Integral Representations for the Vacuum Expectation Value of Three Scalar Local Fields

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Summary. The integral formula of S. BERGMAN and A. WEIL for the representation of an analytic function of several complex variables is applied to the vacuum expectation value of three scalar local fields. Several versions of the result are given. The various formulae are connected with each other by integral transformations (Hankel transforms). One of the versions reads

$$F(z) = \int_0^\infty d^3a \prod_{k=1}^3 \Delta_2^{(+)}(z_k, a_k) \int_0^\infty d\nu \sum_{i=0}^3 \psi_i(a, \nu) f_i(z, \nu),$$

$$f_1(z, \nu) = \frac{\log(-\nu z_1) - \log(\nu - z_2) - \log(\nu - z_3)}{\nu z_1 + (\nu - z_2)(\nu - z_3)} \quad \text{cycl.},$$

$$f_0(z, \nu) = \frac{\log(\nu - z_1) + \log(\nu - z_2) + \log(\nu - z_3) - \log(-z_1) - \log(-z_2) - \log(-z_3)}{\nu^2 - \nu(z_1 + z_2 + z_3) + z_1 z_2 + z_1 z_3 + z_2 z_3}.$$

The functions $\psi_i(a, \nu)$ are arbitrary weight functions. It is assumed that both the weights and the function $F(z)$ are sufficiently bounded at infinity. The three terms above with $i = 1$, 2 or 3 can be related to simple Feynman diagrams, while we have not been able to find such an interpretation of the remaining term.

Introduction

Consider the vacuum expectation value of three scalar fields $A(x)$, $B(x')$ and $C(x'')$

$$F^{ABC}(x - x', x' - x'') = \langle 0 \mid A(x) B(x') C(x'') \mid 0 \rangle =$$

$$= \frac{1}{(2\pi)^6} \iint dp\, dp'\, e^{ip(x-x') + ip'(x'-x'')} G^{ABC}(p^2, p'^2, (p - p')^2) \Theta(p) \Theta(p'). \tag{1}$$

The physical requirement that every eigenvalue of the energy momentum operators of the theory is time-like with positive energy implies that the function $G^{ABC}(p^2, p'^2, (p - p')^2)$ is different from zero only when both p^2 and p'^2 are negative (we use the metric with $p^2 = \overline{p}^2 - p_0^2$, etc.) and when $(p - p')^2 > ((-p^2)^{1/2} - (-p'^2)^{1/2})^2$. These properties of G ensure that the function F is the boundary value of an analytic function $F^{ABC}(z_k)$ regular in a domain bounded by the following analytic hypersurfaces[1][2])

(i) The z_1-cut and the z_2-cut, i.e. the two analytic hypersurfaces $z_k = r$; $r > 0$ and $k = 1$ or 2.

(ii) The analytic hypersurface F_{12} defined by $z_3 = z_1 + z_2 + r + (z_1 z_2)/r$; $r > 0$ (relevant only when Im $z_1 \cdot$ Im $z_2 \geqslant 0$).

(iii) The analytic hypersurface S defined by $z_3 = z_1(1 - k) + z_2(1 - 1/k)$; $k > 0$ (relevant only when Im $z_1 \cdot$ Im $z_2 \leqslant 0$).

Here, $z_1 = -(x - x')^2$, $z_2 = -(x - x'')^2$ and $z_3 = -(x' - x'')^2$ and the physical values of F are obtained as the boundary values for $x - x'$ and $x' - x''$ having infinitesimal imaginary parts in the back ward light cone. The analyticity domain for F is the intersection of the domains bounded by these hypersurfaces and such that the points with $z_k = -x_k$ and $\lambda(x) = x_1^2 + x_2^2 + x_3^2 - 2 x_1 x_2 - 2 x_2 x_3 - 2 x_1 x_3 < 0$ are inside the domain. Obviously, this domain is a natural domain of holomorphy[2]. If we consider a 'local' theory, i.e. a theory where all field operators commute for space-like separations, the analyticity domain of F is much larger than the domain described above. This large domain is also known and is bounded by the following analytic hypersurfaces[1].

(iv) The z_k-cuts with $k = 1, 2$ or 3.

(v) The analytic hypersurfaces F'_{kl} defined by $\sqrt{-r z_m} + \sqrt{r - z_k} \cdot \sqrt{r - z_l} = 0$. Here, r is a positive, real number. All square roots are defined to have positive real parts. This convention is easily shown to be equivalent to the sign conventions for the imaginary parts of z_k given in KW. Further, k, l, and m are all different, and range over 1, 2 and 3 in any permutation.

(vi) The analytic hypersurface $\mathfrak{F}$ defined by $\sqrt{r - z_1} \cdot \sqrt{r - z_2} \cdot \sqrt{r - z_3} + \sqrt{-z_1} \cdot \sqrt{-z_2} \cdot \sqrt{-z_3} = 0$; $r > 0$. The square roots are defined as above.

As earlier, our domain is the intersection of the domains bounded by these hypersurfaces and such that every point inside the earlier domain is also inside the new domain. For brevity, we call the domain described in (i), (ii) and (iii) $\mathfrak{M}$ and the domain described in (iv), (v) and (vi) $\mathfrak{U}$. It can be shown that $\mathfrak{U}$ is the holomorphy envelope of the union of $\mathfrak{M}$ and two other domains obtained from $\mathfrak{M}$ by permutations of the variables z_k[1].

If a function G^{ABC} fulfilling the conditions mentioned above concerning p^2, p'^2 and $(p - p')^2$ but otherwise arbitrary is substituted in (1), the function F^{ABC} obtained in this way is, in general, analytic in $\mathfrak{M}$ but not in $\mathfrak{U}$. In this paper we want to give a representation of the most general function analytic in $\mathfrak{U}$ (and sufficiently bounded at infinity). This representation must contain a completely arbitrary weight function integrated over a certain kernel. Every function analytic in $\mathfrak{U}$ must correspond to at least one possible weight function. Conversely, every weight must give a function analytic in $\mathfrak{U}$.

I. The Bergman-Weil Integral

There is one systematic approach available for a representation of the kind indicated above, viz. the integral representation of S. BERGMAN and A. WEIL for a function of several complex variables, analytic in a domain bounded by analytic hypersurfaces. This integral is a generalization of the conventional Cauchy representation and reads for n variables z_k [3])

$$F(z) = \frac{1}{(2\pi i)^n} \int \cdots \int d\zeta_1 \ldots d\zeta_n \, F(\zeta) \, D(z, \zeta), \qquad (2)$$

$$D(z, \zeta) = \text{Det} \, | \, q_l^k(z, \zeta) \, |. \qquad (2a)$$

The integration in (2) is performed over an n-dimensional manifold, which is a subset of the $2n-1$ dimensional boundary of the domain. This manifold, which is called the 'distinguished boundary', consists of the *common intersection* of n out of all the analytic hypersurfaces that make up the boundary of the analyticity domain. In general, this manifold is made up of several disconnected parts and the integral in (2) is the sum over all such pieces. To each boundary surface labelled by the index k is attached a set of n-functions $q_l^k(z, \zeta)$ fulfilling the identity

$$\sum_{l=1}^{n} (\zeta_l - z_l) \, q_l^k(z, \zeta) = 1, \qquad (3)$$

as long as ζ lies on the k:th analytic hypersurface. Further, all the q_l^k have to be analytic functions of z as long as the point z lies *inside* the analyticity domain and ζ lies on the k:th boundary surface. The existence of such functions q_l^k can be generally proved when the boundary is made up of pieces of analytic hypersurfaces. For the particular boundary we have here, a set of functions q_l^k is explicitly constructed below. Perhaps it is worth while mentioning that these functions are not unique (for $n > 1$) but that there exists an infinite number of sets of functions q_l^k for a given domain. Finally, $F(\zeta)$ in (2) is the boundary value of the function $F(z)$ when z approaches the point ζ on the distinguished boundary.

