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Autor: Stueckelberg, E.C.G.
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Quantum Theory in Real Hilbert Space

by E. C. G. Stueckelberg*)

(Universities of Geneva and Lausanne)

Abstract. Relativistic Quantum Theory is brought to a form, where all operators, including time reversal, are linear: Hilbert space is real. Instead of the imaginary number $i = \sqrt{-1}$, an operator \hat{J} ($\hat{J}^2 = -1$) is introduced, which commutes with all observables and with the orthogonal operators representing ortho-chronous Lorentz transformations, and anti-commutes with the orthogonal representation of pseudo-chronous Lorentz transformations. It is shown, that \hat{J} is necessary in order to have an uncertainty principle (§ 2). Furthermore it follows that momentum-energy and angular momentum-centre of energy are pseudo-chronous quantities. Therefore, the Hamiltonian operator does not change sign under time reversal (§ 5). Lorentz transformations are considered as passive (= coordinate frame-) transformations (§ 7).

In the annexes the following topics are discussed: A possible generalisation of quantum theory involving non linear operators (A-1); The dictionary between conventional theory in complex Hilbert space and the proposed formalism in real Hilbert space (A-2) and (A-3); The dictionary between a quantum theory in quaternion Hilbert space and our real theory (A-4). Also an error, frequently found in literature, concerning the representation of the Lorentz group is pointed out (A-5).

Introduction and Conclusion

This article presents the essential of the lectures on *Relativistic Quantum Theory (QT) of Fields*, given at the universities of Geneva and Lausanne during the past 20 years. The problem was to show students, *why the imaginary unit enters quantum theory*. We start therefore from a theory built entirely upon *real numbers* and are lead to introduce an operator \hat{J} (with $\hat{J}^2 = -1$), in order to have an *uncertainty principle* (UP) between the *mean square errors* $\langle \Delta F^2 \rangle$ and $\langle \Delta G^2 \rangle$ of two observables F and G . Observables are *symmetric tensors* (or *symmetric linear operators*) in real Hilbert space (RHS) $F_{ab} = F_{ba}$, or

$$F^T = F, G^T = G, \dots \quad (0.1)**)$$

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**) A^T is the *transposed operator*: $A_{ab}^T = A_{ba}$, which plays the analogous role as the *hermitian conjugate* \hat{A}^\dagger ($\hat{A}_{pq}^\dagger = \hat{A}_{qp}^*$) in *Complex Hilbert Space* (CHS), see Annex (A-2).

The criterium for the impossibility of measuring F and G simultaneously, is a non vanishing commutator

$$FG - GF = [F, G] = -[F, G]^T \neq 0. \quad (0.2)$$

The expectation value $\langle [F, G] \rangle$ vanishes, because $[F, G]$ is an *antisymmetric tensor*. Therefore, only the *positive definit observable*

$$P = -[F, G]^2 = P^T \quad (0.3)$$

can occur in

$$\langle \Delta F^2 \rangle \langle \Delta G^2 \rangle \geq \lambda^2 \langle P \rangle. \quad (0.4)$$

λ is a real number. *Unless otherwise mentioned* (Annexes (A-2), (A-3) and (A-4)) *all numbers occurring in this paper are real*.

We show, that this uncertainty principle leads to a contradiction, unless $\lambda^2 = 0$, in which case (0.4) is a triviality. We show, in § 2, that the only other possibility consists in introducing an *antisymmetric operator* $J_{(FG)}$ which has an inverse $J_{(FG)}^{-1}$ (and may therefore, without loss of generality, be normalised to -1) and which commutes with F and G .

$$J_{(FG)}^T = -J_{(FG)}, \quad J_{(FG)}^2 = -J_{(FG)}^T J_{(FG)} = -1 \quad (0.5)$$

$$[J_{(FG)}, F] = [J_{(FG)}, G] = 0. \quad (0.6)$$

We may now form the symmetrical operator

$$C_{(FG)} = J_{(FG)} [F, G] = C_{(FG)}^T \quad (0.7)$$

and expect a uncertainty principle of the form

$$\langle \Delta F^2 \rangle \langle \Delta G^2 \rangle \geq \lambda^2 \langle C_{(FG)} \rangle^2. \quad (0.8)$$

Let H be a third observable. $C_{(FG)}$ being an observable, $J_{(CH)}$ has to commute with $C_{(FG)}$ and H . Thus, the simplification to assume but one universal

$$J_{(FG)} = J \quad (0.9)$$

commuting with all observables, seems natural.

FINHELSTEIN, JAUCH and SPEISER³⁾ have shown that only three possibilities: RHS, CHS (Complex Hilbert Space) and QHS (Quaternion Hilbert Space) are possible in Quantum theory (QT). Thus three anti-commuting J 's (J_1, J_2, J_3) may exist. We have analysed QHS in terms of RHS in the annex (A-4).

We begin (§ 1) by an analysis of probability, which leads us necessarily to RHS. The linearity of the operators is a further assumption, which

may eventually be omitted, because classical statistical mechanics does not necessarily require the Jakoben cyclic identity, but may give

$$\overrightarrow{FGH} \{F, \{G, H\}\} \neq 0 \quad (0.10)*)$$

for the generalised Poisson brackets (see annex (A-1) and 2)).

Therefore, the corresponding identity for linear operators

$$\overrightarrow{FGH} J [F, J [G, H]] = 0 \quad (0.10J)$$

may not hold. In § 2 we discuss the uncertainty principle (UP). In § 3 we introduce the representation of the linear group $\{L\}$ (which leaves the metric tensor $g^{\alpha\beta} = g^{(\alpha\beta)**}$ invariant) by orthogonal operators O in RHS:

$$O^T = O^{-1}, \quad L \rightarrow e^{J^\lambda} O. \quad (0.11)$$

In § 4 we show, that the metric $g^{\alpha\beta}$ of the differential manifold $x = \{x^\alpha\}$, $\alpha, \beta \dots = 1, 2, \dots, n$, has necessarily the *thermodynamic signature* (STUECKELBERG and WANDERS⁴⁾⁵⁾) if the existence of *fundamental state* $\Psi^{(0)}$, the *vacuum*, is postulated:

$$\text{signat } (g^{\alpha\beta}) = \pm (1, 1, \dots, 1, -1). \quad (0.12)$$

This gives a preference to one coordinate $x^n = t$, the time. Thus, $\{L\}$ is the *full Lorentz group in n -dimensional space* (including *time reversal* $L = T$).

Furthermore it is shown, that $J = \check{J}$ is a *pseudochronous operator*

$$\check{J} = O^{-1} \check{J} O = \text{sig } (L'^n_n) \check{J} \quad (0.13)***)$$

$$\text{if } 'x'^\alpha = L'^\alpha_\alpha (x^\alpha + L^\alpha) : 'x = Lx; \det (L'^\alpha_\alpha) \neq 0 \quad (0.14)****)$$

$$\text{satisfies } g'^\alpha{}_\beta = L'^\alpha_\alpha L'^\beta_\beta g^{\alpha\beta}. \quad (0.15)$$

Multilocal ortho-chronous observables transform according to

$$\left. \begin{aligned} 'F'^\alpha{}_\beta \dots ('x' y \dots) &= L'^\alpha_\alpha L'^\beta_\beta \dots F^{\alpha\beta} \dots (L^{-1} 'x L^{-1} 'y \dots) \\ &= O^{-1} F'^\alpha{}_\beta \dots ('x' y \dots) O \end{aligned} \right\} \quad (0.16)$$

while *pseudo-chronous quantities* transform according to

$$'F'^\alpha{}_\beta \dots ('x \dots) = \text{sig } (L'^n_n) L'^\alpha_\alpha \dots \check{F}^\alpha (L^{-1} 'x \dots) = O^{-1} \check{F}'^\alpha \dots ('x \dots) O \quad (0.17F)$$

*) \overrightarrow{ABC} stands for the cyclic sum.

**) $F^{(\alpha\beta\gamma\dots)}$ is a totally symmetric tensor, while $F^{[\alpha\beta\gamma\dots]}$ is a totally antisymmetric tensor in α -space.

***) $\text{sig } (\lambda)$ is the sign function $\text{sig } (\lambda) = \pm 1$ for $\lambda \gtrless 0$.

****) Frame transformations are written with the primes to the left:

$'x'^\alpha \leftarrow x^\alpha; ' \Psi'_a \leftarrow \Psi_a$.

We shall later*) also use *pseudo-chorous quantities*, which transform according to ($ik \dots = 1 \ 2 \dots n - 1$)

$$\left. \begin{aligned} \hat{F}'^{\alpha \dots} ('x \dots) &= \text{sig}(\det(L'^i_i)) L'^{\alpha}_{\alpha} \dots \hat{F}^{\alpha \dots} (L^{-1} 'x \dots) \\ &= O^{-1} \hat{F}'^{\alpha \dots} ('x \dots) O \end{aligned} \right\} \quad (0.17\hat{F})$$

and finally *pseudo-quantities* $\overset{\circ}{F} = \overset{\circ}{F}$:

$$\left. \begin{aligned} \overset{\circ}{F}'^{\alpha \dots} ('x \dots) &= \text{sig}(\det(L'^{\alpha}_{\alpha})) L'^{\alpha}_{\alpha} \dots \overset{\circ}{F}^{\alpha \dots} (L^{-1} 'x \dots) \\ &= O^{-1} \overset{\circ}{F}'^{\alpha \dots} ('x \dots) O. \end{aligned} \right\} \quad (0.17\overset{\circ}{F})$$

Let us remark, that we consider (§§ 3, 4 and 5) L always as *passive transformations*. As a matter of fact, we show (§ 7), that this *passive point of view* is perfectly reasonable in QT, because a statistical analysis of observations at two epochs t' and t'' , is independent of whether t'' is later or earlier than t' in the *thermodynamic time scale* ($\check{S}(t'') > \check{S}(t')$; $\check{S}(t) = \text{entropy} > 0$, at epoch t , cf. 4) and 6)7)).

