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Autor: Casimir, H.B.G.

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A Note on Multipole Radiation

by **H. B. G. Casimir**

1. Introduction

In the winter of 1932–1933, after attending a colloquium on internal conversion of γ rays, PAULI and I had a discussion on the question whether a periodically varying charge density with zero static dipole moment could be said to emit dipole radiation. For some reason PAULI at that occasion preferred to restrict the word to the case of a non-vanishing static dipole moment, whereas I was applying this term to any radiation field indistinguishable from the field of a radiating dipole outside a sphere enclosing the radiating charges, irrespective of whether the charge distribution had a non-vanishing static dipole moment. I remember that the discussion became quite violent and confused until we realized that we were mainly disagreeing on matters of terminology. Since then the expansion of a radiation field in a series of electric and magnetic multipole fields has been treated and applied in many papers and texts. Some years ago C. J. BOUWKAMP and I published a note*) in which we showed that this expansion can be conveniently obtained by means of the so-called Debye potentials. In that note the use of group theory and of vector harmonics was systematically avoided. While this makes the treatment in a way more 'elementary', yet to those familiar with the fundamental notions of the theory of representations of the rotation group, the true nature of multipole fields is best expressed by their transformation properties just in the same way as the spherical harmonics of degree l are characterized by the property that under a rotation they transform according to a $(2\,l+1)$-dimensional irreducible representation. In the following I shall show that also the group-theoretical point of view leads in a logical and straightforward way to the use of Debye potentials.

2. Vector harmonics and irreducible representations

Let $Y_l^m(\vartheta, \varphi)$ be the normalized surface harmonics, such that

$$\frac{1}{4\,\pi} \int\!\!\int Y_l^{m*}\, Y_{l'}^{m'} \sin\vartheta\, d\,\vartheta\, d\,\varphi = \delta_{l l'}\, \delta_{m m'} \quad ,$$

*) C. J. BOUWKAMP and H. B. G. CASIMIR, On multipole expansions in the theory of electromagnetic radiation, Physica, 's-Grav. *20*, 539–554, 1954; in the following quoted as *loc. cit.*

and T_l^m the corresponding polynomial

$$T_l^m = Y_l^m r^l \quad .$$

We have

$$\Delta T_l^m = 0 \ ,$$

and also

$$\Delta(r^{-2l-1}\, T_l^m) = 0 \, .$$

Under a rotation, Y_l^m and T_l^m transform with an irreducible representation D_l; an inversion leads to multiplication by the factor $(-)^l$. Consider $3(2\,l+1)$ vector fields $\boldsymbol{F}^m_{s,l}$ with $s = 1, 2, 3$; $-\,l \leq m \leq l$,

$$\boldsymbol{F}^m_{s,l} = \boldsymbol{a}(s)\ T_l^m \ ,$$

where $\boldsymbol{a}(s)$ is the unit vector in the direction of the s-axis. Any vector field whose components are homogeneous polynomials of degree l satisfying Laplace's equation is a linear combination of the $\boldsymbol{F}^m_{s,l}$.

The $\boldsymbol{F}^m_{s,l}$ transform under a rotation according to the direct product

$$D_1 \times D_l \ ;$$

inversion gives a factor $(-)^{l+1}$.

Let us now consider the expressions

(a) $$\nabla T^m_{l+1}\ , \ (-\,l-1 \leq m \leq l+1),$$

(b) $$r^{2l+1}\, \nabla(r^{-2l+1}\, T^m_{l-1}) = r^2\, \nabla T^m_{l-1} - (2l-1)\, \boldsymbol{r}\, T^m_{l-1},\ (-l+1 \leq m \leq l-1),$$

(c) $$\operatorname{curl}\, (\boldsymbol{r}\, T_l^m)\ , \qquad (-\,l \leq m \leq l).$$

