψ^0 exists

$$\langle \breve{G}(x\,y)\rangle_{\Psi^0} = \breve{f}(x\,y) = -\breve{f}(y\,x) \tag{4 bis. 4}$$

requires that

$$'\breve{f}('x\,'y) = c(L)\,\breve{f}(x\,y) = c(L)\,\breve{f}(L^{-1}\,'x\,L^{-1}\,'y) \tag{4 bis. 5}$$

is the same function in every frame, i.e.

$$'\breve{f}('x\,'y) = \breve{f}('x\,'y) \tag{4 bis. 6}$$

because *vacuum is a frame-invariant concept.*
$c(L)$ is a (real) *number*, satisfying the representation condition

$$c(L_2)\,C(L_1) = c(L_2\,L_1)\,. \tag{4 bis. 7}$$

From the identity, following from (4 bis. 5) and (4 bis. 6)

$$c(L)\,\breve{f}(L^{-1}\,'x\,L^{-1}\,'y) = \breve{f}('x\,'y) = -\breve{f}('y\,'x) \tag{4 bis. 8}$$

follows, that $\breve{f}(xy)$ is, except for the factor $c(L)$, an invariant anti-symmetric function. Such a function exists certainly, if we choose a metric

$$\text{signat}\,(g^{\alpha\beta}) = \pm\,(11\ldots1-1)\,. \tag{4 bis. 9}$$

We call $\{x^i\} = \vec{x}$ $(ik\ldots = 12\ldots d,\, d = n-1)$ *space* and $x^n = t$ *time* and define

$$\breve{f}(x\,y) = \text{sig}\,(x^n - y^n)\cdot f\big((x-y)^2\big) = -\breve{f}(y\,x) \tag{4 bis. 10}$$

$$f(z^2) = 0 \quad \text{for} \quad z = x - y = \text{spatial.}$$

We have therefore

$$\begin{aligned} c(L)\,\breve{f}(L^{-1}\,'x\,L^{-1}\,'y) &= c(L)\,\text{sig}\,\big(L^{-1\,n}{}_{'n}('x'^n - 'y'^n)\big)\cdot f\big(('x - 'x)^2\big) \\ &= c(L)\,\text{sig}(L'^n{}_n)\cdot\text{sig}\,('x'^n - 'y'^n)\cdot f\big(('x - 'y)^2\big) \\ &= c(L)\,\text{sig}(L'^n{}_n)\cdot\breve{f}('x\,'y) \equiv \breve{f}('x\,'y)\,, \end{aligned} \tag{4 bis. 11}$$

i.e.

$$c(L) = \text{sig}(L'^n{}_n)\,. \tag{4 bis. 12}$$

This means, that $\breve{f}(x\,y)$ is a *pseudo-chronous number, caracterising the vacuum.* Therefore $\breve{G}(x\,y)$ must be a *pseudochronous observable* i.e.

$$
\begin{aligned}
{}'\breve{G}('x\,'y) &= \operatorname{sig}(L'^{n}{}_{n})\,\breve{G}(x\,y) = \operatorname{sig}(L'^{n}{}_{n})\,\breve{G}(L^{-1}\,'x\,L^{-1}\,'y) = \\
&= O^{-1}\,\breve{G}('x\,'y)\,O = (O^{-1}\,\breve{J}\,O)\,[O^{-1}\,F('x)\,O,\,O^{-1}\,F('y)\,O] = \\
&= (O^{-1}\,\breve{J}\,O)\,[{}'F('x),\,{}'F('y)] = (O^{-1}\,\breve{J}\,O)\,[F(x),\,F(y)] = \\
&= (O^{-1}\,\breve{J}\,O\,\breve{J}^{-1})\,\breve{J}[F(x),\,F(y)] = (O^{-1}\,\breve{J}\,O\,\breve{J}^{-1})\,\breve{G}(x\,y)
\end{aligned}
\tag{4 bis. 13}
$$

where we have made use of (4 bis. 1). Comparing the second and the last member, we have $O^{-1}\,\breve{J}\,O\,\breve{J}^{-1} = \operatorname{sig}(L'^{n}{}_{n}).1$

$$
O^{-1}\,\breve{J}\,O = \operatorname{sig}(L'^{n}{}_{n})\,\breve{J} \equiv {}'\breve{J} \tag{4 bis. 14}
$$