In § 5 we analyse the infinitesimal group $L(\delta \lambda \cdot \delta \omega^{[\dots]})$

$$'x'^{\alpha} = x'^{\alpha} + \delta \lambda'^{\alpha} + \delta \omega'^{\alpha}_{\alpha} x^{\alpha} = x'^{\alpha} + \delta \lambda'^{\alpha} + \frac{1}{2} \delta \omega^{[\mu\nu]} \Sigma_{\mu\nu}^{\alpha} x^{\alpha} \quad (0.18)$$

$$\Sigma_{\mu\nu}^{\alpha} = \Sigma_{[\mu\nu]}^{\alpha} = \delta_{\mu}^{\alpha} g_{\nu\alpha} - \delta_{\nu}^{\alpha} g_{\mu\alpha} \quad (0.19)$$

generating the continuous group $\{L_{(\text{cont})}\}$. The generators of the corresponding Lie group $\{O\}$, with $n + (1/2) n(n-1)$, parameters λ^{μ} and $\omega^{\mu\nu} = \omega^{[\mu\nu]}$ are $-\check{J} \check{H}_{\mu}$ and $\check{J} \check{M}_{\mu\nu} = \check{J} \check{M}_{[\mu\nu]}$. The pseudo-chronous observables are the *pseudo-chronous momentum-energy vector* \check{H}_{μ} and the *pseudo-chronous angular momentum-centre of energy tensor* $\check{M}_{\mu\nu}$. We arrive at the relation

$$(\partial_{\mu}^x + \partial_{\mu}^y + \dots) F^{\alpha \dots} (xy \dots) = -\check{J} [\check{H}_{\mu}, F^{\alpha \dots} (xy \dots)] \quad (0.20)$$

$$\left. \begin{aligned} &([x_{\mu}, \partial_{\nu}^x] + [y_{\mu}, \partial_{\nu}^y] + \dots) \delta_{\alpha'}^{\alpha} \delta_{\beta'}^{\beta} \dots + \Sigma_{\mu\nu}^{\alpha} \delta_{\beta'}^{\beta} \dots \\ &+ \delta_{\alpha'}^{\alpha} \Sigma_{\mu\nu}^{\beta} \delta_{\beta'}^{\beta} \dots + \dots) F^{\alpha' \beta' \dots} (xy \dots) = -\check{J} [\check{M}_{\mu\nu}, F^{\alpha \beta \dots} (xy \dots)]. \end{aligned} \right\} \quad (0.21)$$

From the structure relation of the generators of $\{L_{(\text{cont})}\}$ ($-\partial_{\mu}$ and $N_{\mu\nu}^{\alpha} = [x_{\mu}, \partial_{\nu}] \delta_{\alpha'}^{\alpha} + \Sigma_{\mu\nu}^{\alpha} \delta_{\alpha'}^{\alpha}$), the commutation relations

$$\check{J} [\check{H}_{\mu}, \check{H}_{\nu}] = 0 \quad (0.22)$$

*) In a following article on *real representations of the spinor group* $\{\pm \delta_B^A, \pm \gamma^{\alpha A}_B, \pm \gamma^{[\alpha, \alpha_2] A}_B, \dots, \pm \gamma^{[\alpha, \dots, \alpha_n] A}_B\}$.

$$\check{J}[\check{M}_{\mu\nu}, \check{M}_{\sigma\tau}] = -g_{\mu\sigma} \check{M}_{\nu\tau} - g_{\nu\tau} \check{M}_{\mu\sigma} + g_{\mu\tau} \check{M}_{\nu\sigma} + g_{\nu\sigma} \check{M}_{\mu\tau} \quad (0.23)$$

$$\check{J}[\check{\Pi}_\mu, \check{M}_{\sigma\tau}] = g_{\mu\sigma} \check{\Pi}_\tau - g_{\mu\tau} \check{\Pi}_\sigma \quad (0.24)$$

follow*).

In § 6, we show, that $\check{\Pi}_\mu$ and $\check{M}_{\mu\nu}$ can be expressed in terms of an ortho-chronous observable, the *momentum energy tensor*

$$\Theta^{\alpha\beta}(x) = \Theta^{(\alpha\beta)}(x), \quad \partial_\alpha \Theta^{\alpha\beta}(x) = 0 \quad (0.25)$$

as integrals over a surface $\tau(x) = 0$, whose surface element $d\check{\sigma}_\alpha(x)$ is a *time-like pseudo-chronous vector* ($\text{signat}(g^{\alpha\beta}) = (1, 1 \dots 1 - 1)$)

$$d\check{\sigma}_\alpha(x) d\check{\sigma}^\alpha(x) < 0; \quad d\check{\sigma}_n(x) > 0 \quad (0.26)$$

$$\check{\Pi}^\mu = \int_{\tau(x)=0} d\check{\sigma}_\alpha \Theta^{\alpha\mu}(x); \quad \check{M}^{\mu\nu} = \int_{\tau(x)=0} d\check{\sigma}_\alpha (x^\mu \Theta^{\alpha\nu} - x^\nu \Theta^{\alpha\mu})(x). \quad (0.27)$$

In the annexes, we consider *classical statistical mechanics* with $F_{GH}\{F, \{G, H\}\} \neq 0$ (A-1), *hermitian CHS* (A-2), *unitary and antiunitary transformations* U and V in CHS (A-3), *quaternion Hilbert space* (QHS) in (A-4), and an *error frequently found in literature* due to a wrong definition of the representation O (in RHS) or U (in CHS) (A-5).

§ 1. Analysis of Probability

Let F and G be two observables, whose spectra, assumed discrete, are

$$F: \{F^{(i)}\} = \{F^{(1)} < F^{(2)} < \dots < F^{(i)} < \dots < F^{(\omega_F)}\} \quad (1.1F)$$

$$G: \{G^{(k)}\} = \{G^{(1)} < G^{(2)} < \dots < G^{(k)} < \dots < G^{(\omega_G)}\} \quad (1.1G)$$

and let $W^{(i)}$, resp. $W^{(k)}$ be the probabilities that F takes the value $F^{(i)}$ (resp. G the value $G^{(k)}$)

$$W^{(i)}, W^{(k)} > 0 \quad \sum_i W^{(i)} = \sum_k W^{(k)} = 1. \quad (1.2)$$

Then we may, without loss of generality, write $W^{(i)}$ as a sum of squares

$$W^{(i)} = \sum_{\alpha=1}^{\alpha=\omega} \Psi_{i\alpha}^2; \quad W^{(k)} = \sum_{\beta=1}^{\beta=\omega_k} \Psi_{k\beta}^2, \quad (1.3)$$

which introduces an ω_i - (resp. ω_k -) fold degeneracy of the spectral term $F^{(i)}$ (resp. $G^{(k)}$). Now let us introduce two indices a and a' :

*) Due to a wrong sign in the representation $L \rightarrow e^{\check{J} \lambda O}$, these commutations relation are frequently wrong in several books on QT of fields (see Annex (A-5)).

$$\left. \begin{aligned} a \ b \ \dots &= 1 \ 2 \ \dots \sum \omega_i = 1 \ 2 \ \dots \omega_R \\ 'a \ 'b \ \dots &= '1 \ '2 \ \dots \sum ' \omega_k = '1 \ '2 \ \dots ' \omega_R \end{aligned} \right\} \quad (1.4)$$

and write the degeneracies in the form

$$F^{(a)} = F^{(i\alpha)}, \quad F^{(i\alpha)} = F^{(i)} \quad G^{('a)} = G^{(k\beta)}, \quad G^{(k\beta)} = G^{(k)}. \quad (1.5)$$

Now we may chose the arbitrary large numbers ω_R and $'\omega_R$ equal, and represent $\Psi_a = \Psi_{i\alpha}$, and $'\Psi_a = '\Psi_{k\beta}$ as components of *the same abstract vector Ψ (state vector)*, referred to *two different orthogonal coordinate frames* in an *Eucliden space* of $\omega_R = '\omega_R$ dimensions. In general, this number ω_R will be infinite. Therefore, we call this space the *Real Hilbert Space* (RHS). The two sets of components are related to each other by an orthogonal matrix $O = \{O_{aa'}\}$.

Using the summation convention, we write:

$$' \Psi_a = O_{aa'} \Psi_{a'}; \quad ' \Psi = O \Psi; \quad O^T = O^{-1} \quad (1.6)$$

the expectation values are now

$$\langle F \rangle_\Psi = \sum_i W^{(i)} F^{(i)} = \Psi_a F_{ab} \Psi_b \equiv (\Psi, F \Psi) \quad (1.7)$$

$$F_{ab} = F^{(a)} \delta_{ab} \quad (1.8)$$

$$\langle G \rangle_\Psi = \sum_k W^{(k)} G^{(k)} = ' \Psi_a ' G_{a'b} ' \Psi_b = \Psi_a G_{ab} \Psi_b \equiv (\Psi, G \Psi) \quad (1.9)$$

$$' G_{a'b} = G^{('a)} \delta_{a'b}; \quad G_{ab} = O_{a'a'}^T ' G_{a'b} O_{bb'} \quad (1.10)$$

where

$$(\Phi, \Psi) = (\Psi, \Phi) = \Phi_a \Psi_a = ' \Psi_a ' \Phi_a \quad (1.11)$$

is the scalar product between *vectors in RHS*. F , G and all observables are *symmetrical tensors in RHS*:

$$F^T = F, \quad G^T = G, \quad H^T = H, \dots \quad (1.12)$$

In the a -frame, F is diagonal (1.8) and in the $'a$ -frame $'G$ is diagonal (1.10). The transposed operator of an operator A , $A = \{A_{ab}\}$ is defined by

$$(\Phi, A \Psi) = (A^T \Phi, \Psi); \quad A_{ab}^T = A_{ba}. \quad (1.13)$$

Now, F and G are *two tensor ellipsoids in RHS or a -space*: The length of their principal axes are given by the spectra (1.1F) and (1.1G). The length of the axes are thus independent of the orientation of the a -space vector Ψ . However the *relative orientation of the two ellipsoids F and G* i. e. the *relative orientation of the a -frame and $'a$ -frame* in abstract RHS is not necessarily independent of Ψ . Thus, $O = \{O_{aa'}\}$ may depend on Ψ . This introduces the possibility of assuming F and G to be more general operators than linear ones²⁾ (see (0.10) (0.10J) and Annex (A-1)).

§ 2. The Uncertainty Principle

In order to express the uncertainty principle (UP), we introduce in (0.4) the error operators

$$\Delta F = F - 1 \langle F \rangle_{\Psi}; \quad \Delta G = G - 1 \langle G \rangle_{\Psi} \quad (2.1)$$

from which we form the *mean square errors* $\langle \Delta F^2 \rangle_{\Psi}$ and $\langle \Delta G^2 \rangle_{\Psi}$ in (0.4). There are two possibilities:

$$\langle \Delta F^2 \rangle_{\Psi} \langle \Delta G^2 \rangle_{\Psi} \geq \begin{cases} \lambda^2 \langle P \rangle_{\Psi} & (2.2P) \\ \lambda^2 \langle C \rangle_{\Psi}^2 & (2.2C) \end{cases}$$

where P is a positive observable of dimension $[F]^2 [G]^2$ and C is an observable of dimension $[F] [G]$. λ is a number to be determined.

Let us demonstrate, that the first choice (0.4) or (2.2P) leads to a contradiction: We express (2.2P) or (0.3), (0.4) in the a -frame, where F is diagonal. Then, if $[F, G] \neq 0$, G has nondiagonal elements in this frame. Suppose further that F has the value $F^{(a)}$ i.e. $\Psi_a = \Psi'_a = \pm \delta_{aa}$. Then we have

$$[F, G]_{ab} = (F^{(a)} - F^{(b)}) G_{ab}; \quad G_{ab} \neq 0 \quad (2.3)$$

and (on account of $G^T = G$)

$$P_{ab} = -[F, G]_{ab}^2 = \sum_c (F^{(a)} - F^{(c)}) (F^{(b)} - F^{(c)}) G_{ac} G_{bc}. \quad (2.4)$$

Therefore the expectation value is

$$\langle P \rangle_{\Psi'} = \sum_c (F^{(a')} - F^{(c)})^2 (G_{a'c})^2 = (\text{finite})^2 > 0. \quad (2.5)$$

Now $\langle \Delta F^2 \rangle_{\Psi'} = 0$. Let the spectre of G (1.1 G) be bounded. Then we have $\langle \Delta G^2 \rangle_{\Psi'} \leq (G^{(1)} - G^{(wG)})^2 = (\text{finite})^2$ and (2.2P) (or (0.4)) reads

$$0 \cdot (\text{finite})^2 \geq \lambda^2 (\text{finite})^2 \quad (2.6)$$

which has only the trivial solution $\lambda = 0$, corresponding to the trivial statement

$$\langle \Delta F^2 \rangle_{\Psi} \langle \Delta G^2 \rangle_{\Psi} \geq 0 \quad (2.7)$$

The only other possibility, (2.2C) is to introduce an observable C , linear in F and linear in G . This implies the existence of an antisymmetric tensor in a -space $J_{(FG)} = -J_{(FG)}^T$ commuting with F and G :

$$C = J_{(FG)} [F, G] = C^T \quad (2.8)$$

In order to deduce the UP, we form, with an arbitrary number ξ ,

$$\begin{aligned} |(\Delta F + \xi J \Delta G) \Psi|^2 &= (\Psi, (\Delta F - \xi \Delta G J) (\Delta F + \xi J \Delta G) \Psi) \\ &= \langle \Delta F^2 \rangle_{\Psi} - \langle J^2 \Delta G^2 \rangle_{\Psi} \xi^2 + \langle J [F, G] \rangle_{\Psi} \xi \\ &= f(\xi) \geq f_{\min} (\xi') \geq 0. \end{aligned} \quad (2.9)^*$$

*) We have written J for $J_{(FG)}$.

