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# Domain of Holomorphy of the Three-Point Function

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Abstract: A new derivation of the results of G. Källén and A. Wightman on the domain of holomorphy of the three-point function is given, using essentially a system of coordinates due to R. Jost, and the tube theorem.

#### 1. Preliminaries

As is well known, the domain of analyticity of the three-fold vacuum expectation value in x-space or in p-space which follows from the following axioms: existence and uniqueness of the vacuum, Lorentz invariance, positiveness of the energy, and local commutativity has been computed by G. Källén and A. Wightman<sup>4</sup>). The purpose of the following note is to present a new derivation of the results of K.-W. using a system of coordinates introduced by R. Jost and a technique of analytic completion essentially based on the tube theorem. These tools allow for a much simplified calculation of the domain of holomorphy of the three-point function\*).

Our starting point will be the conclusion from the 'DANAD' formalism of R. Jost and A. Wightman's) that when one expresses the three-fold vacuum expectation values as functions of the scalar products one obtains the same possible results whether one starts from a four-dimensional space-time or from a two-dimensional space-time.

We are thus led to the study of Wightman functions

$$W_{A\ B\ C}(\zeta_{\bf 3},\ \zeta_{\bf 1})$$

where  $\zeta_3$  and  $\zeta_1$  are two complex two-dimensional vectors.

For reasons of symmetry we put  $\zeta_2 = -(\zeta_3 + \zeta_1)$ .

Let us introduce the characteristic coordinates of  $\zeta_i$ :  $\zeta_i = (z_i, z_i')$ . It may be remarked that if one takes  $z_1 = 1$ , W is a function of the three

<sup>\*)</sup> Another derivation, using different methods has also been given by F. J. Dyson (unpublished).

independent variables  $z_1'$ ,  $z_2$ ,  $z_2'$  which are equivalent to the three scalar products  $\zeta_1^2$ ,  $\zeta_2^2$ ,  $\zeta_3^2$ . It is this new system of variables which simplifies to a large extent the calculations. We will however find it practical to keep  $z_1$  as an independent variable, paying due attention to Lorentz invariance. The vectors  $\zeta_i$  transform according to

$$\Lambda \zeta_i = (\lambda z_i, \lambda^{-1} z_i')$$

under two-dimensional complex Lorentz transformations,  $\lambda$  being an arbitrary complex number.

We adopt the following representation of  $(\zeta_i)$  in the complex plane:  $z_i$  is represented by its affix i and  $z'_i$  by the symmetric i' of its affix with respect to the imaginary axis.

We call  $\Delta$  and  $\Delta'$  respectively the triples (i) and (i'). In the following we will only be interested in the Lorentz transformations with  $\lambda = e^{i\alpha}$ ,  $\alpha$  real. These obviously have the simple effect of turning  $\Delta$  and  $\Delta'$  by an angle  $\alpha$  in the same sense around the origin of the complex plane.

The domain in which we initially know that W is analytic as a consequence of the above-mentioned axioms (the union of the permuted extended tubes) is given by the set of all  $(\Delta, \Delta')$  such that one side of  $\Delta$  and the corresponding side of  $\Delta'$  are in a half-plane limited by a straight line through the origin. This domain is limited by pieces of analytic hypersurfaces which we call, following K.-W., cuts,  $F_{ii'}$  and S, S'.

We will now define these and other pieces of analytic hypersurfaces which will appear in the computation of the holomorphy envelope.

- 1. The *i*-cut appears when *i* and *i'* are collinear with and on opposite sides of the origin. Its equation is  $z_i z_i' = \varrho$ .
- 2.  $F_{ij'}(F'_{ij'})$  appears when i and j' are collinear with and on opposite sides (on the same side) of the origin. Its equation is

$$z_i z_i' = \varrho \qquad (z_i z_i' = -\varrho)$$
.