It is easy to show that the components of each of these fields satisfy Laplace's equation and that they are homogeneous polynomials of degree l. Hence these vector fields are linear combinations of the $\boldsymbol{F}^m_{s,l}$. Moreover, since they are derived by invariant operations from quantities transforming as D_{l+1}, D_{l-1} and D_l, respectively, it follows that they transform according to these same representations and therefore are linearly independent. Since the total number is $3(2\,l+1)$ we have solved the problem of finding those linear combinations of the $\boldsymbol{F}^m_{s,l}$ for which the representation $D_1 \times D_l$ is split into its irreducible constituants and simultaneously we have proved that these constituants are D_{l+1}, D_l and D_{l-1}, in agreement with general theorems of group theory. Under inversion all three fields are multiplied by $(-)^{l-1}$. For (a) and (b) this is evident, for (c) one has to bear in mind that under an inversion curl changes into $-$ curl.

It is satisfactory that the irreducible expressions (a), (b) and (c) can be obtained in a rigorous way without making use of the rather intricate formalism of Clebsch-Gordan coefficients. The following property is worthy of note. If $f(r)$ is an arbitrary function of r, then

$$f(r) \operatorname{curl} (\boldsymbol{r}\, T_l^m) = \operatorname{curl} \left(\boldsymbol{r}\, T_l^m f(r)\right).$$

This is the most general form of a vector field transforming as D_l and with inversion factor $(-)^{l+1}$. Further we have

$$\boldsymbol{r} \cdot \operatorname{curl} \left(\boldsymbol{r}\, T_l^m f(r)\right) = 0,$$

$$\boldsymbol{r} \cdot \operatorname{curl} \operatorname{curl} \left(\boldsymbol{r}\, T_l^m f(r)\right) = l(l+1)\, T_l^m f(r).$$

This identity follows from the fact that T_l^m is a homogeneous polynomial of degree l satisfying Laplace's equation.

The most general vector field transforming as D_l with inversion factor $(-)^l$ is

$$\boldsymbol{F} = h(r)\, r^2\, \nabla T_l^m - g(r)\, (2\,l+1)\, \boldsymbol{r}\, T_l^m\ ,$$

where $h(r)$ and $g(r)$ are arbitrary functions of r. If however $\boldsymbol{F}$ has to satisfy the additional condition div $\boldsymbol{F} = 0$ this can always been written in the form

$$\boldsymbol{F} = \operatorname{curl} \operatorname{curl} (\boldsymbol{r}\, T_l^m f).$$

3. Solutions of Maxwell's equations

Let us now study the solutions of Maxwell's equations in the region of empty space outside a sphere of radius R. Since Maxwell's equations

$$\operatorname{curl} \boldsymbol{H} + i\, k\, \boldsymbol{E} = 0$$

$$\operatorname{curl} \boldsymbol{E} - i\, k\, \boldsymbol{H} = 0$$

are invariant under rotation, any solution is transformed by rotation into another solution. It follows from the completeness theorem of irreducible representations that the most general solution can be obtained by superposition of solutions that transform according to irreducible representations. Now if $\boldsymbol{E}$ transforms as D_l also $\boldsymbol{H}$ transforms as D_l, but if the inversion factor of $\boldsymbol{E}$ is $(-)^l$ that of $\boldsymbol{H}$ is $(-)^{l+1}$ and vice versa.

There are hence two types of irreducible fields

(A)

$\boldsymbol{E}$ transforms as D_l with inversion factor $(-)^{l+1}$. Then

$$\boldsymbol{E} = \operatorname{curl} \left(\boldsymbol{r}\, f(r)\, T_l^m\right),$$

and
$$\boldsymbol{H} = \frac{1}{i\,k} \operatorname{curl} \operatorname{curl} \left(\boldsymbol{r}\, f(r)\, T_l^m\right) \quad ,$$

from which follows

$$\boldsymbol{E} \cdot \boldsymbol{r} = 0 \, , \qquad \boldsymbol{H} \cdot \boldsymbol{r} = \frac{l\,(l+1)}{i\,k}\, f(r)\, T_l^m \, .$$