In order to make appear $\langle \Delta G^2 \rangle_\Psi$ in (2.9), it is necessary that $J_{(FG)}^2$ is a number $\neq 0$. Being antisymmetric, this number must be negative. As ξ is an arbitrary number, we lose no generality in normalising $J_{(FG)}^2 = -1$. Thus, we arrive at the conditions (0.5) and (0.6).

The minimum $f_{\min}(\xi')$ of $f(\xi)$ is easily found to be at

$$\xi' = -1/2 \langle J_{(FG)} [F, G] \rangle_\Psi \langle \Delta G^2 \rangle_\Psi^{-1} \quad (2.10)$$

which we insert in the last inequality (2.9). Multiplying with $\langle \Delta G^2 \rangle_\Psi$, we find the inequality (0.8) with $\lambda^2 = 1/4$. Assuming but one universal J (see text following (0.8)), we arrive at the UP:

$$\langle \Delta F^2 \rangle_\Psi \langle \Delta G^2 \rangle_\Psi \geq \frac{1}{4} \langle J[F, G] \rangle_\Psi^2. \quad (2.11)$$

§ 3. The linear inhomogenous group $\{L\}$ in x -space

$\{L\}$ is defined by its general element L (0.14) and the condition (0.15), stipulating the invariance of the metric tensor $g^{\alpha\beta}$. A classical observable transforms according to (0.16) or (0.17). It will be useful to combine the indices'

$$\{\alpha \beta \dots x y \dots\} \equiv X \quad (3.1)$$

and define

$$\begin{aligned} 'F^X &\equiv 'F'^{\alpha\beta\dots} ('x'y\dots) = L^X_X F^X \\ &= L'^\alpha_\alpha L'^\beta_\beta \dots \int d^n x \delta('x - Lx) \int d^n y \delta('y - Ly) \dots F^{\alpha\beta\dots} (xy\dots) \\ &= L'^\alpha_\alpha L'^\beta_\beta \dots F^{\alpha\beta\dots} (L^{-1} 'x L^{-1} 'y \dots). \end{aligned} \quad (3.2)$$

Let us now consider how the expectation value

$$\langle F^{\alpha\dots} (x\dots) \rangle_\Psi = (\Psi, F^{\alpha\dots} (x\dots) \Psi) = \Psi_a F^X_{ab} \Psi_b \quad (3.3)$$

transforms under $\{L\}$. There are *two possibilities*: Either we leave Ψ_a *unchanged* and write

$$\langle 'F'^{\alpha\dots} ('x\dots) \rangle_\Psi = L'^\alpha_\alpha \dots (\Psi, F^{\alpha\dots} (L^{-1} 'x \dots) \Psi) \quad (3.4)$$

which expresses the fact that $'F^X$ is the *transformed operator* (3.2) (0.16) (0.17). Or, we may express the transformed expectation value in terms of the initial operator F^X with the index $'X = (' \alpha \dots 'x \dots)$ and in terms of a *transformed vector* $'\Psi_a$

$$' \Psi_a = O_{aa} \Psi_a = \Psi_a O^T_{a'a} \quad (3.5)$$

in the form

$$\langle 'F'^{\alpha} \dots ('x \dots) \rangle_{\Psi} = \langle F'^{\alpha} \dots ('x \dots) \rangle_{\Psi} = ' \Psi_a F'^X_{a'b} ' \Psi_b. \quad (3.6)$$

Equating (3.4) and (3.6) we obtain an *identity*

$$\Psi_a L'^{\alpha}_{\alpha} \dots F'^{\alpha}_{ab} (L^{-1} 'x \dots) \Psi_b \equiv \Psi_a O^T_{a'a} F'^{\alpha}_{a'b} ('x \dots) O_{b'b} \Psi_b \quad (3.7)$$

between two *quadratic forms* in Ψ_a . Or, these forms are equal, for an arbitrary Ψ , if and only if*)

$$L'^{\alpha}_{\alpha} \dots F'^{\alpha}_{ab} (L^{-1} 'x \dots) = O^T_{a'a} F'^{\alpha}_{a'b} ('x \dots) O_{b'b}. \quad (3.8)$$

We may write this identity (multiplying by $O \dots O^{-1}$)

$$F'^X_{a'b} = L'^X_X O_{a'a} F^X_{ab} O^T_{b'b} \quad (3.9)$$

Thus F^X_{ab} considered as a 'vector' in $X = \{\alpha \dots x \dots\}$ -space and as a *symmetrix tensor* in a -space, is left invariant, if it is transformed with respect to its three indices X and a, b . This is in perfect analogy to metric tensor $g^{\alpha\beta}$ in (0.15) and to the α -vector mixed bispinor $\gamma^{\alpha A}_B$ (cf.¹) and an article, to appear in this journal, on Spinor Calculus in RHS).

Now it is easily seen, that the $\{O\}$ group is a *ray representation* of the $\{L\}$ -group: Write $L_{(1)}$ and $O^{(1)}$ in (3.9) and consider a second transformation $L_{(2)}$ and $O^{(2)}$ leading from the frames $'X 'a$ to the frames $''X ''a$:

$$F''^X_{a''b} = L''^X_{(2)'X} O^{(2)}_{a'a} O^{(2)}_{b'b} F'^X_{a'b} \quad (3.10)$$

and substitute (3.9) in (3.10): From

$$F''^X_{a''b} = (L_{(2)} L_{(1)})''^X_X (O^{(2)} O^{(1)})''_{a'a} (O^{(2)} O^{(1)})''_{b'b} F^X_{ab} \quad (3.11)$$

it is seen that $L \rightarrow O$ and $L \rightarrow (-O)$ is a two valued representation. However, since J commutes with any observable F, G , we may write

$$L \rightarrow e^{\lambda J} O, \quad L^{-1} \rightarrow O^{-1} e^{-\lambda J} \quad (3.12)$$

where the number λ is an arbitrary phase. We see, that O does not necessarily commute with J .

To illustrate the identity

$$\langle 'F'^X \rangle_{\Psi} = \langle F'^X \rangle_{\Psi}$$

we have drawn the Fig. 1 and 2:

*) In CHS, where $\langle \hat{\Psi}, \hat{A} \hat{\Psi} \rangle$ is a *complex number* (cf. Annex (A-2)) the unitary transformation $U^{\dagger} = U^{-1}$ replaces $O^T = O^{-1}$. Thus we have two real identities, and the condition

$$L'^{\alpha}_{\alpha} \hat{A}^{\alpha}_{pq} (L^{-1} 'x) = U^{-1}_{p'p} \hat{A}^{\alpha}_{p'q} ('x) U_{qq} \quad (3.8 \hat{A})$$

is valable for *all operators* \hat{A} , whether hermitian or not.

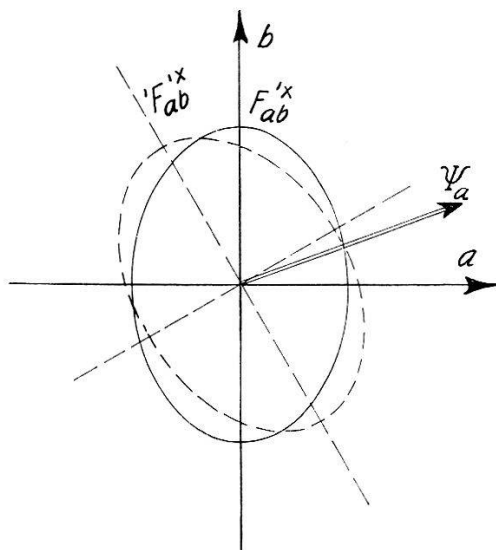


Fig. 1
a-frame

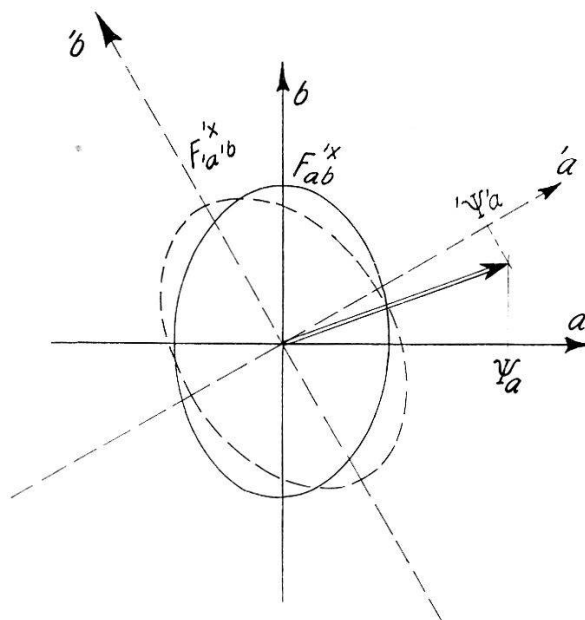


Fig. 2
a-frame and '*a*'-frame

Either (Fig. 1), we form $\Psi_a F_{ab}^{X'} \Psi_b$ in the *a*-frame from the ($'X \leftarrow X$)-transformed tensor

$$'F_{ab}^{X'} = L^X_X F_{ab}^X$$

(which has, on account to the $\{L\}$ -invariance, the same length of the principal axes as the untransformed tensor F_{ab}^X) with respect to Ψ_a .

Or (Fig. 2), we form $'\Psi_a F_{a'b}^{X'} \Psi_b$ in the '*a*'-frame from the tensor $F_{a'b}^{X'}$ (which has, in the '*a*'-frame, the same components as $F_{ab}^{X'}$ has in the *a*-frame) with respect to $'\Psi_a$.