It can be remarked that the formula (2) reduces to the conventional Cauchy integral for the special case $n = 1$. In that case, the sum in (2) consists of only one term and q is uniquely given as $(\zeta - z)^{-1}$. Therefore, (2) becomes just a conventional Cauchy integral. We also remark that in the case when the analyticity domain can be described as a 'topological product' of n one dimensional domains, Eq. (2) reduces to an iterated Cauchy integral. In this case, every boundary surface depends on only one of the n variables z (cf. the boundaries (i) and (iv) in the introduction) and it is possible to choose $q_l^k = \delta_{kl}(\zeta_l - z_l)^{-1}$. Consequently, the determinant (2a) is a product of simple Cauchy denominators and Eq. (2) is

just an n-fold Cauchy integral. In this way, we can understand (2), (2a) and (3) as a generalization of the Cauchy integral to the case when the domain is bounded by general analytic hypersurfaces.

We devote the rest of this section to an explicit construction of functions q_l^k for the boundary surfaces of $\mathfrak{U}^4$).

For the cuts $z_k = r$ we can use the result above and choose $q_l^k = \delta_{kl}(\zeta_l - z_l)^{-1}$.

For the boundaries F_{kl}' we proceed in the following way. When the point ζ lies on the surface we have

$$\sqrt{-r\zeta_m} + \sqrt{r - \zeta_k} \cdot \sqrt{r - \zeta_l} = 0; \tag{4}$$

(Here, and hereafter in this paper m, k and l are all different and range over 1, 2 and 3 in any permutation.)

and, consequently,

$$\left.\begin{aligned}
\sqrt{-rz_m} + \sqrt{r - z_k} \cdot \sqrt{r - z_l} - \sqrt{-r\zeta_m} - \sqrt{r - \zeta_k} \cdot \sqrt{r - \zeta_l} = \\
= \sqrt{-rz_m} + \sqrt{r - z_k} \cdot \sqrt{r - z_l}.
\end{aligned}\right\} \tag{5}$$

We can think of the surface as being parametrized by *two* of the three complex numbers ζ and by the real number r. The remaining coordinate ζ is then a function of the given numbers as shown in Eq. (4). We then proceed to write the left hand side of (5) as

$$\left.\begin{aligned}
&\frac{r(\zeta_m - z_m)}{\sqrt{-rz_m} + \sqrt{-r\zeta_m}} + \frac{1}{2}\left(\sqrt{r - z_k} - \sqrt{r - \zeta_k}\right)\left(\sqrt{r - z_l} + \right. \\
&\left. + \sqrt{r - \zeta_l}\right) + \frac{1}{2}\left(\sqrt{r - z_l} - \sqrt{r - \zeta_l}\right)\left(\sqrt{r - z_k} + \sqrt{r - \zeta_k}\right) = \\
&= (\zeta_m - z_m)\frac{r}{\sqrt{-rz_m} + \sqrt{-r\zeta_m}} + (\zeta_k - z_k)\frac{1}{2}\frac{\sqrt{r - z_l} + \sqrt{r - \zeta_l}}{\sqrt{r - z_k} + \sqrt{r - \zeta_k}} + \\
&+ (\zeta_l - z_l)\frac{1}{2}\frac{\sqrt{r - z_k} + \sqrt{r - \zeta_k}}{\sqrt{r - z_l} + \sqrt{r - \zeta_l}}.
\end{aligned}\right\} \tag{6}$$

Combining (5) and (6) we find, that it is possible to choose

$$q_m^{kl} = \frac{1}{N_{kl}}\frac{\sqrt{r}}{\sqrt{-z_m} + \sqrt{-\zeta_m}}, \tag{7a}$$

$$q_k^{kl} = \frac{1}{N_{kl}}\frac{1}{2}\frac{\sqrt{r - z_l} + \sqrt{r - \zeta_l}}{\sqrt{r - z_k} + \sqrt{r - \zeta_k}}, \tag{7b}$$

$$q_l^{kl} = \frac{1}{N_{kl}}\frac{1}{2}\frac{\sqrt{r - z_k} + \sqrt{r - \zeta_k}}{\sqrt{r - z_l} + \sqrt{r - \zeta_l}}, \tag{7c}$$

$$N_{kl} = \sqrt{-rz_m} + \sqrt{r - z_k} \cdot \sqrt{r - z_l}. \tag{7d}$$

Obviously, this result is not unique. It is, e.g., possible to modify the numerator in (7b) from $1/2\,(\sqrt{r-z_l}+\sqrt{r-\zeta_l})$ to $\sqrt{r-z_l}$ if, at the same time, the numerator in (7c) is changed to $\sqrt{r-\zeta_k}$. However, the choice (7a)–(7d) above is the most symmetric expression that can be obtained in this way. (Other possible functions q_k of entirely different form, e.g., utilizing logarithms instead of square roots, can also be given.)

An entirely similar argument gives the following expressions for the $q_k^{\mathfrak{F}}$ related to the surface $\mathfrak{F}$ mentioned under (vi) in the introduction

$$
\begin{aligned}
q_k^{\mathfrak{F}} = \frac{1}{6\,N_{\mathfrak{F}}} &\left\{ \frac{1}{\sqrt{r-z_k}+\sqrt{r-\zeta_k}} \left[2\left(\sqrt{r-z_l}\sqrt{r-z_m}+\right.\right.\right. \\
&\left.+\sqrt{r-\zeta_l}\sqrt{r-\zeta_m}\right)+\sqrt{r-\zeta_l}\sqrt{r-z_m}+\sqrt{r-z_l}\sqrt{r-\zeta_m}\Big]+ \\
&+\frac{1}{\sqrt{-z_k}+\sqrt{-\zeta_k}}\left[2\left(\sqrt{-z_l}\sqrt{-z_m}+\sqrt{-\zeta_l}\sqrt{-\zeta_m}\right)+\right. \\
&\left.\left.+\sqrt{-z_l}\sqrt{-\zeta_m}+\sqrt{-\zeta_l}\sqrt{-z_m}\right]\right\},
\end{aligned}
\tag{8a}
$$

$$
N_{\mathfrak{F}}=\sqrt{r-z_1}\sqrt{r-z_2}\sqrt{r-z_3}+\sqrt{-z_1}\sqrt{-z_2}\sqrt{-z_3}. \tag{8b}
$$

As above, other expressions which are less complicated from the algebraic point of view but also less symmetric in the variables z and ζ can be found.

II. The Distinguished Boundary of $\mathfrak{U}$ and the Bergman-Weil Integral for $F(z)$

The distinguished boundary of our domain is defined as the common three dimensional intersection of three out of the seven analytic hypersurfaces that constitute our boundary. At this point, a complication arises. E.g., if we ask for the intersection of the z_1-cut, the z_2-cut and one of the surfaces F_{kl}', we find that z_3 is also real on this intersection and that it may be positive. This means, that *four* of our boundary surfaces have a common three dimensional intersection. Normally, this intersection should only have been two dimensional. Therefore, the BERGMAN-WEIL integral formula is not directly applicable in the form (2). We handle this complication by displacing the surfaces F_{kl}' slightly, so as to disentangle these degenerate intersections. Next, we write down the BERGMAN-WEIL integral for the modified domain and, afterwards, go to the limit where the surfaces F_{kl}' are put back in their original positions[5].