which is equation (4.14) of I. $\breve{J}$ is thus a *pseudochronous operator*. Writing $L_{(\mathrm{ochr})} \leftarrow O_{(\mathrm{ochr})}$ and $L_{(\mathrm{pchr})} \leftarrow O_{(\mathrm{pchr})}$ for *ortho-chronous* ($\operatorname{sig}(L'^{n}{}_{n}) = +1$) and *pseudo-chronous* ($\operatorname{sig}(L'^{n}{}_{n}) = -1$) *Lorentz-transformations*, we have

$$
[O_{(\mathrm{ochr})},\,\breve{J}] = [O_{(\mathrm{ochr})},\,\breve{J}]_{-} = 0\,, \tag{4 bis. 15$_{-}$}
$$

$$
(O_{(\mathrm{pchr})},\,\breve{J}) = [O_{(\mathrm{pchr})},\,\breve{J}]_{+} = 0\,. \tag{4 bis. 15$_{+}$}
$$

We are left to show, that (4 bis. 9) are the only two possible signatures of $g^{\mu\nu}$. To show this, let us assume a metric, in which $z = \{z^{\alpha}\}$ has $d = n - r$ *space components* and $r \leqslant 1/2\,n$ *time components* $\vec{t} = \{t^{a}\} = \{z^{a}\}$ ($i\,k\ldots = 12\ldots n-r$, $a\,b\ldots = (n-r+1)\,(n-r+2)\ldots n$).

We have to build a function $\breve{f}(z) = \breve{f}(\vec{x},\,\vec{t})$ which is *invariant* under *homogeneous continuous transformations* $L_{(\mathrm{cont})}$ (= *proper Lorentz-transformations*) and *changes sign* under *reflections* ($'x = (PT)\,x = -x$). Thus, we look for a vector $z_0 = \{z_0^{\alpha}\}$, which changes sign under the transformation $L = PT$ but does not change sign under any continuous transformation. If $r > 1$, it is evident, at least for *space-like* and *time-like vectors* z, that no such vector z_0 exists, because every vector $z = (\vec{t},\,\vec{z})$ can be transformed into $-z$ by rotations in $\vec{z}$-space and $\vec{t}$-space. For null-vectors ($g_{\alpha\beta}\,z^{\alpha}\,z^{\beta} = 0$) we use the theorem[2]), that in n dimensional space time with $r \leqslant 1/2\,n$, there exist exactly r two-by-two orthogonal linearly independent null vectors z. Therefore, for $r > 1$, a rotation of this r-dimensional 'frame', transformes z into $-z$ (for instance z on the 'frame'). Thus only in the case $r = 1$, time-like vectors and null vectors (on the light cone) exist, which change signe under $L = PT$.

The text of I is correct, if we write, on page 748, under b):

'We pose...' instead of 'We try, posing...' and leave out, on p. 749, the text following (A-3.20). (A-3.20) is the correct formula, the text following (A-3.20) is erroneous. However, a more logical deduction, using only observables $F^X = F^{XT} = F^{\alpha\beta\cdots}(xy\ldots)$ $(X = \{\alpha\,\beta\ldots\ xy\ldots\})$ shall be given below in:

Annex 3bis. Unitary ($\widehat{U}$) and Anti-Unitary ($\widehat{V}$) Operators in CHS

We rewrite the operator identity of (I 3.9) in RHS

$$F'^{X}_{'a\,'b} = L'^{X}{}_{X}\, O_{'a\,a}\, F_{a\,b}\, O^{T}_{b\,'b}; \qquad F^{X}_{a\,b} = F^{X}_{(a\,b)}. \qquad \text{(A-3 bis. 1)}$$

For:

a) *Orthochronous Lorentz Transformations*

We decompose the RHS (a-space) in the direct product of a two-dimensional ((r) (i)-space) and a $\omega_C = (1/2)\,\omega_R$ dimensional p-space, using the two-dimensional matrices (in (r) (i)-space)

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \qquad \text{(A-3 bis. 2)}$$