From the two figures follows immediately:

$$' \Psi_a = O_{a a} \Psi_a; F_{a' b}^X = O_{a' a} O_{b b} ' F_{ab}^X \quad (3.13)$$

$$\text{or} \quad ' F_{ab}^X = L^X_X F_{ab}^X = O_{a' a}^{-1} F_{a' b}^X O_{b b}; \quad (3.14)$$

$$' F'^{\alpha \dots} (' x \dots) = L'^{\alpha}_{\alpha} \dots F^{\alpha \dots} (L^{-1} ' x \dots) = O^{-1} F'^{\alpha \dots} (' x \dots) O$$

§ 4. The Thermodynamic Signature of $g^{\alpha\beta}$ and the Pseudo-Chronous Character of $J = \overset{\circ}{J}$

We need the hypotheses that a *particular state* $\Psi^{(0)}$ of the cosmos exists, the vacuum, which is *homogeneous and isotropic*. Let us consider, for the simplicity, a *local scalar observable* $F(x)$, and write the UP for two events x and y in the form

$$\langle J [F(x), F(y)] \rangle_{\Psi^{(0)}} \equiv \overset{\circ}{f}(x y) = - \overset{\circ}{f}(y x). \quad (4.1)$$

Homogeneity requires:

$$\overset{\circ}{f}(x y) = \overset{\circ}{f}(x - y). \quad (4.2)$$

Isotropy would further require:

$$\overset{\circ}{f}(x y) = \overset{\circ}{f}((x - y)^2), \quad (4.3)$$

$$(x - y)^2 = (x - y)_{\alpha} (x - y)^{\alpha}. \quad (4.4)$$

However, (4.3) is in contradiction with the antisymmetry (4.1) of the commutator. There is only one way to turn this difficulty: We have to give to the differentiable manifold $\{x^{\alpha}\}$ *one privileged axis*, $x^n = t$, the time. By this we understand that the metric has the *thermodynamic signature*:

$$\text{signat } (g^{\alpha\beta}) = \pm (1 \ 1 \ \dots \ 1 \ -1). \quad (4.5)$$

We have shown in an earlier paper⁴⁾, that this signature is necessary for a phenomenological relativistic thermodynamics. Thus we may define a function

$$\left. \begin{aligned} \overset{\circ}{f}(x y) &= \text{sig } (x^n - y^n) f((x - x)^2) \\ f((x - y)^2) &= 0 \quad \text{for } x - y = \text{spatial} \end{aligned} \right\} \quad (4.6)$$

which is homogeneous and 'quasi-isotropic'. Now it is easily seen, that $\overset{\circ}{f}$ is a pseudo-chronous bilocal scalar

$$' \overset{\circ}{f}(' x ' y) = \text{sig } (L'^n_n) \overset{\circ}{f}(L^{-1} ' x L^{-1} ' y) \quad (4.7)$$

because, for $L'^n_n > 0$, we have

$$' \overset{\circ}{f}(' x ' y) = \overset{\circ}{f}(L^{-1} ' x L^{-1} ' y) = \overset{\circ}{f}(x y); \quad L'^n_n > 0 \quad (4.8)$$

while, for $L'^n_n < 0$, the relation ($x = L^{-1} 'x$) is

$$\left. \begin{aligned} \check{f}('x 'y) &= -\text{sig}(x^n - y^n) f((x - y)^2) \\ &= -\check{f}(x y); \quad L'^n_n < 0. \end{aligned} \right\} \quad (4.9)$$

Thus we may write, using the homogeneity and the pseudo-chronous isotropy of \check{f} :

$$\check{f}('x 'y) = \text{sig}(L'^n_n) \check{f}(x y). \quad (4.10)$$

Now, consider the transformed value of the observable $J[F(x), F(y)]$, which is, according to (3.5) and (3.6), the expectation value with respect to $\Psi^{(0)}$ of

$$\left. \begin{aligned} O^{-1} (J [F('x), F('y)]) O \\ = O^{-1} J O [F('x), F('y)] \end{aligned} \right\} \quad (4.11)$$

$$\text{in} \quad \langle O^{-1} J O [F('x), F('y)] \rangle_{\Psi^{(0)}} = \check{f}('x 'y) \quad (4.12)$$

or $F('x) = F(L^{-1} 'x) = F(x)$, according to (0.16). Thus, making use of the relation (4.10), we obtain

$$\langle (O^{-1} J O) [F(x), F(y)] \rangle_{\Psi^{(0)}} = \text{sig}(L'^n_n) \check{f}(x y). \quad (4.13)$$

Comparing this relation to (4.1), we find (0.13):

$$J \equiv \check{J}; \quad O^{-1} \check{J} O = \text{sig}(L'^n_n) \check{J} \equiv 'J. \quad (4.14)$$

(4.14) defines the transformed operator $'J$:

The transformation law $\check{J} \rightarrow 'J$ is now analogous to the law (0.17) for an x -independent operator. Note however, that $'J$ is but a *definition*, because we have established the identities expressed in the second equation (0.16) (and (0.17)) by comparing (3.4) and (3.6) in (3.7) only for observables, i. e. for symmetric a -space tensors F, G, \dots and not for antisymmetric a -space tensors like \check{J} .

§ 5. Infinitesimal Lorentz transformations

After having introduced the pseudo-euclidian signature with one privileged axis $x^n = t$ in § 4, our group $\{L\}$ is the *full Lorentz-group in n dimensions*. Writing down the infinitesimal transformation $L = L(\delta\lambda \cdot \delta\omega^{[\cdot\cdot]})$ ((0.18) (0.19)) we find

$$\left. \begin{aligned} L'^\alpha_\alpha F^\alpha(L^{-1} 'x) &= (\delta'^\alpha_\alpha + \frac{1}{2} \delta\omega^{\mu\nu} \Sigma_{\mu\nu}{}'^\alpha_\alpha) F^\alpha('x \cdot - \delta\lambda \cdot - \delta\omega \cdot {}_\nu 'x^\nu) \\ &= F'^\alpha('x) + \delta\lambda^\mu (-' \partial_\mu) F'^\alpha('x) + \frac{1}{2} \delta\omega^{\mu\nu} 'N_{\mu\nu}{}'^\alpha_\alpha F^\alpha('x) \end{aligned} \right\} \quad (5.1)$$

where $-\partial_\mu$ and

$$'N_{\mu\nu}{}^\alpha{}_\alpha = [{}'x_\mu, {}'\partial_\nu] \delta^\alpha{}_\alpha + \Sigma_{\mu\nu}{}^\alpha{}_\alpha \quad (5.2)$$

are the *generators* of the $n + (1/2)n(n-1)$ -parameter *Lie-group* $\{L_{(\text{cont})} = L(\lambda \cdot \omega^{[\cdot\cdot]})\}$, which is the continuous subgroup of $\{L\}$. The generators satisfy the *Lie structure relations*:

$$[-\partial_\mu, -\partial_\nu] = 0 \quad (5.3)$$

$$[N_{\mu\nu}, N_{\sigma\tau}] = -g_{\mu\sigma} N_{\nu\tau} - g_{\nu\tau} N_{\mu\sigma} + g_{\mu\tau} N_{\nu\sigma} + N_{\nu\sigma} g_{\mu\tau} \quad (5.4)$$

$$[-\partial_\mu, N_{\sigma\tau}] = g_{\mu\sigma} (-\partial_\tau) - g_{\mu\tau} (-\partial_\sigma). \quad (5.5)$$

Now the corresponding orthogonal operator $O(\delta\lambda \cdot \delta\omega^{[\cdot\cdot]})$ can be written as

$$O(\delta\lambda \cdot \delta\omega^{[\cdot\cdot]}) = 1 + \delta\lambda^\mu (-\check{J}\check{\Pi}_\mu) + \frac{1}{2} \delta\omega^{\mu\nu} (\check{J}\check{M}_{\mu\nu}). \quad (5.6)$$

The *symmetric operators* $\check{\Pi}_\mu$ and $\check{M}_{[\mu\nu]}$ are *pseudo-chronous observables* and commute with \check{J} , because $O_{(\text{cont})} \rightarrow L_{(\text{cont})}$, contains neither time- nor space-reflections. In particular, the generators of the group $\{O_{(\text{cont})}\}$: $-\check{J}\check{\Pi}_\mu$ and $\check{J}\check{M}_{\mu\nu}$ must satisfy the Lie structure relations (5.3)–(5.5) of $L_{(\text{cont})}$.

Multiplying by \check{J}^{-1} , these are (0.22)–(0.24).

The identity (3.9), which relates L and O is, for the infinitesimal element

$$\left. \begin{aligned} F'^\alpha('x) &= F'^\alpha('x) + \delta\lambda^\mu (-\partial_\mu F'^\alpha('x) - \check{J}[\check{\Pi}_\mu, F'^\alpha('x)]) \\ &+ \frac{1}{2} \delta\omega^{\mu\nu} ({}'N_{\mu\nu}{}^\alpha{}_\alpha F'^\alpha('x) + \check{J}[\check{M}_{\mu\nu}, F'^\alpha('x)]) \end{aligned} \right\} \quad (5.10)$$

and leads thus to (0.20) and (0.21).

The sign of $\check{\Pi}_\mu$ is chosen in order to give, for signat $(g^{\alpha\beta}) = (1 \ 1 \dots 1 \ -1)$, the relation $(\partial_n = \partial_t, \check{\Pi}^n = -\check{\Pi}_n \equiv \check{H})$

$$\partial_t F(\vec{x}, t) = \check{J}[\check{H}, F(\vec{x}, t)], \quad (5.11)$$

where the Heisenberg operator $F(\vec{x}, t)$ and the Schrodinger operator $\bar{F}(\vec{x})$ are related by

$$\langle F(\vec{x}, t) \rangle_\Psi = \langle \bar{F}(\vec{x}) \rangle_{\bar{\Psi}(t)}; \quad \bar{\Psi}(t) = e^{-\check{J}\check{H}t} \Psi \quad (5.12)$$

or

$$F(x) = F(\vec{x}, t) = e^{\check{J}\check{H}t} F(\vec{x}) e^{-\check{J}\check{H}t}. \quad (5.13)$$

We write \check{H} with the pseudo-chronous sign $\check{}$, because it is the n -th component of $\check{\Pi}^\alpha$. For time reflection, $({}'x^i = x^i, {}'x^n = -x^n)$ we have therefore:

$${}'\check{H} = \check{H}, \quad {}'\check{\Pi} = -\check{\Pi}. \quad (5.14)$$

The *energy operator* $\overset{\circ}{H} = \overset{\circ}{\Pi}^n$ does not change its sign, while the *momentum operator* $\overset{\circ}{\Pi}$ changes sign, because velocities change sign.

In order to show that $\overset{\circ}{M}^{ik}$ is the *angular momentum operator*, we consider the transformation for an infinitesimal displacement of the origine:

$$'x'^{\mu} = x'^{\mu} + \delta\lambda'^{\mu} \quad (5.15)$$

$$'M'^{\mu\nu} = O^{-1} \overset{\circ}{M}^{\mu\nu} O = \overset{\circ}{M}^{\mu\nu} + \delta\lambda^{\sigma} \overset{\circ}{J}[\overset{\circ}{\Pi}_{\sigma}, \overset{\circ}{M}^{\mu\nu}] \quad (5.16)$$

or, using (0.24)

$$'M'^{\mu\nu} = \overset{\circ}{M}^{\mu\nu} + (\delta\lambda'^{\mu} \overset{\circ}{\Pi}'^{\nu} - \delta\lambda'^{\nu} \overset{\circ}{\Pi}'^{\mu}). \quad (5.17)$$

This shows that the arm-length of the moment with respect to the primed frame (α -frame) is larger by the amount $\delta\lambda'^{\mu}$ than the arm-length with respect to the α -frame.