3. S(S') appears when  $\Delta(\Delta')$  shrinks to a line. Its equation is

$$z_1=\pm\,\varrho\,\,z_2$$
 or  $z_2=\pm\,\varrho\,\,z_3$  or  $z_3=\pm\,\varrho\,\,z_1$   $(z_1'=\pm\,\varrho\,\,z_2')$  or  $z_2'=\pm\,\varrho\,\,z_3'$  or  $z_3'=\pm\,\varrho\,\,z_1')$ 

4.  $\mathfrak{F}(\mathfrak{F}')$  has the equation

$$z_1 z_1' - z_2 z_3' \equiv z_2 z_2' - z_3 z_1' \equiv z_3 z_3' - z_1 z_2' = \varrho$$

$$(z_1 z_1' - z_3 z_2' \equiv z_2 z_2' - z_1 z_3' \equiv z_3 z_3' - z_2 z_1' = \varrho)$$

 $\varrho$  represents in all these definitions a real non-negative parameter.

For the computation of the holomorphy envelope we will essentially use the following lemma: If the funtion  $f(\alpha, \beta, \gamma)$  of three complex variables is analytic in the product P of the three upper half-planes, except maybe for  $\arg \alpha \leqslant \arg \beta \leqslant \arg \gamma$ , then f is analytic in P.

To prove this lemma let us write  $\log \alpha = u_1$ ,  $\log \beta = u_2$ ,  $\log \gamma = u_3$ . Expressed in the variables  $u_i$ , the hypothesis of the lemma is that f is analytic in a certain tube, and the thesis of the lemma is that f is analytic in the convex hull of this tube, a fact which results from the tube theorem (2), Th. 9, p. 92).

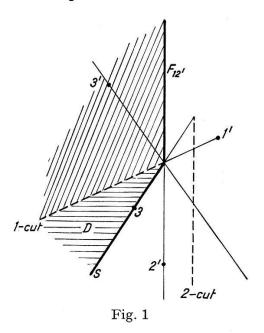
In applications of this lemma, we will not exactly have a tube to start with, because of some extra singularities (cuts). These extra singularities will not however intersect with the singularities to be eliminated. From this it follows that, since the tube theorem may be proved by means of the continuity theorem\*), it may be applied to the latter singularities. This process of course does not eliminate the extra singularities.

## 2. Analytic completion

Following K.-W. we perform the analytic completion in three steps

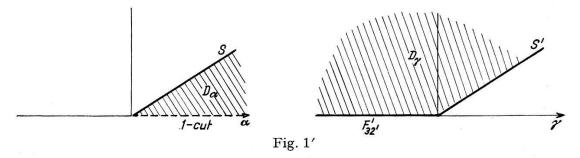
# 1. Continuation through S and S'

Figure 1 represents the original domain of analyticity in the variable  $z_1$ . We are interested in the part D of the excluded domain.



<sup>\*)</sup> Such a proof has been given by R. Jost and was reproduced in <sup>5</sup>). Similar proofs had been given earlier by K. Stein<sup>6</sup>) (for the case of two complex variables, see Hilfsatz 1, p. 557) and by G. Källén and A. Wightman (private communication of Prof. A. Wightman to Prof. R. Jost).

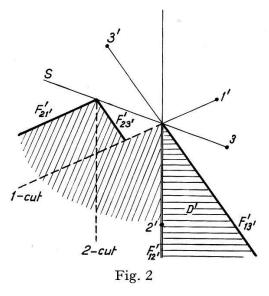
In order to show that D is contained in the holomorphy envelope we apply the lemma with the following variables  $\alpha = z_1 z_1'$ ,  $\beta = z_3 z_1'$ ,  $\gamma = z_2' z_3$ . For fixed  $\beta$ , D is the product of the domains  $D_{\alpha}$  and  $D_{\gamma}$  shown on Fig. 1'.



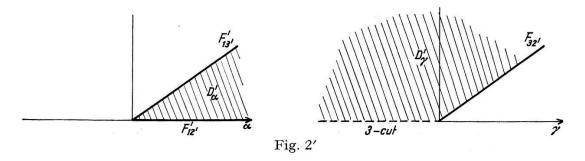
After this partial completion, the new boundary of the domain of analyticity is  $F'_{32}$  because if arg  $\gamma$  becomes bigger than  $\pi$ , the 2-cut intersects with D.