On account of (4 bis. 15_), we have ($O = O_{(\mathrm{ochr})}$):

$$\begin{aligned} F'^{X}_{'a\,'b} &= 1 \times F'^{X}_{(r)\,'p\,'q} + j \times F'^{X}_{(i)\,'p\,'q} \\ &= L'^{X}{}_{X}\,(1 \times O_{(r)\,'p\,p} + j \times O_{(i)\,'p\,p})\,(1 \times F^{X}_{(r)\,p\,q} + j \times F^{X}_{(i)\,p\,q})\,\cdot \\ &\quad \cdot\,(1 \times O^{T}_{(r)\,q\,'q} - j \times O^{T}_{(i)\,q\,'q}, \end{aligned}$$

$$F^{X}_{(r)\,p\,q} = F^{X}_{(r)\,(p\,q)}; \quad F^{X}_{(i)\,p\,q} = F^{X}_{(i)\,[p\,q]} \qquad \text{(A-3 bis. 3)}$$

formula, which, may be rewritten in CHS, using the correspondence RHS $\rightleftarrows$ CHS, $A \rightleftarrows \widehat{A}$ given in (I A-2.3) $\rightleftarrows$ (I A-2.5) in the form

$$\widehat{F}'^{X}_{'p\,'q} = L'^{X}{}_{X}\,\widehat{O}_{p\,'p}\,\widehat{F}^{X}_{p\,q}\,\widehat{O}^{\dagger}_{q\,'q} = L'^{X}{}_{X}\,\widehat{U}_{'p\,p}\,\widehat{F}^{X}_{p\,q}\,\widehat{U}^{\dagger}_{q\,'q} \qquad \text{(A-3 bis. 4)}$$

with:

$$\widehat{F}^{\dagger} = \widehat{F} = \widehat{F}^{T*} = \widehat{F}^{*T},$$

$$\widehat{U} = \widehat{O}; \quad \widehat{U}^{\dagger}\,\widehat{U} = \widehat{U}\,\widehat{U}^{\dagger} = 1; \quad \widehat{U}^{\dagger} = \widehat{U}^{-1} \qquad \text{(A-3 bis. 5)}$$

where $\widehat{A}^*$ is the *complex conjugate operator* $((\widehat{A}\widehat{B})^* = \widehat{A}^*\widehat{B}^*)$ and $\widehat{A}^\dagger = \widehat{A}^{*T} = \widehat{A}^{T*}$ is the *hermitian conjugate operator* $((\widehat{A}\widehat{B})^\dagger = \widehat{B}^\dagger\widehat{A}^\dagger)$ in CHS. Explicitly written, and omitting the indices $pq\ldots = 12\ldots\omega_C$ referring to a frame in ω_C-dimensional CHS, the identity

$$\widehat{F}'^{\alpha'\beta\cdots}('x\,'y\ldots) = c(L)\; L'^{\alpha}{}_{\alpha}\, L'^{\beta}{}_{\beta}\ldots\widehat{U}\, F^{\alpha\beta\cdots}(L^{-1}\,'x\, L^{-1}\,'y\ldots)\,\widehat{U}^{-1}$$

holds, with $c(L) = 1$, $\operatorname{sig}(\det(L'^{i}{}_{i}))$ and $\operatorname{sig}(\det(L'^{\alpha}{}_{\alpha}))$ for *ortho* $(\widehat{F})$, *pseudo-chronous* $(\overset{\smile}{\widehat{F}})$, *pseudochorous* $(\overset{\frown}{\widehat{F}})$ *and pseudo-*$(\overset{\circ}{\widehat{F}})$ observables. The *transformed operators* are given by

$$\boxed{\begin{aligned} {}'\widehat{F}'^{\alpha'\beta\cdots}('x\,'y\ldots) &= c(L)\; L'^{\alpha}_{(\mathrm{ochr})\,\alpha}\; L'^{\beta}_{(\mathrm{ochr})\,\beta}\ldots\widehat{F}^{\alpha\beta\cdots}(L^{-1}\,'x\, L^{-1}\,'y\ldots) \\ &= \widehat{U}^{-1}\, F'^{\alpha'\beta\cdots}('x\,'y\ldots)\,\widehat{U}\,. \qquad \text{(A-3 bis. 6)} \end{aligned}}$$