§ 6. The momentum-energy density operator

$\overset{\circ}{\Pi}^{\mu}$ and $\overset{\circ}{M}^{[\mu\nu]}$ can be expressed as integrals over an arbitrary time-like surface $'\tau(x) = 0$, whose covariant n -component $d\overset{\circ}{\sigma}_n(x)$ of the surface-element $d\sigma_{\alpha}(x)$ is positive in every α -frame, if we choose the signature $+(1\ 1\ \dots\ 1\ -1)$. This means that $d\overset{\circ}{\sigma}_{\alpha}(x)$ is a *pseudo-chronous time-like vector* (0.26). Then it follows from Gauss' theorem, that the pseudo-chronous quantities $\overset{\circ}{\Pi}^{\mu}$ and $\overset{\circ}{M}^{\mu\nu}$ are independant of the surface $'\tau(x) = 0$ chosen, if (0.25) holds.

To verify the transformation law, let us transform (6.2) according to

$$' \overset{\circ}{\Pi}^{\mu} = \int_{'\tau(x)=0} d\overset{\circ}{\sigma}_{\alpha}(x) O^{-1} \Theta^{\alpha\mu}(x) O = L^{\mu}_{\sigma} \int_{'\tau(x)=0} d\overset{\circ}{\sigma}_{\alpha}(x) L^{\alpha}_{\beta} \Theta^{\beta\sigma}(L^{-1}x) \quad (6.5)$$

and write $y = L^{-1}x$. Then, from the pseudo-chronous character of $d\overset{\circ}{\sigma}_{\alpha}$ follows

$$d\overset{\circ}{\sigma}_{\alpha}(Ly) L^{\alpha}_{\beta} = \text{sig}(L^n_n) d\overset{\circ}{\sigma}_{\beta}(y) \quad (6.6)$$

where $d\overset{\circ}{\sigma}_n(Ly)$ is orientated parallel to $x^n = (Ly)^n$, while $d\overset{\circ}{\sigma}_n(y)$ is orientated parallel to $y^n = (L^{-1}x)^n$.

Thus we have finally, writing $'\mu$ for μ and μ for σ

$$' \overset{\circ}{\Pi}'^{\mu} = \text{sig}(L^n_n) L'^{\mu}_{\mu} \int_{'\tau(Ly)=\tau(y)=0} d\overset{\circ}{\sigma}_{\beta}(y) \Theta^{\beta\mu}(y). \quad (6.7)$$

The integral being independent of the particular surface $\tau(x) = 0$ or $'\tau(x) = 0$ chosen, we may write:

$$' \overset{\circ}{\Pi}'^{\mu} = \text{sig}(L^n_n) L'^{\mu}_{\mu} \overset{\circ}{\Pi}^{\mu}. \quad (6.8)$$

Analogously, we transform

$$\overset{\circ}{M}{}^{\mu\nu} = O^{-1} \overset{\circ}{M}{}^{\mu\nu} O = \int_{\tau(x)=0} d\overset{\circ}{\sigma}_\alpha(x) L^\alpha_\beta(x) L^\mu_\nu(x) \Theta^{\beta\tau}(L^{-1}x) - x^\nu L^\mu_\sigma \Theta^{\beta\sigma}(L^{-1}x) \quad (6.9)$$

and substitute $x = Ly$: $x^\mu = L^\mu_\sigma(y^\sigma + L^\sigma)$. Using again (6.6), we obtain

$$\overset{\circ}{M}{}^{\mu\nu} = \text{sig}(L'^n_n) L'^\mu_\mu L'^\nu_\nu \int_{\tau(y)=0} d\overset{\circ}{\sigma}_\beta(y) ((y^\mu + L^\mu) \Theta^{\beta\nu}(y) - (y^\nu + L^\nu) \Theta^{\beta\mu}(y)) = \text{sig}(L'^n_n) L'^\mu_\mu L'^\nu_\nu (\overset{\circ}{M}{}^{\mu\nu} + L^\mu \overset{\circ}{\Pi}{}^\nu - L^\nu \overset{\circ}{\Pi}{}^\mu) \quad (6.10)$$

Thus we have verified the pseudo-chronous character of momentum-energy $\overset{\circ}{\Pi}{}^\mu$ and of angular momentum-centre of energy $\overset{\circ}{M}{}^{[\mu\nu]}$, expressed as surface integrals of an ortho-chronous momentum-energy density $\Theta^{(\alpha\beta)}(x)$ over a surface $\tau(x) = 0$, with a pseudo-chronous time-like surface element $d\overset{\circ}{\sigma}_\alpha(x)$.

§ 7. Physical Meaning of the Passive Time Reversal

To our *passive interpretation* of time reversal, it has been objected, that only the *active interpretation* has a physical sense, because an observation at an epoch t' changes the earlier state (in the thermodynamic sense) of the system Ψ into a later state Ψ' corresponding to the measure of an observable $F = F^{(i)}$. However, we may consider an observer which makes only correlation experiments:

This observer makes a great number, say N , of experiments, at two epochs $t'_{(1)}$ and $t''_{(1)}$, $t'_{(2)}$ and $t''_{(2)}$ etc. ..., separated always by Δt . Let us suppose first that t measures the thermodynamic time order, and that

$$t'' - t' = t''_{(1)} - t'_{(1)} = t''_{(2)} - t'_{(2)} = \dots = t''_{(N)} - t'_{(N)} = \Delta t > 0 \quad (7.1)$$

Every time, an observer observes $F^{(i)}$ at the earlier epoch t' , he will observe $G = \{G^{(k)}\}$ at the later epoch and thus be able to make a statistics

$$F^{(i)} \rightarrow G^{(1)}, G^{(2)}, \dots, G^{(k)}, \dots G^{(\omega_G)} \quad (7.2)$$

giving transition probabilities

$$W(\overleftarrow{k}, i') \geq 0; \quad \sum_k W(\overleftarrow{k}, i') = 1. \quad (7.3)$$

However, he is free to evaluate his statistics the other way round: Every time he registers $G^{(k)}$ at the later epoch t'' , he makes statistics of the corresponding measures of $F = F^{(i)}$ at the earlier epoch t' . Thus, he obtains transition probabilities

$$W(\overrightarrow{k}, i) \geq 0, \quad \sum_i W(\overrightarrow{k}, i) = 1. \quad (7.4)$$

The coefficients (7.3) and (7.4) are of course equal

$$W(\overrightarrow{k}, i) = W(\overleftarrow{k}, i) = W(k, i). \quad (7.4b)$$

The arrows \leftarrow (evolution in the thermodynamic sense) and \rightarrow (evolution in the opposite sense) are thus superfluous. This means, that quantum-mechanical 'predictions' can be made for the future as well as for the past. If the system is not degenerate, we may write $F^{(a)}$ and $G^{(a)}$ for $F^{(i)}$ and $G^{(k)}$ (see (1.5)) and our correlation coefficients are

$$W(a, a) = (O_a)_a^2 = (O_a^T)_a^2. \quad (7.5)$$

They correspond to the doubly normalised transition amplitudes:

$$W(k, i) \geq 0; \quad \sum_k W(k, i) = 1; \quad \sum_i W(k, i) = 1 \quad (7.6)$$

used by INAGAKI, PIRON and WANDERS⁹⁾ to prove the Boltzmann H theorem for the most general case (STUECKELBERG⁸⁾), while the usual proof assumes, instead of (7.6), detailed balancing

$$W(k, i) = W(i, k) \quad (7.7)$$

which is known to be insufficient⁸⁾⁹⁾.

Therefore, passive time reversal can be verified experimentally.

§ 8. Acknowledgments

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Annex 1. Remark on Generalised Poisson Brackets

In order to establish the *Boltzmann H-theorem* in classical statistical mechanics, we have to start from a covariant theory of motion in *phase space* $x = \{x^\alpha\}$ ($\alpha \beta \dots = 1 \ 2 \dots \omega$), which satisfies the *theorem of Liouville*. The *conservation of energy* $H = H(x)$ leads to an equation of motion for $x^\alpha = z^\alpha(t)$

$$\dot{z}^\alpha(t) = \partial_i z^\alpha(t) = (\partial_\beta H(z(t)) j^{\beta\alpha}(z(t))), \quad (A-1.1)$$

where

$$j^{\alpha\beta}(x) = j^{[\alpha\beta]}(x) \quad (A-2.2)$$

is an *antisymmetrical tensor in phase space*. The *scalar density* of the Gibbs ensemble is $w(x, t) \geq 0$. It satisfies the equation of continuity

$$\int d^\omega x \ w(x, t) = 1; \quad \partial_t w(z, t) + \partial_\alpha (\dot{z}^\alpha w(z, t)) = 0 \quad (A-2.2)$$

and transforms according to

$$\mathfrak{w}'(x, t) = |\det(A'^{\mu}_{\mu}(x))| \mathfrak{w}(x, t) \geq 0. \quad (\text{A-2.3})$$

$$x'^{\alpha} = \psi'^{\alpha}(x); \quad A'^{\alpha}_{\alpha}(x) = \partial_{\alpha} \psi'^{\alpha}(x)$$

Let $d\Omega(x) = \mathfrak{g}(x) |d\varphi^{[12 \dots \omega]}| = \mathfrak{g} d^{(\beta)}x$ be the invariant scalar volume element, where $d\varphi^{[\alpha_1 \alpha_2 \dots \alpha_{\omega}]} = \det(d^{(\beta)}x^{\alpha})$ is the antisymmetric tensor of the parallelipiped, formed from ω non coplanar vectors $d^{(\beta)}x^{\alpha}$. Then we may introduce the *scalar of the density* $w(x, t)$

$$w(x, t) = \frac{dW(x, t)}{d\Omega(x)} = \frac{\mathfrak{w}}{\mathfrak{g}}(x, t) > 0 \quad (\text{A-2.4})$$

where $\mathfrak{g}(x)$ is the 'density of volume'. To form such a density we have only the antisymmetric contravariant tensor $j^{[\mu\nu]}$ at our disposal, which is the *fundamental tensor in phase space*, analogous to the metric tensor $g^{(\alpha\beta)}$ in Riemann space.

$$\text{Therefore we put} \quad \mathfrak{g} = |\det(j^{\mu\nu})^{-1/2}| > 0 \quad (\text{A-2.5})$$

because it has the right transformation property. It is $\neq 0$ if, and only if, $\omega = 2f$ is an *even number*. In terms of $w(x, t)$, the continuity equation (A-2.2) takes the form

$$\partial_t w + D_{\alpha}(z^{\alpha}w) = 0; \quad D_{\alpha} = \partial_{\alpha} + G_{\alpha}; \quad G_{\alpha} = \partial_{\alpha} \log \mathfrak{g}, \quad (\text{A-2.6})$$

where D_{α} is the *covariant divergence operator*. The *theorem of Liouville* states, that the *scalar of the density* w remains constant, if we follow an orbit $x^{\alpha} = z^{\alpha}(t)$:

$$\frac{d}{dt} w(z(t), t) \equiv \dot{w}(z, t) = (\partial_t w + \dot{z}^{\alpha} \partial_{\alpha} w)(z, t) = 0. \quad (\text{A-2.7})$$

This implies (see (A-2.6)):

$$D_{\alpha} \dot{z}^{\alpha} = D_{\alpha}((\partial_{\beta} H) j^{\beta\alpha}) = (\partial_{\alpha} \partial_{\beta} H) j^{[\alpha\beta]} + (\partial_{\beta} H) D_{\alpha} j^{[\alpha\beta]} = 0 \quad (\text{A-2.8})$$

and is a covariant condition*) for the fundamental tensor:

$$D_{\alpha} j^{[\alpha\beta]} = q^{\beta} = 0. \quad (\text{A-2.9.j})^{*}$$

We may express it in terms of the density $j^{\alpha\beta} = \mathfrak{g} j^{[\alpha\beta]}$

$$\partial_{\alpha} j^{[\alpha\beta]} = q^{\beta} = 0. \quad (\text{A-2.9.j})^{*}$$

(A-2.9) is formally analogous to the *second set of Maxwell's equations*, if no electric charges q^{β} are present.