We then remark, that, e.g., the surface F_{12}' is relevant only when the imaginary parts of z_1 and z_2 have the same sign while the imaginary part of z_3 has the opposite sign. Further, when this is the case, F_{12}' never intersects any other boundary surface except the cuts[1]. Finally, there are

really two disconnected pieces of F'_{12} in the boundary, one piece with the imaginary part of, say, z_1 positive and the other piece with the imaginary part of z_1 negative. As these two pieces are entirely disconnected, we are at liberty to displace the surface F'_{12} in different ways in the two cases. We choose to take as our displaced surface

$$z_3 (1 + i\varepsilon) = z_1 + z_2 - r - \frac{z_1 z_2}{r} \; ; \quad \text{Im } z_1 \cdot \text{Im } z_2 > 0,$$

$$\text{Im } z_1 \cdot \text{Im } z_3 < 0, \quad \text{Im } z_1 \cdot \text{Im } [z_3 (1 + i\varepsilon)] < 0, \tag{9}$$

where the real number ε is positive when Im z_1 is negative and negative otherwise[6]). The intersection of this displaced surface and the z_1- and z_2-cuts is shown in Figure 1 for the case, that Im $\zeta_1 > 0$. The relevant part
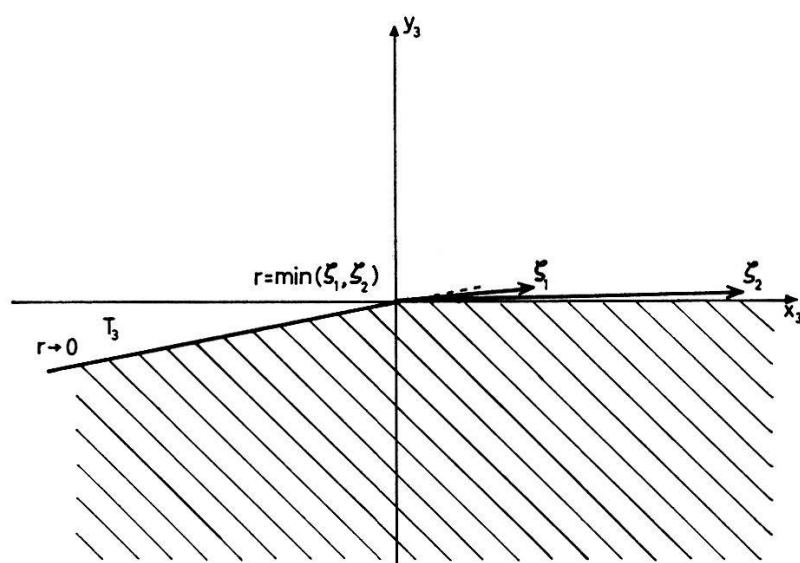
Fig. 1

The intersection ζ_1-cut $\cap$ ζ_2-cut $\cap$ F'_{12} shown in the ζ_3-plane

of the surface F'_{12} is here given by a straight line from the origin and into the lower half plane with a slope given by $-(1 + i\varepsilon)^{-1}$. As the two q's belonging to the two cuts are diagonal, the determinant in (2a) reduces to the simple product

$$D(z, \zeta) = \frac{1}{(\zeta_1 - z_1)(\zeta_2 - z_2)} q_3^{12}(z; \zeta; r), \tag{10}$$

$$r = -\frac{1}{2} [\zeta_3 - \zeta_1 - \zeta_2 + \sqrt{\lambda(\zeta)}], \tag{10a}$$

$$\lambda(\zeta) = (\zeta_3 - \zeta_1 - \zeta_2)^2 - 4 \zeta_1 \zeta_2, \tag{10b}$$

where we have explicitly indicated the dependence of q_3^{12} on r. Note, in particular, the sign in front of the square root in (10a). In the limit when

ε goes to zero we get the following contribution to the BERGMAN-WEIL integral

$$I = \frac{1}{(2\pi i)^3} \int\limits_{T_3} \frac{d^3\zeta}{(\zeta_1 - z_1)(\zeta_2 - z_2)} q_3^{12}(z; +++; r) F(+++). \tag{11}$$

The symbol $F(+++)$ indicates the boundary value of $F(z)$ which is obtained when all three complex numbers ζ_k have an infinitesimal, positive imaginary part. The three plus signs in q_3^{12} have a similar meaning for the variables ζ. The domain of integration in (11) is given by ζ_1 and ζ_2 each ranging independently over the positive real axis from 0 to ∞, while ζ_3 ranges over the negative real axis from $-\infty$ to 0. We denote this domain by T_3 and are later going to use the notations T_1 and T_2 for domains defined in an analogous way. In a similar way we also get a contribution from the other piece of F'_{12} where all signs of the imaginary parts are changed. However, this term appears with a negative sign as all complex integrations have to be done in a positive sense. The total contribution I_{12}^{12} from this part of the distinguished boundary, viz. the common intersection of F'_{12}, the z_1-cut and the z_2-cut becomes

$$I_{12}^{12} = \frac{1}{(2\pi i)^3} \int\limits_{T_3} \frac{d^3\zeta}{(\zeta_1 - z_1)(\zeta_2 - z_2)} [q_3^{12}(z; +++; r) F(+++) - \left.\right\} \tag{12}$$
$$\left.- q_3^{12}(z; ---; r) F(---)].\right\}$$

For the particular function q_3^{12} given in Eq. (7a), the boundary value does not depend on the way in which the variables ζ_k approach their boundaries (at least not as long as $\zeta \in T_3$). Therefore, Eq. (12) can be simplified to

$$I_{12}^{12} = \frac{1}{(2\pi i)^3} \int\limits_{T_3} \frac{d^3\zeta}{(\zeta_1 - z_1)(\zeta_2 - z_2)} q_3^{12}(z; \zeta; r) [F(+++) - F(---)]. \tag{12a}$$

The expression (12) is always correct while (12a) may be wrong for another set of functions q_k^{12}.

We next consider the intersection z_1-cut $\cap$ z_3-cut $\cap$ F'_{12}. Two typical cases of this intersection are shown in Figure 2 and Figure 3. A detailed discussion similar to the one given above yields the following contribution from this part of the distinguished boundary

$$I_{13}^{12} = \frac{1}{(2\pi i)^3} \int\limits_{T_2+\mathfrak{D}_1} \frac{d^3\zeta}{(\zeta_1 - z_1)(\zeta_3 - z_3)} [q_2^{12}(z; ++-; r') F(++-; r') - \left.\right\}$$
$$- q_2^{12}(z; --+; r') F(--+; r')] - \left.\right\} \tag{13}$$
$$- \frac{1}{(2\pi i)^3} \int\limits_{\mathfrak{D}_1} \frac{d^3\zeta}{(\zeta_1 - z_1)(\zeta_3 - z_3)} [q_2^{12}(z; ++-; r) F(++-; r) - \left.\right\}$$
$$- q_2^{12}(z; --+; r) F(--+; r)], \left.\right\}$$

$$r' = -\frac{1}{2}[\zeta_3 - \zeta_1 - \zeta_2 - \sqrt{\lambda(\zeta)}] = \frac{\zeta_1 \zeta_2}{r} > r. \tag{13a}$$

The first term in Eq. (13) contains an integration over the domain T_2 defined above and an integration over the domain $\mathfrak{D}_1$ defined as follows. In $\mathfrak{D}_1$ all the ζ_k are positive and fulfill the inequality $\zeta_1 > (\sqrt{\zeta_2} + \sqrt{\zeta_3})^2$. Other domains $\mathfrak{D}_k$ are defined in a similar way and used below. The appearance of the domain T_2 in the integration requires no further ex-
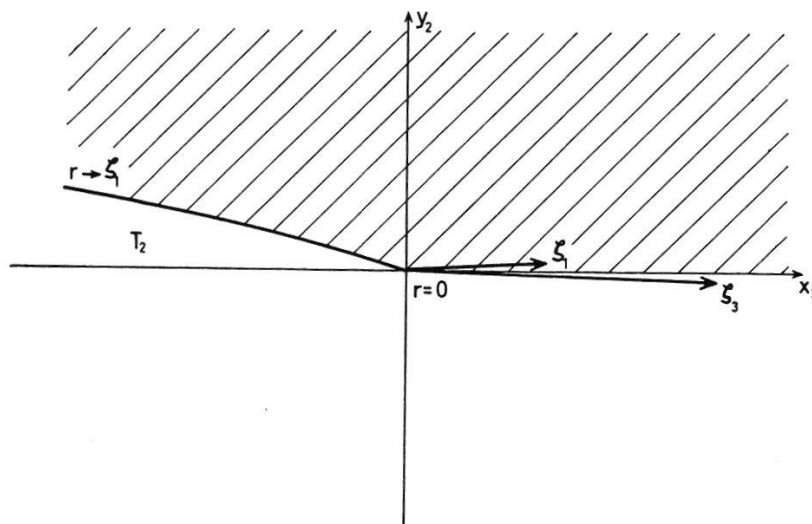
Fig. 2