They lead to the identity, for expectation values:

$$\begin{aligned} \langle\widehat{\Psi}, {}'\widehat{F}^{\alpha\cdots}('x\ldots)\,\widehat{\Psi}\rangle &= c(L)\; L'^{\alpha}{}_{\alpha}\ldots\langle\widehat{\Psi}, \widehat{F}^{\alpha\cdots}(L^{-1}\,'x\ldots)\,\widehat{\Psi}\rangle \\ &= \langle{}'\widehat{\Psi}, \widehat{F}'^{\alpha\cdots}('x\ldots)\,{}'\widehat{\Psi}\rangle \qquad \text{(A-3 bis. 7)} \end{aligned}$$

with

$$'\widehat{\Psi} = \widehat{U}\,\Psi\,; \qquad L_{(\mathrm{ochr})} \leftarrow \widehat{U}_{(\mathrm{ochr})}\,. \qquad \text{(A-3 bis. 8)}$$

b) *Pseudo-Chronous Lorentz-Transformations*

We need an operator

$$K = k \times 1\,; \quad K^T = K = K^{-1}; \quad K^2 = 1 \qquad \text{(A-3 bis. 8)}$$

which transformes

$$'\breve{J} = K^{-1}\,\breve{J}\,K = -\breve{J}; \quad \breve{J} = j \times 1\,. \qquad \text{(A-3 bis. 9)*)}$$

Such an operator can be given in terms of the pseudoquaternions (I A-4.8). We chose, for exemple, the two-dimensional matrix in (r) (i)-space

$$k = k^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad k^2 = 1,\; k^T = k,\; (k, j) = 0\,. \qquad \text{(A-3 bis. 10)}$$

) In CHS $K \rightleftarrows \widehat{K} : \widehat{K}\, i\, \widehat{K} = -i$ (A-3 bis 9).

The operator $O_{(\mathrm{pchr})} = O$ can now be given in terms of

$$O = O' K; \quad O^T = O^{-1} = K^T O'^T = K O'^T; \quad [O', \breve{J}] = 0,$$
$$O'^T O' = O' O'^T = 1 \qquad \text{(A-3 bis. 11)}$$

and the identity (A-3 bis. 1) can be written, using the decomposition

$$\begin{aligned} F'^X_{'a'b} &= 1 \times F'^X_{(r)'p'q} + j \times F'^X_{(i)'p'q} \\ &= L'^X{}_X (1 \times O'_{(r)p'p} + j \times O'_{(i)'pp}) (k \times 1) (1 \times F^X_{(r)pq} + j \times F^X_{(i)pq}) \cdot \\ &\quad \cdot (k \times 1) (1 \times O'^T_{(r)q'q} - j \times O'^T_{(i)q'q}) \\ &= L'^X{}_X (1 \times O'_{(r)'pp} + j \times O'_{(i)'pp}) (1 \times F^X_{(r)pq} - j \times F^X_{(i)pq}) \cdot \\ &\quad \cdot (1 \times O'^T_{(r)q'q} - j \times O'^T_{(i)'qq}). \end{aligned} \qquad \text{(A-3 bis. 12)}$$

We have the correspondence RHS $\rightleftarrows$ CHS.

$$1 \times F^X_{(r)pq} - j \times F^X_{(i)pq} \rightleftarrows F^X_{pq} - i F^X_{pq} = \widehat{F}^{X*}_{pq}. \qquad \text{(A-3 bis. 13)}$$

Now, using again the RHS $\rightleftarrows$ CHS correspondence (I A-2.3) $\rightleftarrows$ (I A-2.5), we may write (A-3 bis. 12) in CHS, omitting again the p-space indices,

$$\begin{aligned} \widehat{F}'^X = F'^X_{(r)} + i F'^X_{(i)} &= L'^X{}_X \widehat{O}' \widehat{F}^{X*} \widehat{O}'^{-1} \\ &= L'^X{}_X \widehat{U}' \widehat{F}^{X*} \widehat{U}'^{-1}; \quad \widehat{U}'^{-1} = U'^\dagger \end{aligned} \qquad \text{(A-3 bis. 14)}$$

or, explicitly

$$\widehat{F}'^{\alpha'\beta\cdots}('x\,'y\ldots) = c(L)\, L'^\alpha{}_\alpha L'^\beta{}_\beta \ldots \widehat{U}' \widehat{F}^{\alpha\beta\ldots *}(L^{-1}\,'x\, L^{-1}\,'y\ldots) \widehat{U}'^{-1}. \qquad \text{(A-3 bis. 15)}$$

