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# Systems of Observables in Quantum Mechanics

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*Abstract.* The self adjoint linear operators which represent the observables of a physical system are in general not an irreducible system. Because a complete set of commuting observables must determine the state of a physical system unambiguously the observables generate an algebra of operators which must contain a maximal abelian subalgebra. The structure of such algebras is investigated and it is shown by applying the theory of the direct integral of Hilbert spaces that there exists always a unique canonical representation of the Hilbert space as a direct integral in such a way that the transformations which are induced by the observables in the component subspaces are irreducible.

## 1. Introduction

In this paper we shall analyze the mathematical structure of the sets of operators which represent observables in a quantum mechanical system. Until recently it has generally been assumed, either tacitly or overtly, that such operators form an irreducible system in the space of the state vectors. And indeed the simple systems usually encountered in ordinary atomic physics are of this kind. Yet as Wick, Wightman, and Wigner have first pointed out, there are important physical systems for which this is not the case[1]). In such systems not every self-adjoint operator is an observable. In $W^3$ the example given for such operators which are not observables are the field operators of a spinor field. Another example are the 1- and 2-components of the isotopic spin operators[2]). When this phenomenon occurs we speak of *superselection rules*.

Under these circumstances one may well ask the question what one can state in general about the structure of the sets of operators which represent the physical observables. In this general form the question does not seem to be well set since there are no known criteria by which one might characterize sets of operators as observables of a real physical system. However there is an important restriction on such systems which stems from the fact that there must exist a 'complete set of commuting observables'. Roughly speaking the existence of such a set means that the set of all observables cannot be too small, although it may not have to be an irreducible set. It will be shown in this paper that this condition leads to a complete structure theory of the systems of physical observables with the very simple result that to every physical system belongs a unique decomposition of the Hilbert space in such a manner that the observables act irreducibly on the component spaces.

## 2. Complete Sets of Commuting Observables

The need for this concept arises from the problem of assigning state vectors to sets of measurements on a physical system. The state vectors are elements in a separable Hilbert space $\mathfrak{H}$ and the observables are represented as linear operators on $\mathfrak{H}$.

The measurements which we are considering here are those which PAULI has called measurements of the first kind[3]). They are characterized by the property that a repetition of a measurement of the same kind immediately afterwards gives the same result as the preceding measurement with probability one. They alone are suitable for the determination of the state of a system.

For instance if $A$ is the operator, representing such a measurement and the result of an experiment was an eigen value $a$ of $A$ then the state vector $\psi$ after this measurement is an eigen-state of $A$ with the eigen value $a$:

$$A\psi = a\psi \tag{2.1.}$$

If the eigen value $a$ is non-degenerate, that is if, apart from normalization there belongs only one eigen vector to the eigen value $a$, then the measurement of $A$ with the result $a$ is just sufficient for the complete determination of the state. Any additional measurement on this system will either give no new information or it will destroy the information already obtained by the first measurement.

In general one cannot expect that a single measurement will determine the state. Thus if the eigen value $a$ is degenerate then additional independent measurements are possible which do not interfere with each other. For example the momentum measurement of an electron does not determine the state of the electron no matter how accurately it has been performed. The reason is that there exist internal degrees of freedom, the spin, which doubles each momentum state of the electron and only a measurement of the spin together with the momentum is sufficient for the full determination of the state of the electron.

In order to have an unambiguous description of the states of a physical system in terms of state vectors it is thus necessary to postulate the existence of a complete set of compatible independent measurements which provide the maximum amount of possible information about the physical system. Such observables are represented by a set of self adjoint operators in a Hilbert space $\mathfrak{H}$ and one is naturally led to the problem of characterizing a set of self-adjoint operators which correspond to such a maximal set of observables.

In case we are dealing with operators in a finite dimensional space this is easy and can be given adequately in the words of DIRAC[4]) as 'a set of observables which all commute with one another and for which there is only one simultaneous eigenstate belonging to any set of eigenvalues.'