) $D_{\alpha} F^{[\alpha\beta\gamma \dots]} = G^{[\beta\gamma \dots]}$ or $\partial_{\alpha} \mathfrak{F}^{[\alpha\beta \dots]} = \mathfrak{G}^{[\beta\gamma \dots]}$ (A-2.9)

and $\partial_{[\alpha} F_{\beta\gamma \dots]} = G_{[\alpha\beta\gamma \dots]}$ are covariant relations. (A-2.13*)

An observable $F(x)$ varies with time, according to

$$\dot{F}(z(t)) = \dot{z}^\alpha \partial_\alpha F = (\partial_\beta H) j^{[\beta\alpha]} \partial_\alpha F \equiv \{H, F\} = -\{F, H\} \quad (\text{A-2.10})$$

where $\{H, F\}$ defines a *generalised Poisson bracket*.

It immediately follows, from

$$\begin{aligned} \{F, \{G, H\}\} &= (\partial_\alpha F) j^{[\alpha\beta]} \partial_\beta ((\partial_\gamma G) j^{[\gamma\delta]} \partial_\delta H) \\ &= (\partial_\alpha F) j^{[\alpha\beta]} (\partial_\beta \partial_\gamma G) j^{[\gamma\delta]} \partial_\delta H + (\partial_\alpha F) j^{[\alpha\beta]} (\partial_\gamma G) j^{[\gamma\delta]} \partial_\beta \partial_\delta H \\ &\quad + (\partial_\alpha F) (\partial_\beta G) (\partial_\gamma H) j^{[\alpha\varrho]} \partial_\rho j^{[\beta\gamma]} \end{aligned}$$

that the cyclic sum is

$$\underset{FGH}{\hookrightarrow} \{F, \{G, H\}\} = (\partial_\alpha F) (\partial_\beta G) (\partial_\gamma H) \underset{\alpha\beta\gamma}{\hookrightarrow} j^{[\alpha\varrho]} \partial_\rho j^{[\beta\gamma]}. \quad (\text{A-2.11})$$

Thus, the Jacobi identity is not necessarily satisfied.

If we require this identity, we must have

$$\underset{\alpha\beta\gamma}{\hookrightarrow} j^{\alpha\varrho} \partial_\rho j^{\beta\gamma} \equiv q^{[\alpha\beta\gamma]} = 0, \quad (\text{A-2.12})$$

to which we may give the form of the *first set of Maxwell's equations* in the absence of magnetic charge $q_{[\alpha\beta\gamma]}$

$$\underset{\alpha\beta\gamma}{\hookrightarrow} \partial_\alpha j_{[\beta\gamma]} \equiv q_{[\alpha\beta\gamma]} = 0. \quad (\text{A-2.13}^*)$$

if we introduce the inverse tensor $j_{\alpha\beta}$:

$$j_{\alpha\rho} j^{\beta\varrho} = \delta_\alpha^\beta; \quad j_{\alpha\beta} = \min(j^{\alpha\beta}) / \det(j^{\mu\nu}). \quad (\text{A-2.14})$$

From this definition follows

$$j_{\alpha\rho} j_{\beta\sigma} j^{\varrho\sigma} = j_{\alpha\rho} \delta_\beta^\varrho = j_{\alpha\beta}. \quad (\text{A-2.15})$$

We may thus raise and lower indices, with these antisymmetric fundamental tensors $j^{[\alpha\beta]}$ and $j_{[\alpha\beta]}$

$$F_{\alpha\beta\dots} = j_{\alpha\alpha'} j_{\beta\beta'} \dots F^{\alpha'\beta'} \dots \quad F^{\alpha\beta} = j^{\alpha\alpha'} j^{\beta\beta'} \dots F_{\alpha'\beta'} \dots \quad (\text{A-2.16})$$

In particular, it follows from

$$\partial_\rho \delta_{\gamma'}^\gamma = (\partial_\rho j^{\beta\gamma}) j_{\beta\gamma'} + j^{\beta\gamma} \partial_\rho j_{\beta\gamma'} = 0$$

that

$$\partial_\rho j^{\beta\gamma} = j^{\beta\beta'} j^{\gamma\gamma'} \partial_\rho j_{\beta'\gamma'}. \quad (\text{A-2.17})$$

Introducing this expression into (A-2.12), we arrive at the first set of 'Maxwell's equations' (A-2.13), with the conditions that 'magnetic charges' $q_{[\alpha\beta\gamma]}$ vanish.

This totally antisymmetric tensor of, magnetic charge' $q^{[\alpha\beta\gamma]}$ in phase space with a fundamental tensor $j^{[\alpha\beta]}$ plays an analogous role to the Riemann-Christoffel Tensor $R_{([\alpha\beta][\gamma\delta])}$ in a space with a metric $g_{(\alpha\beta)}$: If $q^{[\alpha\beta\gamma]}$ vanishes, a coordinate frame $'x'^\alpha$ exists, where $j^{[\alpha\beta]}$ has the form

$$\{j'^{\alpha'\beta'}\} = \left\{ \begin{array}{c|c|c} \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} & 0 & 0 \\ \hline 0 & \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} & 0 \\ \hline 0 & 0 & \ddots \end{array} \right\}; \det (j'^{\alpha'\beta'}) = 1. \quad (\text{A-2.19})$$

This is analogous to the existence of an euclidian or pseudo-euclidian frame $'g_{\alpha'\beta'} = \pm \delta_{\alpha'\beta'}$ in the flat space, if $R_{([\alpha\beta][\gamma\delta])} = 0$. The proof of this theorem is given for instance by WHITTAKER¹⁰ (see also PAULI¹²):

It states, that the Pfaff differential expression formed from a_α in

$$j_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha \quad (\text{A-2.20})$$

consisting of ω terms

$$a_\alpha(x) dx^\alpha = \sum_{i=1}^{i=f} p_i(x) dq_i(x) + \begin{cases} 0; 2f \leq n \\ dq_{l+1}; 2f+1 \leq n \end{cases} \quad (\text{A-2.21})$$

can always be expressed in the form of the right-hand side.

Now put (as $\omega = 2f$)

$$p_i(x) = 'x^{2i-1}, q_i(x) = 'x^{2i}, i = 1 \ 2 \ \dots \ f. \quad (\text{A-2.22})$$

Then we have

$$'a_{2i}('x) = 'x^{2i-1}; \quad 'a_{2i-1}('x) = 0 \quad (\text{A-2.22})$$

and the only non zero component of $'j'^{\alpha'\beta'}$

$$' \partial_{2i-1} 'a_{2i}('x) = 'j_{2i-1, 2i} = +1, \quad 'j^{2i, 2i-1} = \min ('j_{2i, 2i-1}) = -1. \quad (\text{A-2.23})$$

Introducing

$$'H('x) = H(x) = 'H(p, q) \quad (\text{A-2.24})$$

we have

$$\left. \begin{aligned} \dot{z}^{2i-1} = \dot{p}_i = ' \partial_{2i} 'H('z) 'j^{2i, 2i-1} &= - ' \partial_{2i} 'H('z) = - \frac{\partial}{\partial q_i} 'H(p, q) \\ \dot{z}^{2i} = \dot{q}_i = ' \partial_{2i-1} 'H('z) 'j^{2i-1, 2i} &= ' \partial_{2i-1} 'H('z) = + \frac{\partial}{\partial p_i} 'H(p, q). \end{aligned} \right\} \quad (\text{A-2.25})$$

Therefore we see, that the Jakobi identity is *by no means necessary to establish the H-theorem*: Only the 'second Maxwell set' (absence of "electric charges' q^α) has to be satisfied. The presence of 'magnetic charges' $q_{[\alpha\beta\gamma]}$ in the "first Maxwell set' does *not invalidate* Liouville's theorem (from which the Boltzmann H-theorem follows). To this generalised statistical mechanics corresponds aQT, in which the observables are no more linear operators, because of:

$$\overset{\curvearrowright}{FGH} J [F, J [G, H]] \neq 0. \quad (\text{A-2.26})$$

Annex 2. Complex Hilbert Space (CHS)

If we restrict ourselves to the ortho-chronous subgroup $\{L_{(\text{ochr})}\}$ with $L'_{(\text{ochr})n} > 0$, all operators $\{F, G, \dots, \tilde{J}, O_{(\text{ochr})}\}$ commute with \tilde{J} :

$$[\tilde{J}, O_{(\text{ochr})}] = [\tilde{J}, F] = \dots = 0; \quad \tilde{J}^T = -\tilde{J}; \quad \tilde{J}^2 = -1. \quad (\text{A-2.1})$$

We may now establish a relation (dictionary) between our QT in RHS and the conventional QT in CHS. To do this, we consider the ω_R -dimensional RHS as a product space between a 2-dimensional RHS and an $\omega_C = 1/2 \omega_R$ -dimensional RHS. We write

$$\Phi = (\Phi_{(r)} \Phi_{(i)}); \quad \Psi = \begin{pmatrix} \Psi_{(r)} \\ \Psi_{(i)} \end{pmatrix} \quad (\text{A-2.2})$$

and the arbitrary operator A ($[\tilde{J}, A] = 0$) as the Kronecher product (\times)

$$A = 1 \times A_{(r)} + j \times A_{(i)}; \quad \tilde{J} = j \times 1 \quad (\text{A-2.3})$$

$$A_{(r)} = \{A_{(r) pq}\} \quad \text{and} \quad A_{(i)} = \{A_{(i) pq}\}$$

are $\omega_C = 1/2 \omega_R$ -dimensional matrices ($p, q \dots = 1, 2 \dots \omega_C$) and

$$j = \begin{pmatrix} 0 & -\lambda \\ +\lambda & 0 \end{pmatrix}; \quad j^2 = -1; \quad \lambda^2 = 1. \quad (\text{A-2.4})$$