The intersection ζ_1-cut $\cap$ ζ_3-cut $\cap$ F'_{12} shown in the ζ_2-plane for the case
Re $\zeta_1 <$ Re ζ_3;   Im $\zeta_1 > 0$
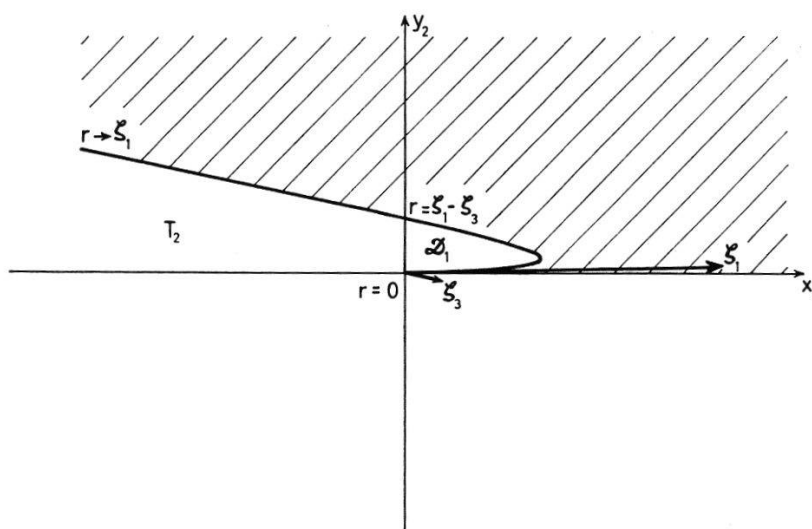
Fig. 3

The intersection ζ_1-cut $\cap$ ζ_3-cut $\cap$ F'_{12} shown in the ζ_2-plane for the case
Re $\zeta_1 >$ Re ζ_3;   Im $\zeta_1 > 0$

planation. The appearance of this term is clearly shown both in Figure 2 and Figure 3. The domain $\mathfrak{D}_1$ is shown as a 'sling' in Figure 3 for $\zeta_1 > \zeta_3$. As is seen in Figure 3, this domain appears twice in the integration, one contribution corresponding to the upper 'shore' in Figure 3 and the other

contribution steming from the lower part of the sling. The values of r for the two parts differ in the sign in front of the square root $\sqrt{\lambda(\zeta)}$ in Eqs. (13a) and (10a). Obviously, the upper part of the sling corresponds to the bigger value of r (cf. Fig. 3). The second integral in Eq. (13) corresponds to the lower part of the sling. In Eq. (13) we have denoted the two boundary values of $F(z)$ with $F(++-;r')$ and $F(++-;r)$ respectively. This is done to take into consideration the possibility that the function $F(z)$ might have different boundary values on the two parts of the sling. In practical applications, this might not be the case. When the boundary value of $F(z)$ is the same on both cases, Eq. (13) can be somewhat simplified in so far as the terms containing q_2^{12} with the same value of ζ but different values of r can be lumped together.

For the particular functions q_m^{kl} shown in Eqs. (7) it is not possible to simplify Eq. (13) in a way similar to the simplification which led from Eq. (12) to Eq. (12a). In $\mathfrak{D}_1$ one has $\zeta_2 < r < \zeta_1$. Therefore, the square root $\sqrt{r - \zeta_1}$ in the denominator has different values when ζ_1 is located on different sides of its cut.

The two expressions (12) and (13) can be considered as typical examples of contributions $I_{k'l'}^{kl}$ to the BERGMAN-WEIL integral. All in all, there are nine integrals of this kind. By symmetry arguments one finds that they can be written in the following two standard forms

$$
I_{kl}^{kl} = \frac{1}{(2\pi i)^3} \int_{T_m} \frac{d^3\zeta}{(\zeta_k - z_k)(\zeta_l - z_l)} \left[q_m^{kl}(z; \zeta_+; r) F(\zeta_+) - q_m^{kl}(z; \zeta_-; r) F(\zeta_-) \right], \tag{14a}
$$

and

$$
\begin{aligned}
I_{km}^{kl} = {} & \frac{1}{(2\pi i)^3} \int_{T_l + \mathfrak{D}_k} \frac{d^3\zeta}{(\zeta_k - z_k)(\zeta_m - z_m)} \left[q_l^{kl}(z; \zeta_+; r') F(\zeta_+; r') - q_l^{kl}(z; \zeta_-; r') F(\zeta_-; r') \right] - \\
& - \frac{1}{(2\pi i)^3} \int_{\mathfrak{D}_k} \frac{d^3\zeta}{(\zeta_k - z_k)(\zeta_m - z_m)} \left[q_l^{kl}(z; \zeta_+; r) F(\zeta_+; r) - q_l^{kl}(z; \zeta_-; r) F(\zeta_-; r) \right].
\end{aligned} \tag{14b}
$$

The symbol ζ_+ in (14a) and (14b) indicates that the point ζ approaches the distinguished boundary in such a way that ζ_k and ζ_l both lie above the real axis while ζ_m is located just below the real axis. For ζ_- the situation is reversed. When the integration goes over the domain T_l it does not matter from which side ζ_l approaches the negative real axis.

We now turn to a discussion of that part of the distinguished boundary which is related to the surface $\mathfrak{F}$. For $\mathfrak{F}$ to be relevant one has to have the same sign for all imaginary parts of z_k. As before, the two cases when all imaginary parts are positive and when they all are negative, correspond

to two disconnected pieces of the boundary. In this case also, we have the complication that the surface $\mathfrak{F}$ and the three cuts have a common, three-dimensional intersection. Therefore, we displace $\mathfrak{F}$ as before and do it in different ways in the two cases. We find it convenient to define our displaced surface from

$$z_3(1 + i\varepsilon) = \frac{(r - z_1)\,(r - z_2)}{r - z_1 - z_2} \; ; \quad \mathrm{Im}\, z_k \cdot \mathrm{Im}\, z_1 > 0,$$

$$\mathrm{Im}\,[z_3\,(1 + i\varepsilon)] \cdot \mathrm{Im}\, z_1 > 0, \tag{15}$$

where the sign of ε is the same as the sign of the imaginary parts of z_k.

The different intersections z_k-cut $\cap$ z_l-cut $\cap$ $\mathfrak{F}$ are illustrated in Figures 4–6. A discussion, entirely similar to the argument above, gives
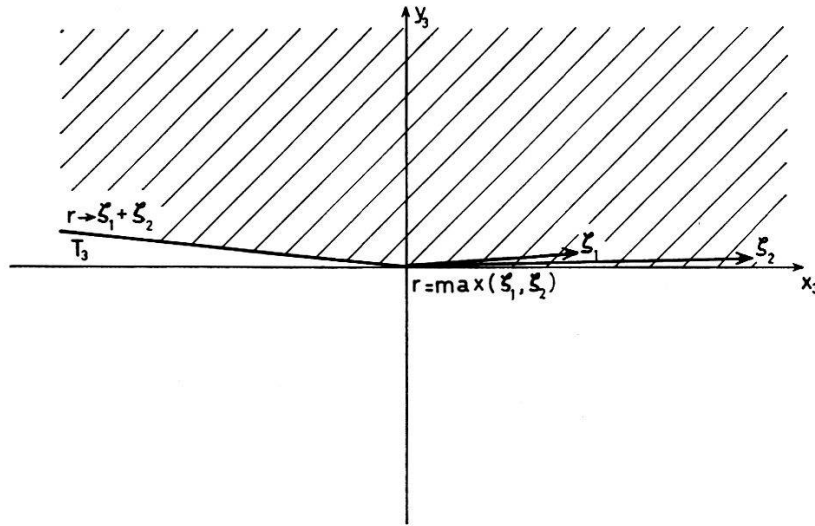