It follows from Maxwell's equations and the radiation condition at infinity that we can write

$$f(r) = \frac{1}{r^l}\, C\, i\, k\, h_l(k\, r)\, e^{-i\omega t} \quad ,$$

with
$$h_l(k\, r) = \left(\frac{\pi}{2\, k\, r}\right)^{1/2} H_{l+1/2}^{(1)}\,(k\, r) \, .$$

I shall denote these fields (for $C = 1$) by

$${}_M\boldsymbol{E}_l^m \, , \quad {}_M\boldsymbol{H}_l^m \, ;$$

they are the magnetic 2^l-pole fields. Any superposition of such magnetic multipole fields can be written in the form

$$\boldsymbol{E} = i\, k \operatorname{curl}\, (\boldsymbol{r}\, \Pi_2),$$
$$\boldsymbol{H} = \operatorname{curl} \operatorname{curl}\, (\boldsymbol{r}\, \Pi_2).$$

(B)

$\boldsymbol{H}$ transforms as D_l with inversion factor $(-)^{l+1}$. Then

$$\boldsymbol{H} = \operatorname{curl} \left(\boldsymbol{r}\, f(r)\, T_l^m\right),$$
$$\boldsymbol{E} = -\frac{1}{i\,k} \operatorname{curl} \operatorname{curl} \left(\boldsymbol{r}\, f(r)\, T_l^m\right) \quad ,$$

from which follows

$$\boldsymbol{r} \cdot \boldsymbol{H} = 0 \quad , \qquad \boldsymbol{r} \cdot \boldsymbol{E} = -\frac{l(l+1)}{i\,k}\, f(r)\, T_l^m \quad .$$

Again
$$f(r) = \frac{1}{r^l}\, C\, i\, k \left(\frac{\pi}{2\, k\, r}\right)^{1/2} H_{l+1/2}^{(1)}\,(k\, r)\, e^{-i\omega t} \quad .$$

Such fields are called *electric* 2^l-pole fields and will be denoted (for $C = 1$) as

$${}_E\boldsymbol{E}_l^m \, , \; {}_E\boldsymbol{H}_l^m \, .$$

Any superposition of electric multipole fields can be written as

$$\boldsymbol{H} = -\, i\, k \operatorname{curl}\, (\boldsymbol{r}\, \Pi_1),$$
$$\boldsymbol{E} = \operatorname{curl} \operatorname{curl}\, (\boldsymbol{r}\, \Pi_1).$$

Our group-theoretical analysis leads therefore automatically to the Debye potentials.

4. Source representations

The most general solution of Maxwell's equations for $r > R$ is

$$\boldsymbol{E} = \sum_{l,m} a_l^m \, {}_E\boldsymbol{E}_l^m + \sum_{l,m} b_l^m \, {}_M\boldsymbol{E}_l^m \,,$$

$$\boldsymbol{H} = \sum_{l,m} a_l^m \, {}_E\boldsymbol{H}_l^m + \sum_{l,m} b_l^m \, {}_M\boldsymbol{H}_l^m \,.$$

The coefficients can be determined by the method of *loc. cit.* We have

$$\boldsymbol{r} \cdot \boldsymbol{H} = \sum_{l,m} b_l^m \, l\,(l+1)\, \Pi_l^m \,,$$

$$\boldsymbol{r} \cdot \boldsymbol{E} = \sum_{l,m} a_l^m \, l\,(l+1)\, \Pi_l^m \,,$$

where

$$\Pi_l^m = h_l\,(k\,r)\; T_l^m \,.$$

It follows that the scalar quantities $\boldsymbol{r}.\boldsymbol{H}$ and $\boldsymbol{r}.\boldsymbol{E}$ satisfy the equation of D'ALEMBERT, a result that came somewhat as a surprise in *loc. cit.* In *loc. cit.* a source representation for these quantities was obtained by expansion of explicit expressions for $\boldsymbol{r}.\boldsymbol{H}$ and $\boldsymbol{r}.\boldsymbol{E}$ obtained from the differential equations