Now let us consider the ω_C -dimensional complex Matrix ($i = +\sqrt{-1}$)

$$\hat{A} = A_{(r)} + i A_{(i)}; \quad \hat{A}^\dagger = A_{(r)}^T - i A_{(i)}^T \quad (\text{A-2.5})*$$

and the two ω_C -dimensional complex vectors

$$\hat{\Phi} = \Phi_{(r)} + i \Phi_{(i)}; \quad \hat{\Psi} = \Psi_{(r)} + i \Psi_{(i)} \quad (\text{A-2.6})$$

where $\Phi_{(r)}, \Phi_{(i)}, \Psi_{(r)}, \Psi_{(i)}, A_{(r)}$ and $A_{(i)}$ are the real ω_C -dimensional vectors and matrices. Now, we define the usual complex matrix element in CHS by

$$\left. \begin{aligned} \langle \hat{\Phi}, \hat{A} \hat{\Psi} \rangle &= ((\Phi_{(r)} - i \Phi_{(i)}), (A_{(r)} + i A_{(i)}) (\Psi_{(r)} + i \Psi_{(i)})) \\ &= ((\Phi_{(r)}, A_{(r)} \Psi_{(r)}) + (\Phi_{(i)}, A_{(r)} \Psi_{(i)}) - (\Phi_{(r)}, A_{(i)} \Psi_{(i)}) \\ &\quad + (\Phi_{(i)}, A_{(i)} \Psi_{(r)})) + i((\Phi_{(r)}, A_{(i)} \Psi_{(r)}) + (\Phi_{(i)}, A_{(i)} \Psi_{(i)}) + \\ &\quad + (\Phi_{(r)}, A_{(r)} \Psi_{(i)}) - (\Phi_{(i)}, A_{(r)} \Psi_{(r)})) = (\Phi_{(r)} \Phi_{(i)}) \\ &\quad \left(\begin{pmatrix} A_{(r)} - A_{(i)} \\ A_{(i)} \quad A_{(r)} \end{pmatrix} \begin{pmatrix} \Psi_{(r)} \\ \Psi_{(i)} \end{pmatrix} - i(\Phi_{(r)} \Phi_{(i)}) \begin{pmatrix} -A_{(i)} - A_{(r)} \\ A_{(r)} - A_{(i)} \end{pmatrix} \begin{pmatrix} \Psi_{(r)} \\ \Psi_{(i)} \end{pmatrix} \right) \\ &= \langle \hat{A} \hat{\Psi}, \hat{\Phi} \rangle^*. \end{aligned} \right\} \quad (\text{A-2.6})*$$

*) An * denotes the conjugate complex number. An \dagger signifies the Hermitian conjugate operator:

$$\hat{A}_{pq}^\dagger = \hat{A}_{qp}^*; \quad \hat{A}^\dagger = (\hat{A}^*)^T = (\hat{A}^T)^*. \quad (\text{A-2.9})$$

This expression is equal to

$$\langle \widehat{\Psi}, \widehat{A} \widehat{\Phi} \rangle = (\Phi, A \Psi) - i(\Phi, \overset{\circ}{J} A \Psi) \quad (\text{A-2.7})$$

if we choose $\lambda = +1$ in (A-2.3), i. e.

$$j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (\text{A-2.8})$$

(A-2.7) is the dictionary we proposed to establish.

It is worth noting, that the definition of j in (A-2.3) is univoque: (A-2.8).

Annex 3. Unitary (U) and Anti-Unitary Operators (V) in CHS

We write our dictionary between CHS and RHS (A-2.7) in the form

$$\langle \widehat{\Phi}, \widehat{A}^\alpha(x) \widehat{\Psi} \rangle = (\Phi, (1 - i \overset{\circ}{J}) A^\alpha(x) \Psi) \quad (\text{A-3.1})$$

where the left hand side is the scalar product in complex Hilbert space (= CHS) and the right hand side is the scalar product in RHS (i. e. all symbols, except i , stand for real quantities (vectors, operators)). Let us consider the transformed quantity (matrix element of $A^\alpha(x)$ between the states $\Phi \leftarrow \Psi$)

$$\left. \begin{aligned} \langle \widehat{\Phi}, \widehat{A}'^\alpha(x) \widehat{\Psi} \rangle &= L'^\alpha_\alpha (\Phi, (1 - i \text{sig}(L'^n_n) \overset{\circ}{J}) A^\alpha(L^{-1} x) \Psi) \\ \equiv \langle \widehat{\Phi}, \widehat{A}'^\alpha(x) \widehat{\Psi} \rangle &= (\Phi, (1 - i \overset{\circ}{J}) A'^\alpha(x) \Psi) \end{aligned} \right\} \quad (\text{A-3.2})$$

a) Orthochronous transformations

($L'^n_{(\text{ochr})n} > 0$): We have

$$\Psi'_a = O_{aa} \Psi_a; \begin{pmatrix} \Psi'_{(r)} \\ \Psi'_{(i)} \end{pmatrix} = (1 \times O_{(r)} + j \times O_{(i)}) \begin{pmatrix} \Psi_{(r)} \\ \Psi_{(i)} \end{pmatrix} \quad (\text{A-3.4})$$

$$\widehat{\Psi}'_p = U_{pp} \widehat{\Psi}_p; \widehat{\Psi}' = (O_{(r)} + i O_{(i)}) \widehat{\Psi} \equiv U \widehat{\Psi} \quad (\text{A-3.5})$$

$$\widehat{\Phi}'^*_p = \widehat{\Phi}^*_p U^{*T}_{p'p} = \widehat{\Phi}^*_p U^\dagger_{p'p}; \widehat{\Phi}'^* = (O_{(r)} - i O_{(i)}) \widehat{\Phi}^* \equiv U^* \widehat{\Phi}^* \equiv \widehat{\Phi}^* U^\dagger \quad (\text{A-3.5*})$$

From the orthogonality of O

$$\left. \begin{aligned} O^T O &= (1 \times O^T_{(r)} - j \times O^T_{(i)}) (1 \times O_{(r)} + j \times O_{(i)}) = \\ 1 \times (O^T_{(r)} O_{(r)} + O^T_{(i)} O_{(i)}) &+ j \times (O^T_{(r)} O_{(i)} - O^T_{(i)} O_{(r)}) = 1 \times 1 = 1 \end{aligned} \right\} \quad (\text{A-3.6})$$

follows

$$O^T_{(r)} O_{(r)} + O^T_{(i)} O_{(i)} = 1; \quad O^T_{(r)} O_{(i)} - O^T_{(i)} O_{(r)} = 0. \quad (\text{A-3.8})$$

Thus, according to the definition of U and U^\dagger in (A-3.5), and (A-3.5*),

$$U^\dagger U = 1 \quad (\text{A-3.9})$$

$U = \{U_{pq}\}$ is an *unitary matrix* in $\omega_C = 1/2 \omega_R$ -dimensional CHS ($pq \dots = 1 \ 2 \dots \omega_C$; $a \ b \dots = 1 \ 2 \dots \omega_R$). Because of $\text{sig}(L'_n) = 1$ and of (A-3.1), we may write the second member of (A-3.2) in the form $L'^\alpha_\alpha \langle \hat{\Phi}, \hat{A}^\alpha(L^{-1} 'x) \hat{\Psi} \rangle$. Thus we obtain finally (see (A-2.5)) the identity

$$\hat{A}'^\alpha('x) = L'^\alpha_\alpha U \hat{A}^\alpha(L^{-1} 'x) U^\dagger \quad (\text{A-3.10})$$

i. e. the tensor $\hat{A}_{pq}(x) = \hat{A}_{pq}^X$ is invariant if it is transformed with respect to all its indices:

$$\hat{A}'^X_{p'q} = L'^X_{(\text{ochr})X} U_{p'p} U_{q'q}^* \hat{A}_{pq}^X \quad (\text{A-3.11})$$

and U is an unitary representation of $L_{(\text{ochr})}$

$$L_{(\text{ochr})} \rightarrow e^{i\lambda} U. \quad (\text{A-3.12})$$

b) *Pseudochronous Transformations* ($L'_{(\text{pchr})n} < 0$)

We try (A-3.2), posing for the transformed matrix element (A-3.2)

$$\left. \begin{aligned} \langle \hat{\Phi}, \hat{A}'^\alpha('x) \hat{\Psi} \rangle &= \langle \hat{\Phi}, \hat{A}'^\alpha('x) ' \hat{\Psi} \rangle = \\ L'^\alpha_\alpha \langle \hat{\Phi}, (1 + i \check{J}) A^\alpha(L^{-1} 'x) \Psi \rangle &= L'^\alpha_\alpha \langle \hat{\Phi}, \hat{A}^\alpha(L^{-1} 'x) \hat{\Psi} \rangle^*. \end{aligned} \right\} \quad (\text{A-3.13})$$

It is, in virtue of the definition (A-3.1), a *linear function of the untransformed conjugate complex element*. We shall see however, that the identity in $\hat{A}'^\alpha_{p'q}('x)$ leads now to a contradiction. In order to show that, write, introducing a *non-linear operator* V

$$' \hat{\Psi} = (V \hat{\Psi}), ' \hat{\Phi} = (V \hat{\Phi})$$

the second and fourth member of (A-3.13) in the Form:

$$\langle (V \hat{\Phi}), V(V^{-1} (\hat{A}'^\alpha('x) (V \hat{\Psi}))) \rangle = L'^\alpha_\alpha \langle \hat{A}^\alpha(L^{-1} 'x) \hat{\Psi}, \hat{\Phi} \rangle. \quad (\text{A-3.14})$$

Thus, we must have a relation

$$(V \hat{\Psi})_p = U_{p'p} \hat{\Psi}_{p'}^*; \quad (V^{-1} \hat{\Psi})_p = U_{p'p}^T \hat{\Psi}_{p'}^* \quad (\text{A-3.15})$$

$$(V \hat{\Phi})_p^* = U_{p'p}^* \hat{\Phi}_{p'} = \hat{\Phi}_{p'} U_{p'p}^\dagger; \quad (V^{-1} \hat{\Phi})_p^* = U_{p'p}^\dagger \hat{\Phi}_{p'}. \quad (\text{A-3.15}^*)$$

where U is an unitary matrix, in order to have

$$\left. \begin{aligned} \langle V(\hat{\Phi}), V(\hat{\Psi}) \rangle &= \langle \hat{\Phi}, \hat{\Psi} \rangle^* = \langle \hat{\Psi}, \hat{\Phi} \rangle; \\ U^\dagger U &= U^T U^* = 1. \end{aligned} \right\} \quad (\text{A-3.16})$$

On account of (A-3.16), the identity (A-3.14) (second = forth member in (A-3.13)) has the form:

$$(V^{-1} (\hat{A}'^\alpha('x) (V \hat{\Psi})))_p^* \hat{\Phi}_p = (L'^\alpha_\alpha A^\alpha(L^{-1} 'x) \hat{\Psi})_p^* \hat{\Phi}_p, \quad (\text{A-3.17})$$

which must hold for all $\hat{\Phi}_q$. Thus we have

$$(V^{-1} (\hat{A}'^\alpha('x) (V \hat{\Psi})))_p = (L'^\alpha_\alpha \hat{A}^\alpha(L^{-1} 'x) \hat{\Psi})_p \quad (\text{A-3.18})$$

from this identity in $\hat{\Psi}_p$ (see (A-3.15)) follows

$$U_{p'p}^T \hat{A}_{p'q}^{*'}('x) U_{qq}^* = L'^\alpha_\alpha \hat{A}_{pq}^\alpha(L^{-1} 'x) \quad (\text{A-3.19})$$

$$\text{or} \quad A_{p'q}^{*'} = L'^X_X U_{p'p}^* U_{qq} \hat{A}_{pq}^X. \quad (\text{A-3.20})$$

This identity can evidently not be satisfied in general, because no linear transformation $(L'^\alpha_\alpha U^* \dots U^T)$ exists, which transforms *all tensors* \hat{A}_{pq}^X in hermitian CHS into their conjugate complex. Thus we conclude, that pseudo-chronous transformations are not given by (A-3.13), but we must have

$$\langle \hat{\Phi}, \hat{A}'^\alpha('x) \hat{\Psi} \rangle = \langle \hat{\Phi}, \hat{A}^{*'}^\alpha('x) \hat{\Psi} \rangle = L'^\alpha_\alpha \langle \hat{\Phi}, \hat{A}^\alpha(L^{-1} 'x) \hat{\Psi} \rangle^*. \quad (\text{A-3.13}^*)$$

In this case, $\hat{A}'^\alpha('x)$ (in the third member of (A-3.2) and in the first members of equations (A-3.14), (A-3.17)–(A-3.20)) has to be replaced by $\hat{A}^{*'}^\alpha('x)$.