Fig. 4

The intersection ζ_1-cut $\cap$ ζ_2-cut $\cap$ $\mathfrak{F}$ shown in the ζ_3-plane for the case $\mathrm{Im}\, \zeta_k > 0$

the following contribution to the BERGMAN-WEIL integral from this part of the distinguished boundary

$$I_{12}^{\mathfrak{F}} = \frac{-1}{(2\pi i)^3} \int\limits_{T_3} \frac{d^3\zeta}{(\zeta_1 - z_1)\,(\zeta_2 - z_2)}\, [q_3^{\mathfrak{F}}(z;\zeta_+;r')\, F(\zeta_+) - \\ - q_3^{\mathfrak{F}}(z;\zeta_-;r')\, F(\zeta_-)], \tag{16a}$$

$$I_{23}^{\mathfrak{F}} = \frac{1}{(2\pi i)^3} \int\limits_{T_1 + \mathfrak{D}_3 - \mathfrak{D}_1} \frac{d^3\zeta}{(\zeta_2 - z_2)\,(\zeta_3 - z_3)}\, [q_1^{\mathfrak{F}}(z;\zeta_+;r')\, F(\zeta_+;r') - \\ - q_1^{\mathfrak{F}}(z;\zeta_-;r')\, F(\zeta_-;r')] + \\ + \frac{1}{(2\pi i)^3} \int\limits_{\mathfrak{D}_3 - \mathfrak{D}_1} \frac{d^3\zeta}{(\zeta_2 - z_2)\,(\zeta_3 - z_3)}\, [q_1^{\mathfrak{F}}(z;\zeta_+;r)\, F(\zeta_+;r) - \\ - q_1^{\mathfrak{F}}(z;\zeta_-;r)\, F(\zeta_-;r)], \tag{16b}$$
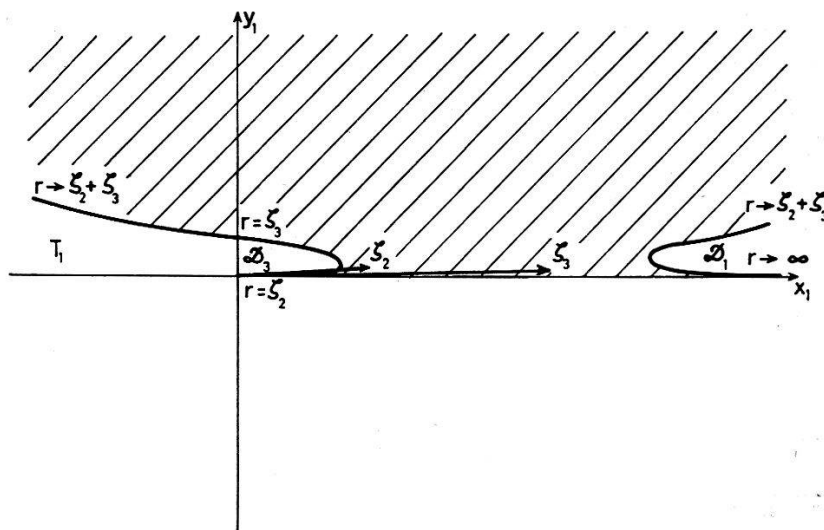
Fig. 5

The intersection ζ_2-cut $\cap$ ζ_3-cut $\cap$ $\mathfrak{F}$ shown in the ζ_1-plane for the case $\mathrm{Im}\ \zeta_k > 0$:
$$\mathrm{Re}\ \zeta_3 > \mathrm{Re}\ \zeta_2$$
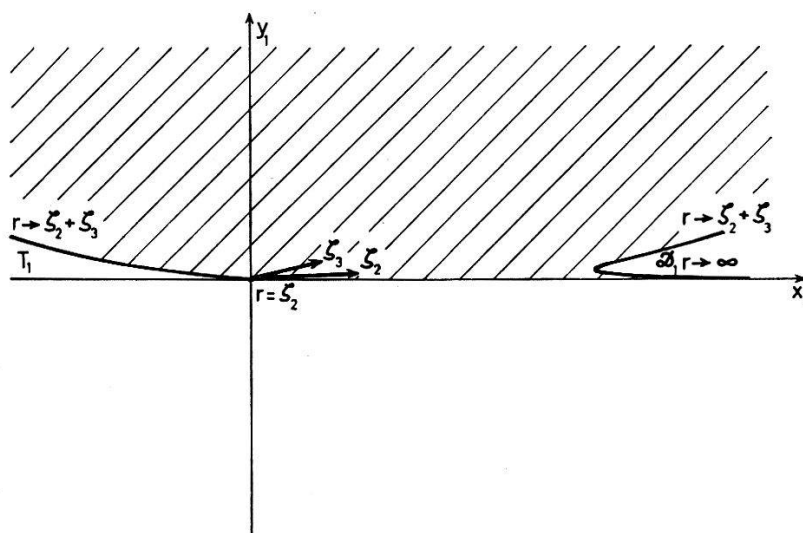
Fig. 6

The intersection ζ_2-cut $\cap$ ζ_3-cut $\cap$ $\mathfrak{F}$ shown in the ζ_1-plane for the case $\mathrm{Im}\ \zeta_k > 0$;
$$\mathrm{Re}\ \zeta_3 < \mathrm{Re}\ \zeta_2$$

$$r = \frac{1}{2}\left[\zeta_1 + \zeta_2 + \zeta_3 - \sqrt{\lambda(\zeta)}\right], \tag{16c}$$

$$r' = \frac{1}{2}\left[\zeta_1 + \zeta_2 + \zeta_3 + \sqrt{\lambda(\zeta)}\right] > r. \tag{16d}$$

Apart from the two terms exhibited in (16a) and (16b) there is one more contribution to the BERGMAN-WEIL integral from this part of the distinguished boundary, which should be denoted by $I_{13}^{\mathfrak{F}}$. This term is completely analogous to the contribution $I_{23}^{\mathfrak{F}}$ in (16b) and is obtained from that expression by a permutation of the indices 1 and 2. The two terms in

(16b) are integrated over a domain which is denoted by $\mathfrak{D}_3 - \mathfrak{D}_1$. By this we mean that we integrate over the two domains $\mathfrak{D}_3$ and $\mathfrak{D}_1$ but that the contribution from the last domain is given a negative sign. The other notations in Eqs. (16) are then self explanatory.

We note that there are two integrals over the domain $\mathfrak{D}_3$ in $I_{23}^{\mathfrak{F}}$ and that there are two corresponding integrals in $I_{13}^{\mathfrak{F}}$. If we add corresponding terms in $I_{13}^{\mathfrak{F}}$ to each term in $I_{23}^{\mathfrak{F}}$ and then use the identity (3) under the integral sign, once for the terms for r and again for the terms r', we find that the sum of the two integrals reduces to an expression which is completely analogous to the two terms which are integrated over $\mathfrak{D}_1$ and $\mathfrak{D}_2$. In this way, the sum of the terms $I_{kl}^{\mathfrak{F}}$ is completely symmetric under a permutation of the indices 1, 2 and 3 in spite of the unsymmetric displacement (15) of the surface $\mathfrak{F}$. The total contribution from this part of the distinguished boundary now becomes a sum of six terms, all of which can be written in one or the other of the following two standard forms

$$
I_{T_k}^{\mathfrak{F}} = \frac{-1}{(2\pi i)^3} \int\limits_{T_k} \frac{d^3\zeta}{(\zeta_l - z_l)(\zeta_m - z_m)} \left[q_k^{\mathfrak{F}}(z; \zeta_+; r')\, F(\zeta_+) - \right.
$$
$$
\left. - q_k^{\mathfrak{F}}(z; \zeta_-; r')\, F(\zeta_-) \right], \qquad (17a)
$$

$$
I_{\mathfrak{D}_k}^{\mathfrak{F}} = \frac{1}{(2\pi i)^3} \int\limits_{\mathfrak{D}_k} \frac{d^3\zeta}{(\zeta_l - z_l)(\zeta_m - z_m)} \left[q_k^{\mathfrak{F}}(z; \zeta_+; r')\, F(\zeta_+; r') - \right.
$$
$$
- q_k^{\mathfrak{F}}(z; \zeta_+; r)\, F(\zeta_+; r) - q_k^{\mathfrak{F}}(z; \zeta_-; r')\, F(\zeta_-; r') +
$$
$$
\left. + q_k^{\mathfrak{F}}(z; \zeta_-; r)\, F(\zeta_-; r) \right]. \qquad (17b)
$$

Note that the symbols ζ_+ and ζ_- in Eqs. (16) and (17) have a different meaning from before and now indicate, that *all* variables ζ_k lie either above or below the real axis.