$$\boldsymbol{r} \cdot \boldsymbol{H} = (i\,k/c) \sum_{l,m} \Pi_l^m \int \boldsymbol{i} \cdot \mathrm{curl}' \, (\boldsymbol{r}' \, \Pi_l^{m*}) dV',$$

$$\boldsymbol{r} \cdot \boldsymbol{E} = -\,(1/c) \sum_{l,m} \Pi_l^m \int \boldsymbol{i} \cdot \mathrm{curl}' \, \mathrm{curl}' \, (\boldsymbol{r}' \, \Pi_l^{m*}) \, d\,V',$$

from which we obtain immediately

$$a_l^m = -\frac{1}{c}\,\frac{1}{l\,(l+1)} \int \boldsymbol{i}.\mathrm{curl}' \, \mathrm{curl}' \left(\boldsymbol{r}' \, j_l\,(k\,r')\; Y_l^m\,(\vartheta', \varphi')\right) dV',$$

$$b_l^m = \frac{i\,k}{c}\,\frac{1}{l\,(l+1)} \int \boldsymbol{i}.\mathrm{curl} \left(\boldsymbol{r}' \, j_l\,(k\,r')\; Y_l^m\,(\vartheta', \varphi')\right) dV',$$

where we have followed the notations of *loc. cit.*:

$$j_l(x) = \left(\frac{\pi}{2\,x}\right)^{1/2} J_{l+1/2}\,(x) \,.$$

Of course, the general form of the integrand in these expressions follows also from group-theoretical arguments. Group theory does not provide us with the constants of proportion, although it does indicate the existence and general form of the addition theorem that is at the basis of the expansions for $\boldsymbol{r}.\boldsymbol{E}$ and $\boldsymbol{r}.\boldsymbol{H}$.

Let us now return to the question mentioned at the beginning of this note. It seems reasonable to say that a charge-current distribution emits pure electric dipole radiation if $b_l^m = 0$ for all l, m and $a_l^m = 0$ for $l \neq 1$. In that case the radiation field for $r > R$ is identical with the field of a radiating dipole. This condition is satisfied if $\boldsymbol{i}$ transforms as D_1 with inversion factor -1. For a dipole field with axis in the z-direction the most general current distribution of this type is

$$\boldsymbol{i} = \{g(r) + h(r)\}\, \boldsymbol{e}_z - 3\, \boldsymbol{r} . z\, h(r)/r^2,$$

where $g(r)$ and $h(r)$ are zero for $r \geq R$ but are otherwise arbitrary continuous functions of r.

The static dipole moment is given by

$$q_{stat} = -\frac{1}{i\,\omega} \iiint_{r<R} i_z\, dV = -\frac{4\,\pi}{i\,\omega} \int_0^R g(r)\, r^2\, dr\,.$$

If $g_0(r)$ is a function satisfying

$$g_0(r) = 0 \quad \text{for} \quad r \geq R \quad \text{and} \quad \int_0^R g_0(r)\, r^2\, dr = 0,$$

then

$$\boldsymbol{i} = \{g_0(r) + h(r)\}\, \boldsymbol{e}_z - 3\, \boldsymbol{r}\, z\, h(r)/r^2$$

is the most general current distribution having no static dipole moment and emitting pure electric dipole radiation. Exactly the same radiation field would be emitted by a dipole of infinitesimal dimensions with a dipole moment

$$q_{eq} = -\frac{4\,\pi}{i\,\omega} \int \Big[\frac{\sin k\,r}{k\,r}\, g_0(r) + \\ + \frac{1}{k\,r}\Big\{\sin k\,r - 3\Big(\frac{\sin k\,r}{k^2\,r^2} - \frac{\cos kr}{kr}\Big)\Big\}\, h(r)\Big]\, r^2\, dr\,.$$