(A-3.20) is now analogous to (A-3.11)

$$\hat{A}_{p'q}^{*'} = L'^X_{(\text{pchr})X} U_{p'p}^* U_{qq} \hat{A}_{pq}^X. \quad (\text{A-3.20}^*)$$

This rather complicated formalism for time reversal in CHS shows clearly the advantage of RHS, where all transformations $L \rightarrow e^{\lambda J} O$ are linear.

Annex 4. Quaternion Hilbert Space (QHS)

QHS has been introduced by FINKELSTEIN, JAUCH and SPEISER³⁾. They pose, with $i_\alpha^2 = -1$; $i_1 i_2 = -i_2 i_1 = i_3$ cycl.:

$$\hat{\Phi} = \Phi_{(r)} + i_1 \Phi_{(1)} + i_2 \Phi_{(2)} + i_3 \Phi_{(3)} = \{\hat{\Phi}_p\} \quad (\text{A-4.1})$$

$$\hat{\Psi} = \Psi_{(r)} + i_1 \Psi_{(1)} + i_2 \Psi_{(2)} + i_3 \Psi_{(3)} = \{\hat{\Psi}_q\} \quad (\text{A-4.2})$$

$$\hat{A} = A_{(r)} + i_1 A_{(1)} + i_2 A_{(2)} + i_3 A_{(3)} = \{\hat{A}_{pq}\}. \quad (\text{A-4.3})$$

The QHS has $\omega_Q = 1/4 \omega_R$ -dimensions: $p, q, \dots = 1, 2, \dots, \omega_Q$. The scalar product is defined by

$$\langle \hat{\Phi}, \hat{A} \hat{\Psi} \rangle = \hat{\Phi}_p^* \hat{A}_{pq} \hat{\Psi}_q \quad (\text{A-4.4})$$

where $\hat{\Phi}_p^*$ is the conjugated quaternion of $\hat{\Phi}_p$ ($i_\alpha \rightarrow -i_\alpha$). All numbers, $\hat{\Phi}_p, \hat{A}_{pq}, \hat{\Psi}_q$ and $\langle \hat{\Phi}, \hat{A} \hat{\Psi} \rangle$ are now quaternions ($\Phi_{(r)} \dots A_{(r)} \dots$ are real vectors and tensors). A straight forward calculation gives

$$\langle \hat{\Phi}, \hat{A} \hat{\Psi} \rangle_{(r)} = (\Phi, A \Psi) \quad (\text{A-4.5})$$

with

$$\Phi = (\Phi_{(r)} \Phi_{(1)} \Phi_{(2)} \Phi_{(3)}); \Psi = \begin{pmatrix} \Psi_{(r)} \\ \Psi_{(1)} \\ \Psi_{(2)} \\ \Psi_{(3)} \end{pmatrix} \quad (\text{A-4.6})$$

$$A = 1 \times A_{(r)} + j_1 \times A_{(1)} + j_2 \times A_{(2)} + j_3 \times A_{(3)}. \quad (\text{A-4.7})$$

The j_α 's are four by four matrices and satisfy the same commutation laws as the quaternions i_α 's ($j^2 = -1$; $j_1 j_2 = -j_2 j_1 = j_3$ cycl). They are KRONECKER products of the 'pseudo-quaternions':

$$\left. \begin{aligned} 1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad l = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ -j^2 &= k^2 = l^2 = 1; \quad jk = -kj = l, \quad kl = -lk = -j \\ lj &= -jl = k \end{aligned} \right\} \quad (\text{A-4.8})$$

$$j_1 = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} = 1 \times j; \quad j_2 = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix} = j \times k; \quad j_3 = \begin{pmatrix} 0 & -l \\ l & 0 \end{pmatrix} = j \times l. \quad (\text{A-4.9})$$

However, in order to form the vector components of the quaternion, we need three further four by four matrices

$$\left. \begin{aligned} k_1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = k \times 1; \quad k_2 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} = 1 \times k; \quad k_3 = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix} = k \times k \\ k_1 k_2 &= k_2 k_1 = k_3 \text{ cycl.}; \quad k_\alpha^2 = 1 \\ [k_\alpha, j_\alpha] &= 0; \quad (k_\alpha, j_\beta) = 0; \quad \alpha \neq \beta \end{aligned} \right\} \quad (\text{A-4.10})*$$

in order to write

$$\langle \hat{\Phi}, \hat{A} \hat{\Psi} \rangle_\alpha = -(\Phi, (k_\alpha j_\alpha \times 1) A \Psi). \quad (\text{A-4.11})$$

Introducing, analogous to (A-2.3)

$$J_\alpha = j_\alpha \times 1; \quad K_\alpha = k_\alpha \times 1, \quad (\text{A-4.12})$$

we find

$$\left. \begin{aligned} \langle \hat{\Phi}, \hat{A} \hat{\Psi} \rangle &= (\Phi, A \Psi) - i_1 (\Phi, K_1 J_1 \Psi) - i_2 (\Phi, K_2 J_2 \Psi) \\ &\quad - i_3 (\Phi, K_3 J_3 \Psi). \end{aligned} \right\} \quad (\text{A-4.13})$$

Thus, QT in QHS is not equivalent to QT in RHS, because we need — in addition to the three anticommuting operators

$$J_\alpha^2 = -1; \quad J_1 J_2 = -J_2 J_1 = J_3 \text{ cycl.} \quad (\text{A-4.14})$$

— three further operators

$$\left. \begin{aligned} K_\alpha^2 &= 1; \quad K_1 K_2 = K_2 K_1 = K_3 \text{ cycl.} \\ [K_\alpha, J_\alpha] &= 0; \quad (K_\alpha, J_\beta) = 0, \quad \alpha \neq \beta. \end{aligned} \right\} \quad (\text{A-4.15})*$$

*) We write $[A, B] = AB - BA$ for the commutator and $(A, B) = AB + BA$ for the anticommutator. This is in strict analogy to our notation for antisymmetric tensors $j^{[\alpha\beta]} = -j^{[\beta\alpha]}$ and for symmetric tensors $g^{(\alpha\beta)} = g^{(\beta\alpha)}$.

Annex 5. An Error in the Representation Theory Frequently Found in Literature

Instead of our identity, following from (3.4) and (3.5)

$$(\Psi, 'F'^{\alpha}('x) \Psi) = (' \Psi, F'^{\alpha}('x) ' \Psi) \quad (\text{A-5.1})$$

many authors start from the identity

$$(' \bar{\Psi}, ' \bar{F}'^{\alpha}('x) ' \bar{\Psi}) = (\bar{\Psi}, \bar{F}'^{\alpha}('x) \bar{\Psi}), \quad (\text{A-5.2}^*)$$

which is analogous to the relation (5.2) between the Schroedinger and the Heisenberg representation. We refer particularly to the otherwise excellent book by JAUCH and ROHRLICH¹¹) (to be referred to as *J R*). They define

$$' \bar{\Psi} = \bar{O} \bar{\Psi}, \quad \bar{O}^T = \bar{O}^{-1} \quad (\text{A-5.3})$$

and, as we have done,

$$' \bar{F}'^{\alpha}('x) = L'^{\alpha}_{\alpha} \bar{F}^{\alpha}(L^{-1} 'x). \quad (\text{A-5.4})$$

From the identity in $' \bar{\Psi}'_a$ they find, of course

$$' \bar{F}'^{\alpha}('x) = \bar{O} \bar{F}'^{\alpha}('x) \bar{O}^{-1}. \quad (\text{A-5.5})$$

Now, they pretend that $\{e^{\check{J}\lambda} \bar{O}\}$ is a ray representation of $\{L\}$. The identity which follows from (A-5.4) and (A-5.5) is, explicitly written:

$$L'^{\alpha}_{\alpha} \bar{F}'^{\alpha}_{a'b}(L^{-1} 'x) = \bar{O}_{aa} \bar{F}'^{\alpha}_{ab}('x) \bar{O}^{-1}_{b'b} \quad (\text{A-5.6})$$

(Eq. (1-43) and (1-42) p. 11 of *JR*). According to *JR*, \bar{O} (and not $\bar{O}^{-1} = \bar{O}^T \equiv O$, as we found in (3.8)) is a representation of L . They give no proof of their statement.

We shall give an argument, which may have lead them to this contradictory statement: Write $L_{(1)}$ in (A-5.4) and $\bar{O}^{(1)}$ in (A-5.5). $L_{(1)}$ transforms from the $X = \{x\}$ -frame to the $'X$ -frame. Let $L_{(1)}$ be followed by $L_{(2)}$ transforming $''X \leftarrow 'X$:

$$'' \bar{F}''^{\alpha}(''x) = L_{(2)}''^{\alpha}_{\alpha} \bar{F}'^{\alpha}(L_{(2)}^{-1} ''x) \quad (\text{A-5.4}^{(2)})$$

$$'' \bar{\Psi}''_a = \bar{O}^{(2)}_{a'a} \bar{\Psi}'_a \quad (\text{A-5.3}^{(2)})$$

$$'' \bar{F}''^{\alpha}(''x) = \bar{O}^{(2)} \bar{F}'^{\alpha}('x) \bar{O}^{(2)-1}. \quad (\text{A-5.5}^{(2)})$$

Now, substitute (5.4⁽¹⁾) into (5.4⁽²⁾) i. e.

$$'' \bar{F}''^{\alpha}(''x) = (L_{(2)} L_{(1)})''^{\alpha}_{\alpha} \bar{F}^{\alpha}((L_{(2)} L_{(1)})^{-1} ''x) \quad (\text{A-5.6})$$

and (A-5.5⁽¹⁾) into (A-5.5⁽²⁾)

$$'' \bar{F}''^{\alpha}(''x) = (\bar{O}^{(2)} \bar{O}^{(1)}) \bar{F}'^{\alpha}('x) (\bar{O}^{(2)} \bar{O}^{(1)})^{-1}. \quad (\text{A-5.7})$$

*) We denote vectors and operators satisfying (A-5.2) by a bar, in order to distinguish them from the vectors and operators in the text and in (A-5.1).

If we eliminate ${}^{\prime\prime}\bar{F}^{\alpha}({}^{\prime\prime}x)$ from the last two equations, we find writing $X = \{\alpha x\}$

$$(L_{(2)} L_{(1)})^{\prime\prime X}{}_X \bar{F}^X_{a{}^{\prime\prime}b} = (\bar{O}^{(2)} \bar{O}^{(1)})_{aa} \bar{F}^X_{ab} (\bar{O}^{(2)} \bar{O}^{(1)})^{-1}_{b{}^{\prime\prime}b}. \quad (\text{A-5.8})$$

Thus, it seems to follow from (A-5.8) that $\{\overset{\circ}{e}J^{\lambda}\bar{O}\}$ is a representation of $\{L\}$. This is of course contradictory to the theory presented in the text (§ 3). C. PIRON and H. RUEGG shall publish a note in this journal, which shows how the two contradictory points of view can be understood.

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