The rest of the distinguished boundary is empty as neither the relevant parts of the three cuts nor any two of the more complicated surfaces ever intersect except in manifolds of lower dimension. (E.g., the three cuts do intersect on a three dimensional set, but this intersection is irrelevant since it lies outside the domain $\mathfrak{U}$ and touches the domain only on the two dimensional subset where these three cuts intersect one of the other surfaces, even after distortions, that is where one of the z_k is zero). We thus have the following representation of the most general function $F(z)$ regular in $\mathfrak{U}$ and sufficiently bounded at infinity.

$$
F(z) = \sum_{k<l} \left(I_{kl}^{kl} + I_{km}^{kl} + I_{lm}^{kl} \right) + \sum_{k=1}^{3} \left(I_{T_k}^{\mathfrak{F}} + I_{\mathfrak{D}_k}^{\mathfrak{F}} \right), \qquad (18)
$$

where the various integrals I are defined in Eqs. (14) and (17).

The representation (18) contains the boundary value of the function $F(z)$ on the distinguished boundary as a weight function. As has already been remarked earlier[1]), the distinguished boundary consists of only 'physical points' and (18) allows an explicit computation of the value of the function $F(z)$ in an arbitrary point inside the regularity domain $\mathfrak{U}$ when the values in these physical points are known. Therefore, no knowledge of a function in some 'unphysical region' is necessary for an application of this formula. The actual evaluation of the integrals I above might, of course, be very complicated.

We also want to remark that, if we substitute an arbitrary weight function (decreasing sufficiently rapidly at infinity to make the integrals convergent) in (18) instead of the boundary values of $F(z)$, the resulting function can have no singularities except on the cuts and where the functions q_l^k are non-analytic, i. e. on or beyond the boundary of $\mathfrak{U}$. Therefore, an arbitrary weight, substituted in (18) yields a function regular at least in $\mathfrak{U}$. However, it must not be supposed that this function has the given weight as boundary value. In general, this will not be the case. This phenomenon is quite analogous to the behaviour of the conventional Cauchy integral where an arbitrary weight function gives an analytic function, the boundary value of which is, in general, not equal to the given weight.

III. Alternative Versions of the Integral Representation (18)

The discussion in the previous section has provided us with an integral representation of the most general function $F(z)$ regular in our domain $\mathfrak{U}$ and sufficiently bounded at infinity. However, for practical applications this formula (Eq. (18)) is somewhat involved, but it can be simplified to a certain extent if we are willing to introduce one more integration and to write our representation as an integral over four variables instead of three. To see this, we observe that the dependence on z_k in every term of (18) can be written in the form $(\zeta_k - z_k)^{-1} (\zeta_l - z_l)^{-1} q(z, \zeta)$ where the q's are given by Eqs. (7) and (8). (Cf. Eqs. (14) and (17).) Closer inspection shows that the q's themselves can be written in the form $N^{-1} Q(z, \zeta)$ where $Q(z, \zeta)$ is a function which is regular, analytic in the topological product of the cut planes in the variables z_k. The denominators N are given by Eqs. (7d) and (8b). It follows that the kernel in each term of (18) can be written as $N^{-1} \tilde{Q}(z, \zeta)$ where $\tilde{Q}(z, \zeta)$ is another function regular in the whole cut planes and behaving at infinity not worse than $(z_1 z_2 z_3)^{-1/2}$. Such a function can be represented in the form

$$\tilde{Q}(z, \zeta) = \int_0^\infty \int_0^\infty \int_0^\infty \frac{da_1 \, da_2 \, da_3 \, P(a_k, \zeta)}{(a_1 - z_1)(a_2 - z_2)(a_3 - z_3)} , \qquad (19)$$

where the weight $P(a, \zeta)$ is given by an 'iterated jump' of the function $\tilde{Q}(z, \zeta)$. As an example, we can look at the kernel in Eq. (12), where q_3^{12} is given by Eq. (7a). For this case we get

$$\tilde{Q}(z, \zeta) = \frac{1}{(\zeta_1 - z_1)(\zeta_2 - z_2)} \frac{\sqrt{r}}{\sqrt{-z_3} + \sqrt{-\zeta_3}}, \tag{20a}$$

$$P(a, \zeta) = \frac{\sqrt{ra_3}}{\pi} \frac{\delta(a_1 - \zeta_1)\,\delta(a_2 - \zeta_2)}{a_3 - \zeta_3}\,\Theta(-\zeta_3). \tag{20b}$$

The factor $\Theta(-\zeta_3)$, which is 1 for negative ζ_3 and vanishes for positive ζ_3, is inserted so that this particular $P(a, \zeta, r)$ will be non-zero only in T_3, which is the only domain in which q_3^{12} appears in the integral. Similarly the terms in (18) involving $q_2^{12}(z; ++-; r)$ can be taken care of by defining $P(a, \zeta_+, r)$ appropriately in the domain where $\zeta \in \mathfrak{D}_1$ and $0 < r < \sqrt{\zeta_1 \zeta_2}$. The terms involving $q_2^{12}(z; ++-; r')$ can be treated in the same way by setting $r' = r$ and defining $P(a, \zeta_+, r)$ appropriately in the domain $\zeta \in T_2 + \mathfrak{D}_1$ and $r > \sqrt{\zeta_1 \zeta_2}$. The other terms in (18) can be handled in a similar way and the corresponding weights computed, but the results are uninteresting for our applications and we do not want to give the details. After a treatment of this kind, all terms involving, e.g., q_k^{12} can be represented with the aid of the expression $(N_{12})^{-1}\tilde{Q}$. The function $\tilde{Q}$ is given by (19) above, where, however, the weight P is the sum of the expression (20b) and other similar contributions coming from the other terms in (18). The net result is that the whole function $F(z)$ in (18) can now be represented in the following form

$$F(z) = \int_0^\infty \frac{d^3a}{(a_1 - z_1)(a_2 - z_2)(a_3 - z_3)} \int_0^r dr \left[\frac{\varphi_1(a_k, r)}{N_1(z_k, r)} + \right. $$
$$\left. + \frac{\varphi_2(a_k, r)}{N_2(z_k, r)} + \frac{\varphi_3(a_k, r)}{N_3(z_k, r)} + \frac{\varphi_0(a_k, r)}{N_0(z_k, r)} \right], \tag{21}$$

$$N_1(z_k, r) = N_{23}(z_k, r) \text{ cycl.}, \tag{21a}$$

$$N_0(z_k, r) = N_{\mathfrak{F}}(z_k, r), \tag{21b}$$

$$\varphi_1(a_k, r) = \frac{1}{(2\pi i)^3} \int d^3\zeta \left[\delta\left(r + \frac{1}{2}\left(\zeta_3 - \zeta_1 - \zeta_2 + \sqrt{\lambda(\zeta)}\right)\right) + \right.$$
$$\left. + \delta\left(r + \frac{1}{2}\left(\zeta_3 - \zeta_1 - \zeta_2 - \sqrt{\lambda(\zeta)}\right)\right) \right] \cdot$$
$$\cdot \left[P(a_k, \zeta_+, r)\, F(\zeta_+, r) + P(a_k, \zeta_-, r)\, F(\zeta_-, r) \right], \text{ etc.} \tag{21c}$$

The integration over ζ in (21c) goes over the whole distinguished boundary. This expression has the same property as the formula (18), viz. that every function $F(z)$ regular in $\mathfrak{U}$ and sufficiently bounded at infinity can

be represented in this way. Further, an arbitrary set of weight functions $\varphi_i(a_k, r)$ (also sufficiently bounded at infinity) yields a function $F(z)$ regular in $\mathfrak{U}$ when substituted into (21). However, we presumably have the same lack of uniqueness in (21) as in (18), viz. that the *same* function $F(z)$ can presumably be represented by an infinite number of weights $\varphi_i(a_k, r)$.