# 2. Continuation through F and F'

Figure 2 represents the new domain of analyticity in the variable  $z_1$ , and we are interested in the part D' of the excluded domain.



In order to show that D' is contained in the holomorphy envelope, we again apply the lemma with the variables  $\alpha = -z_1 z_2'$ ,  $\beta = -z_2'/z_3'$ ,  $\gamma = -1/z_3 z_3'$ . For fixed  $\beta$ , D' is the product of  $D'_{\alpha}$  and  $D'_{\gamma}$  (cf. Fig. 2').



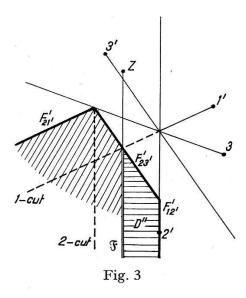
# 3. Final completion

After the second step, Figure 2 becomes Figure 3 and we are interested in the part D'' of the excluded domain.

Figure 3 is essentially identical to Figure 26 of K.-W. Following K.-W. we remark that the point  $z=z_3z_3'/z_2'$  is on the continuation of the bounary  $\mathfrak{F}$  of D''. Adding the equations  $z_1z_1'=\varrho$  and  $z_1z_3'+z_3z_3'=\varrho'$  of the 1-cut and  $F_{23}'$  respectively one finds indeed

$$z_1 = rac{z_3 \, z_3'}{z_2'} - rac{arrho + arrho'}{z_2'}$$

so that F coincides with the hypersurface defined in the preliminaries.



In order to show that D'' is contained in the holomorphy envelope, we apply once more the lemma with the variables

$$lpha = - (z_1 - z) \ z_2' \qquad eta = - \ rac{z_2'}{z_3'} \qquad \gamma = rac{-1}{z_3 \ z_3'} \ .$$

For fixed  $\beta$ , D'' is contained in the product of  $D''_{\alpha}$  and  $D''_{\gamma}$  as depicted in Figure 3' and it is important to remark that the 1-cut and the 2-cut do not intersect with D''.

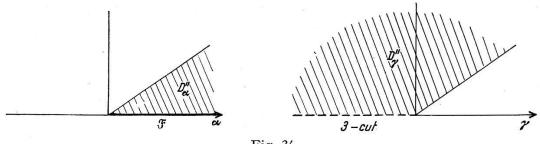


Fig. 3'

### 4. Conclusion

It is easy to show, using the Kantensatz (see <sup>1</sup>), Satz 20, p. 52) that the domain of analyticity finally obtained is a natural domain of holomorphy. It may be described as follows\*) using our  $z_1$ -plots with fixed  $\Delta'$  and  $z_3$ .

- (a) If the 1-cut and the 2-cut do not intersect, they are the only present singularities.
- (b) If the 1-cut and the 2-cut intersect, the situation is depicted either by Figure 3 or by Figure 4.

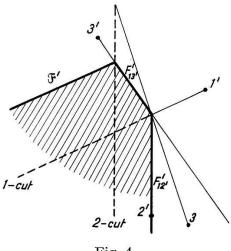


Fig. 4

In conclusion I wish to thank Prof. R. Jost for the suggestion of this problem and many valuable discussions and comments as well as for his hospitality at the Physics Institute of the E.T.H. I am also grateful to the Université Libre de Bruxelles for its financial support during the period in which this work was completed.

## **Bibliography**

- 1) H. Behnke and P. Thullen, Ergebn. der Math. 3, Nr. 3, Berlin 1934.
- <sup>2</sup>) S. Bochner and W. T. Martin, Several Complex Variables, Princeton Mathematical Series No. 10 (1948).
- 3) R. Jost, Lecture notes, Naples (1959).
- 4) G. Källen and A. Wightman, Mat. Fys. Skr. Dan. Vid. Selsk. 1, No. 6 (1958).
- <sup>5</sup>) D. Ruelle, Thèse, Université Libre de Bruxelles (1959).
- 6) K. STEIN, Math. Ann. 114, 543 (1937).
  - \*) Private communication of Prof. R. Jost.