The transformed operator is defined by

$$\boxed{\begin{aligned} '\widehat{F}'^{\alpha\cdots}('x\ldots) &= c(L_{(\mathrm{pchr})})\, L'^\alpha_{(\mathrm{pchr})\alpha} \ldots \widehat{F}^{\alpha\cdots}(L^{-1}\,'x\ldots) \\ &= \widehat{U}'^{-1*} \widehat{F}^{\alpha\ldots *}('x\ldots) \widehat{U}'^*. \end{aligned} \qquad \text{(A-3 bis. 16)}}$$

The identity for expectation values takes the form

$$\begin{aligned} \langle \widehat{\Psi}, '\widehat{F}'^{\alpha\cdots}('x\ldots) \widehat{\Psi}\rangle &= c(L)\, L'^\alpha{}_{\alpha\ldots} \langle \widehat{\Psi}, \widehat{F}^{\alpha\cdots}(L^{-1}\,'x\ldots) \widehat{\Psi}\rangle \\ &= \langle '\widehat{\Psi}, \widehat{F}'^{\alpha\cdots}('x\ldots)\, '\widehat{\Psi}\rangle. \end{aligned} \qquad \text{(A-3 bis. 17)}$$

Defining the anti-unitary operator $\widehat{V}$ by

$$'\widehat{\Psi} = \widehat{V}\,\widehat{\Psi} = \widehat{U}\,\widehat{\Psi}^* \to {}'\Psi_{'p} = U'_{'p\,p}\,\Psi^*_p\,; \quad L_{(\mathrm{pchr})} \leftarrow \widehat{V}\,, \qquad \text{(A-3 bis. 18)*)}$$

we have

$$\langle {}'\widehat{\Psi}, \widehat{F}'^{\alpha\cdots}('x\ldots)\,'\Psi\rangle = c(L)\,L'^{\alpha}{}_{\alpha}\ldots\langle\widehat{\Psi}, \widehat{F}^{\alpha\cdots}(L^{-1}\,'x\ldots)\,\widehat{\Psi}\rangle =$$

$$= \widehat{\Psi}_p\,\widehat{U}'^{T*}_{p\,'p}\,\widehat{F}'^{\alpha\ldots}_{'p\,'q}('x)\,\widehat{U}'_{'q\,q}\,\Psi^*_q = \widehat{\Psi}^*_q\,\widehat{U}'^{T}_{q\,'q}\,\widehat{F}'^{\alpha\ldots T}_{'q\,'p}('x)\,\widehat{U}'^{*}_{'p\,p}\,\widehat{\Psi}_p =$$

$$= \langle\widehat{\Psi}, \widehat{U}'^{\dagger *}\,\widehat{F}'^{\alpha\ldots T}('x)\,\widehat{U}'^{*}\,\widehat{\Psi}\rangle\,. \qquad \text{(A-3 bis. 19)}$$

By definition, $F^{\alpha\cdots}(x\ldots)$ is an observable. Therefore it follows, from (A-3 bis. 5), that in (A-3 bis. 19), we have

$$\widehat{F}'^{\alpha\ldots T}('x\ldots) = \widehat{F}'^{\alpha\ldots *}('x\ldots) \qquad \text{(A-3 bis. 20)}$$

and, from (A-3 bis. 16) ($\widehat{U}'^{\dagger} = U'^{-1}$)

$$\widehat{U}'^{\dagger *}\,\widehat{F}'^{\alpha\ldots T}('x\ldots)\,\widehat{U}'^{*} = c(L)\,L'^{\alpha}{}_{\alpha}\ldots\widehat{F}^{\alpha\cdots}(L^{-1}\,'x\ldots)\,. \qquad \text{(A-3 bis. 21)}$$

Thus, formula (A-3 bis. 19) is identical with the second equation (A-3 bis. 17) and with (A-3 bis. 16).