We can give several alternative versions of the expression above. First of all, the Cauchy denominators can be replaced by functions $\varDelta_2^{(+)}(z, a)$ if we perform a Hankel transform in the variable a_k. The function $\varDelta_2^{(+)}$ is the conventional singular function with positive frequencies defined by

$$\varDelta_2^{(+)}(z, a) = - \frac{a}{8\pi} \frac{H_1^{(1)}(\sqrt{az})}{\sqrt{az}} = \frac{1}{i\pi} \int_0^\infty \frac{db\, \bar{\varDelta}(b, a)}{b - z}, \tag{22a}$$

$$\bar{\varDelta}(a, b) = \frac{\delta(a)}{4\pi} - \frac{\Theta(ab)}{8\pi} b \frac{J_1(\sqrt{ab})}{\sqrt{ab}}, \tag{22b}$$

where $H_1^{(1)}$ and J_1 are the conventional Hankel and Bessel functions of order one. Formally, the function $\bar{\varDelta}(a, b)$ fulfills the relation

$$\int_0^\infty d\zeta\, \bar{\varDelta}(a, \zeta)\, \bar{\varDelta}(\zeta, b) = \frac{1}{16\pi^2}\, \delta(a - b); \quad a \geqslant 0, b \geqslant 0. \tag{23}$$

Eq. (23) expresses the orthonormality relation between Bessel functions usually exploited in connection with Hankel transforms[7]).

From (23) and (22a) we get the following representation of $(a - z)^{-1}$

$$\frac{1}{a - z} = 16\pi^3 i \int_0^\infty db\, \varDelta_2^{(+)}(z, b)\, \bar{\varDelta}(b, a). \tag{23a}$$

If this representation of the Cauchy denominator is used in Eq. (21) we find that the whole formula can be written as

$$\left. \begin{aligned} F(z) = \int_0^\infty d^3a\, \varDelta_2^{(+)}(z_1, a_1)\, \varDelta_2^{(+)}(z_2, a_2)\, \varDelta_2^{(+)}(z_3, a_3) \cdot \\ \cdot \int_0^\infty dr \left[\frac{\tilde{\varphi}_1}{N_1} + \frac{\tilde{\varphi}_2}{N_2} + \frac{\tilde{\varphi}_3}{N_3} + \frac{\tilde{\varphi}_0}{N_0} \right]. \end{aligned} \right\} \tag{24}$$

The new weight functions $\tilde{\varphi}_i(a_k, r)$ are related to the weights $\varphi_i(a_k, r)$ in Eq. (21) with the aid of the following set of formulae

$$\tilde{\varphi}_i(a, r) = (16\pi^3 i)^3 \int_0^\infty d^3b \prod_{k=1}^3 \bar{\varDelta}(a_k, b_k)\, \varphi_i(b, r), \tag{25a}$$

and

$$\varphi_i(a, r) = \frac{i}{\pi^3} \int_0^\infty d^3b \prod_{k=1}^3 \bar{\varDelta}(a_k, b_k)\, \tilde{\varphi}_i(b, r). \tag{25b}$$

This alternative version is particularly interesting because of the following remark. The expressions N_i^{-1} are themselves regular in $\mathfrak{U}$ and, consequently, have Fourier transforms with support only in or on the forward light cones. Further, the Fourier transform of the product

$$\prod_{k=1}^{3} \Delta_2^{(+)}(z_k, a_k)$$

is different from zero only when the two vectors p and p' fulfill the conditions $-p^2 \geqq (\sqrt{a_1} + \sqrt{a_3})^2$; $-p'^2 \geqq (\sqrt{a_2} + \sqrt{a_3})^2$. The Fourier transform of $F(z)$ is then a convolution integral of these two expressions. Obviously, this convolution has the same support properties as the Fourier transform of the product of the $\Delta_2^{(+)}$-functions. Consequently, if the range of integration of the variables a_k is restricted to go from certain lower limits $a_k^{(0)}$ to infinity, the Fourier transform of the function $F(z)$ has the support property that it vanishes unless $-p^2 \geqq (\sqrt{a_1^{(0)}} + \sqrt{a_3^{(0)}})^2$ and $-p'^2 \geqq (\sqrt{a_2^{(0)}} + \sqrt{a_3^{(0)}})^2$. This remark is of interest if one is asking for a representation of a function corresponding to the vacuum expectation value of a product of three local fields where the details of the mass spectrum is also taken into consideration. (Cf. KW p. 12.) Eq. (24) allows us to write down examples of functions $F(z)$ with the correct support property in p-space and with singularities over the whole boundary of $\mathfrak{U}$ in x-space. This verifies that the restrictions following from the mass spectrum do not change the domain of analyticity in x-space. However, we have not been able to prove that *every* function with these support properties in p-space and regular in $\mathfrak{U}$ can necessarily be represented with the aid of (24) with at least one set of weight functions $\varphi_i(a, r)$.

One further modification can also be mentioned. The denominators N_i an be sreplaced by other expressions obtained by multiplying N_i by cfunction X_i regular and different from zero in the cut planes (and suitably bounded at infinity). The following factors are of particular interest in this connection.

$$X_1 = \frac{1}{r}\left(\sqrt{-rz_1} - \sqrt{r - z_2}\,\sqrt{r - z_3}\right)[\log(-rz_1) - \\ - \log(r - z_2) - \log(r - z_3)]^{-1}. \qquad (26a)$$

Two other functions X_2 and X_3 are defined in a similar way and can be obtained from X_1 after obvious permutations. Further,

$$X_0 = \frac{1}{r}\left(\sqrt{r - z_1}\,\sqrt{r - z_2}\,\sqrt{r - z_3} - \\ - \sqrt{-z_1}\,\sqrt{-z_2}\,\sqrt{-z_3}\right)[\log(r - z_1) + \log(r - z_2) + \\ + \log(r - z_3) - \log(-z_1) - \log(-z_2) - \log(-z_3)]^{-1}. \qquad (26b)$$

The square roots are defined as before and the logarithms are made unique by the prescription $-\pi \leqq \operatorname{Im} \log (\ldots) \leqq \pi$. As these factors X_i have neither zeros nor singularities except on the positive real axes, the whole expression (24) can be replaced by

$$F(z) = \int_0^\infty d^3a \prod_{k=1}^3 \Delta_2^{(+)}(z_k, a_k) \int_0^\infty dr \sum_{i=0}^3 \psi_i(a, r)\, f_i(z, r), \qquad (27)$$

$$f_1(z, r) = \frac{\log(-rz_1) - \log(r - z_2) - \log(r - z_3)}{rz_1 + (r - z_2)(r - z_3)} \text{ cycl.} \qquad (27a)$$

$$f_0(z, r) = \frac{\log(r - z_1) + \log(r - z_2) + \log(r - z_3) - \log(-z_1) - \log(-z_2) - \log(-z_3)}{r^2 - r(z_1 + z_2 + z_3) + z_1 z_2 + z_1 z_3 + z_2 z_3}.$$
$$(27b)$$

Various other modifications of these formulae are also possible. One can, e.g., make a Hankel transform with respect to the variables a in (27) and replace the Hankel functions $\Delta_2^{(+)}$ by Cauchy denominators. One can also make a Hankel transform with respect to the variable r in Eq. (27), Eq. (24) or in Eq. (21). We do not want to write down in detail all the possible modifications which can be obtained in this way. Of course, one always assumes that the weight functions obtained after such formal operations are sufficiently bounded to make the resulting integrals convergent. If that is not the case, certain counter terms have to be added in our representations, so as to make the integrals meaningful.

IV. Comparison with Perturbation Theory

The representation given above for a function $F(z)$ analytic in the domain $\mathfrak{U}$ and sufficiently bounded at infinity has been obtained on the basis of very general arguments. We have used the result of KW concerning the shape of the domain $\mathfrak{U}$ and the BERGMAN-WEIL integral technique for the representation of our function. This means that our whole argument is based only on the assumptions of Lorentz invariance, local commutativity and reasonable mass spectrum (cf. KW). As is well known, the results obtained in this way are compatible with perturbation theory, but the examples one has been able to construct from perturbation theory have a domain of analyticity which is larger than the domain $\mathfrak{U}$. In particular, it has so far been impossible to find any trace of the surface $\mathfrak{F}$ in perturbation theory examples, while singularities can be obtained in arbitrary points on the surfaces F'_{kl} (cf. appendix III of KW).

In view of these facts it is tempting to try to relate the terms with $i = 1, 2$ and 3 in our representations to perturbation theory expressions. As a matter of fact, it is possible to recognize the Hankel transform with respect to the variable r of those three terms in the representation (27),

which stem from the surfaces F'_{kl}, as perturbation theory examples. For this purpose, we consider nine scalar fields $\phi_1, \ldots, \phi_6, \psi_1, \ldots, \psi_3$ interacting with the aid of the following Lagrangian

$$\mathfrak{L} = g_1 \, \phi_4(x) \, \phi_5(x) \, \phi_6(x) + g_2 \, [\psi_1(x) \, \phi_1(x) \, \phi_3(x) \, \phi_4(x) + \\ + \psi_2(x) \, \phi_1(x) \, \phi_2(x) \, \phi_5(x) + \psi_3(x) \, \phi_2(x) \, \phi_3(x) \, \phi_6(x)] . \tag{28}$$

The fields ϕ_i have masses given by $\sqrt{a_i}$ while the masses of the fields ψ_i are irrelevant. We next define 'currents' $j_i(x)$ by

$$j_i(x) = \frac{\delta \, \mathfrak{L}}{\delta \, \psi_i(x)} \; ; \quad i = 1, 2 \text{ or } 3, \tag{29}$$

and consider the vacuum expectation value of the product of these three currents. Further, we use perturbation theory and expand everything in powers of g_1 and g_2. The leading term is of order $g_2^3 \, g_1$ and is given by

$$F(z) = \langle 0 \mid j_1(x) \, j_2(x') \, j_3(x'') \mid 0 \rangle = \\ = g_2^3 \, g_1 \, \Delta_2^{(+)} \, (x - x', a_1) \, \Delta_2^{(+)} \, (x' - x'', a_2) \, \Delta_2^{(+)} \, (x - x'', a_3) \cdot \\ \cdot \int dy \, [\Delta_R \, (x - y, a_4) \, \Delta_2^{(+)} \, (y - x', a_5) \, \Delta_2^{(+)} \, (y - x'', a_6) + \\ + \Delta_R \, (x' - y, a_5) \, \Delta_2^{(+)} \, (x - y, a_4) \, \Delta_2^{(+)} \, (y - x'', a_6) + \\ + \Delta_R \, (x'' - y, a_6) \, \Delta_2^{(+)} \, (x - y, a_4) \, \Delta_2^{(+)} \, (x' - y, a_5)] . \tag{30}$$

The complex variables z_k are defined by $z_1 = - (x - x')^2$; $z_2 = - (x' - x'')^2$ and $z_3 = - (x - x'')^2$. The first three factors in (30) correspond exactly to the product of the three $\Delta_2^{(+)}$ functions in (27). The expression inside the square bracket in (30) is the same as the perturbation theory example treated in appendix III of KW in Eqs. (A. 54)–(A. 59). It was there shown that the Hankel transform of this expression with the aid of the function $\bar{\Delta}(a, b)$ can be written as

$$\int d^3b \prod_{k=4}^{6} \bar{\Delta}(a_k, b_k) \int dy \, [\ldots] = \\ = \text{const} \iiint\limits_{0}^{1} \frac{d\alpha_1 \, d\alpha_2 \, d\alpha_3 \, \delta \, (1 - \alpha_1 - \alpha_2 - \alpha_3)}{z_1 \alpha_2 \alpha_3 + z_2 \alpha_3 \alpha_1 + z_3 \alpha_1 \alpha_2 - \alpha_1 a_6 - \alpha_2 a_4 - \alpha_3 a_5} . \tag{31}$$

We now consider the special case that two of the quantities a_4, a_5 and a_6 are zero, while the third one is put equal to r. This can be done in three different ways. If we further perform a differentiation in (31) with respect to the mass that is equal to r, we get three integrals, one of which is given by

$$J_3 = \iiint\limits_{0}^{1} \frac{d\alpha_1 \, d\alpha_2 \, d\alpha_3 \, \delta \, (1 - \alpha_1 - \alpha_2 - \alpha_3) \, \alpha_3}{[z_1 \alpha_2 \alpha_3 + z_2 \alpha_1 \alpha_3 + z_3 \alpha_1 \alpha_2 - r \alpha_3]^2} . \tag{32}$$

The other two terms are obtained after obvious permutations. The integration over the variables α in (32) is elementary and straight forward. The result is

$$J_3 = -\frac{\log(-r z_3) - \log(r - z_1) - \log(r - z_2)}{(r - z_1)(r - z_2) + r z_3} = -f_3(z, r) \,. \tag{33}$$

The function $f_3(z, r)$ in (33) is the same as the function f_3 defined in Eq. (27a). The other functions $f_i(z, r)$ in (27a) are obtained from (31) for the other choices of the relation between r and the masses a_4, a_5 and a_6. In this way we see that the Hankel transform of the perturbation theory example given here, if differentiated once with respect to the variable r, can be identified with three of the terms in our representation (27). The corresponding FEYNMAN diagram is shown in Figure 7. We have not been able to find a similar connection between perturbation theory results and the remaining term in our representation.
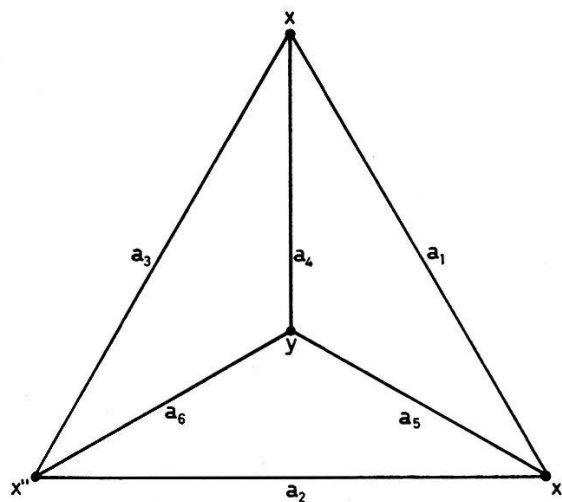

Fig. 7

Feynman diagram corresponding to Eq. (30) and related to three of the terms in the representation (27)

References

[1] G. KÄLLÉN and A. WIGHTMAN, Mat. Fys. Skr. Dan. Vid. Selsk. *1*, No. 6 (1958). This paper, which is referred to as KW below, also contains references to earlier work.

[2] The reader who is unfamiliar with the theory and the terminology of functions of several complex variables is referred to H. BEHNKE and P. THULLEN, Ergebn. d. Mathem. *3*, Nr. 3, Berlin 1934, and to S. BOCHNER and W. T. MARTIN, Several Complex Variables, Princeton 1948. A short «physicist's summary» of some of these concepts is given in section VI in KW.

[3] For a derivation of this formula we refer to F. SOMMER, Mathem. Ann. *125*, 172 (1952). We follow the terminology and notation of this paper.

[4] This construction was originally done by one of us (G. K.) in collaboration with A. WIGHTMAN in the course of some preliminary investigation of the BERGMAN-

WEIL integral technique to the three fold vacuum expectation value. We want to thank Prof. WIGHTMAN for his kind permission to use these results here.

[5]) This procedure has been suggested to us by Prof. F. SOMMER. We are very much indebted to Prof. SOMMER for this suggestion as well as for a correspondence about the BERGMAN-WEIL integral.

[6]) This new surface is easily shown to give exactly the relevant part of the F'_{12} curve in the limit when $\varepsilon \to 0$. It might appear that this distortion could open gaps in the boundary of the domain; however it can be verified that the displaced contour (9) never crosses the real axis of z_3 $(1 + i\varepsilon)$ nor the negative real z_3-axis. The similar property also holds for the corresponding displacements of F'_{13}, F'_{23} and $\mathfrak{F}$ described below, so that the four displaced surfaces plus the z_1-cut, z_2-cut and z_3-cut define a complete and continuous boundary for a domain of analyticity.

[7]) The Hankel transform for ordinary functions is extensively treated in the mathematical literature, cf. e. g. I. SNEDDON, Fourier integrals, New York 1951, esp. pp. 48–70. Formula (23) applied to Lorentz-invariant, tempered distributions, which we are also interested in here, has been discussed and justified by L. GÅR-DING, Suppl. Nuovo Cimento *14*, 9 (1959), esp. pp. 45–54.