In order to show the physical significance of (A-3 bis. 16) let us consider the *'real' scalar free field***) considered as an observable:

$$w(x) = w^T(x)\,;\ \widehat{w}(x) = \widehat{w}^{\dagger}(x)\,, \qquad \text{(A-3 bis. 22)}$$

expanded in plane waves (signat $(g^{\alpha\beta}) = (11\ldots 1-1)$).

$$\widehat{w}(x) = (2)^{-1/2}\,(2\,\pi)^{-d/2}\int d\sigma(\breve{k})\,(\widehat{a}(\breve{k})\,e^{i\,(\breve{k},\,x)} + \widehat{a}^{\dagger}(\breve{k})\,e^{-i\,(\breve{k},\,x)})\,,$$

$$\breve{k}^2 + M^2 = 0\,; \quad \breve{k}^n > |M|\,,$$

$$d\sigma(\breve{k}) = (\breve{k}^n)^{-1}\,d^d\,\breve{k}\,; \quad d^d\,\breve{k} = \prod_{i=1}^{d} d\breve{k}^i\,, \qquad \text{(A-3 bis. 23)***)}$$

---

*) $'\widehat{\Psi} = \widehat{U}'\,\widehat{K}\,\Psi = \widehat{U}'\,\widehat{\Psi}^*$.

**) *'Field Quantization in Real Hilbert Space'* will be the object of a forthcoming paper in this journal (referred to as III).

***) The surface integral is to be extended over the *positive shell* of the hyperboloid $\breve{k}^2 + M^2 = 0$. $\breve{k} = \{\breve{k}^\alpha\}$ is therefore a *pseudochronous vector* (see III).

where, if the sum over plane waves is made denumerable, $\widehat{a}(\breve{k})$ and $\widehat{a}^{\dagger}(\breve{k})$ are the usual annihilation and creation operators of quanta in a state $\breve{k}$. Using (A-3 bis. 16) for the transformation $L = PT$

$$'x = (PT)\, x = -\, x\,; \quad '\acute{x}^{\alpha} = -\, \acute{x}^{\alpha} \tag{A-3 bis. 24}$$

we have (A-3 bis. 16):

$$\begin{aligned}
'\widehat{w}('x) &= \widehat{w}(x) \equiv \widehat{w}(-'x) \\
&= (2)^{-1/2}\,(2\,\pi)^{-d/2} \int d\sigma(\breve{k})\, (\widehat{a}(\breve{k})\, e^{-i\,(\breve{k},\,'x)} + \widehat{a}^{\dagger}(\breve{k})\, e^{i\,(\breve{k},\,'x)}) \\
&= \widehat{U}'^{-1*}\, \widehat{w}^{*}('x)\, \widehat{U}'^{*} \\
&= (2)^{-1/2}\,(2\,\pi)^{-d/2} \int d\sigma(\breve{k})\, \big((\widehat{U}'^{*-1}\, \widehat{a}^{*}(\breve{k})\, \widehat{U}'^{*})\, e^{-i\,(\breve{k},\,'x)} + \\
&\quad + \widehat{U}'^{*-1}\, \widehat{a}^{\dagger *}(\breve{k})\, \widehat{U}'^{*})\, e^{i\,(\breve{k},\,'x)}\big)\,.
\end{aligned} \tag{A-3 bis. 25}$$

Thus, we have

$$\widehat{U}'^{-1*}\, \widehat{a}^{*}(\breve{k})\, \widehat{U}'^{*} = \widehat{a}(\breve{k})\,, \tag{A-3 bis. 26}$$

$$\widehat{U}'^{-1*}\, \widehat{a}^{\dagger *}(k)\, \widehat{U}'^{*} = \widehat{a}^{\dagger}(\breve{k})\,. \tag{A-3 bis. 26}$$

As the relations

$$[\widehat{a}(\breve{k}), \widehat{a}^{\dagger}(\breve{k}')] = \delta(\breve{k}, \breve{k}')\,; \quad \delta(\breve{k}, \breve{k}') = \breve{k}^{n}\, \delta(\vec{k} - \vec{k}')\,, \tag{A-3 bis. 27}$$

$$[\widehat{a}(\breve{k}), \widehat{a}(\breve{k}')] = 0\,, \tag{A-3 bis. 28}$$

are invariant, if we go over to the conjugate complex operators, an unitary matrix $\widehat{U}'^{*}$ exists always, satisfying (A-3 bis. 26). In particular, if the annihilation and creation operators are chosen real, $\widehat{U}'^{*} = 1$ is the unit operator.

References

[1]) E. C. G. Stueckelberg, Helv. Phys. Acta *33*, 727 (1960) to be referred to as I.
[2]) R. Jost and M. Guenin, unpublished.