In the more important infinite dimensional case this definition is not suitable since operators may have continuous spectrum and then there exist no eigen functions. This means an equation such as (2.1) for a state vector $\psi$ in $\mathfrak{H}$ has no solution if $a$ is in the continuous part of the spectrum of $A$. A simple example is the operator $q$ associated with one of the Cartesian coordinates of a particle. Its effect on a Schrödinger function $\psi(x)$ is defined as

$$(q\,\psi)(x) = x\,\psi(x) \tag{2.2}$$

and it is easy to verify that there exists no square integrable function $\psi(x)$ which satisfies an equation

$$(q\,\psi)(x) = a\,\psi(x)$$

for any value of $a$.

It is therefore clear that a different definition is needed for such a complete set of observables if it should be applicable in the infinite dimensional case also.

To this end we shall first redefine the concept in the finite dimensional case in a new but equivalent way which can be generalized to the infinite dimensional case with the addition of only a few measure theoretic and topological technicalities.

The first step consists in replacing the set of observables $A_1, A_2, \ldots$ etc. by the algebra $\mathfrak{A}$ of all polynomials $p(A_1, A_2, \ldots)$ with complex coefficients. This is a natural step also from the physical point of view. For we must remember that a measurement of the set of observables $A_1, A_2, \ldots$ etc. is also simultaneously a measurement of every polynomial $p(A_1, A_2, \ldots)$. Indeed if the result of such a measurement is the set of values $a_1, a_2, \ldots$ respectively then the polynomial $p(A_1, A_2 \ldots)$ may be considered as measured too with the value $p(a_1, a_2, \ldots)$. A determination of $A_1, A_2, \ldots$ is therefore at the same time a determination of every operator in the algebra $\mathfrak{A}$.

Next we detach ourselves from the particular basis $A_1, A_2, \ldots$ used for generating the algebra $\mathfrak{A}$. There are obviously many different choices of operators in $\mathfrak{A}$ which generate the same algebra. It is therefore natural to replace the original base by another one which is characterized by the greatest simplicity. A little reflection shows that it is always possible to choose one single operator $A \in \mathfrak{A}$ which alone generates the algebra in the sense that every element in $\mathfrak{A}$ can be written as a polynomial $p(A)$ of the one operator $A$. If the algebra $\mathfrak{A}$ is generated by a *complete* system of observables $A_1, A_2, \ldots$ then the single operator $A$ which generates $\mathfrak{A}$ must have a non-degenerated (or simple) spectrum. We have thus succeeded in replacing the property of the completeness of a set of operators by that of a single operator.

We can further simplify this by replacing the property of the operator $A$ to have a simple spectrum by a property of the algebra $A$ which it generates. To this end it is convenient to consider the commutator algebra $\mathfrak{A}'$, that is the set of all operators which commute with $\mathfrak{A}$. Because $\mathfrak{A}$ is abelian we have obviously

$$\mathfrak{A} \subset \mathfrak{A}' \tag{2.3}$$

But because $\mathfrak{A}$ is generated by an operator $A$ with simple spectrum we also have

$$\mathfrak{A}' \subseteq \mathfrak{A} \tag{2.4}$$

and hence

$$\mathfrak{A} = \mathfrak{A}' \tag{2.5}$$

An algebra $\mathfrak{A}$ which satisfies (2.5) is called maximal abelian.

The verification of (2.4) is based on the observation that any operator $B$ which commutes with $A$ is necessarily a function of $A$ if $A$ has simple spectrum. In the coordinate system which diagonalizes $A$ the operator $B$ is also diagonal because it commutes with $A$. Let $a_r$ and $b_r$ be the corresponding eigenvalues of $A$ and $B$. Define the polynomial $p(x)$ in such a way that $p(a_r) = b_r$ then $p(A) = B$. Thus $B$ is a function of $A$ and hence $\in \mathfrak{A}$.

A final remark on the finite dimensional case will be of use later. We have generated the algebra $\mathfrak{A}$ by considering all complex polynomials of the generating operators $A_1, A_2, \ldots$. We could have proceeded differently by using the known fact that the set of such polynomials can also be obtained as the set of all operators $A$ which commute with all operators $T$ which in turn commute with $A_1, A_2, \ldots$. This seems a roundabout way of getting at the algebra $\mathfrak{A}$ but it turns out that in the infinite dimensional case this way of generating the algebra $\mathfrak{A}$ is far simpler than by a generalization of the functions $p(A_1, A_2, \ldots)$.

With these remarks we arrive at the following equivalent definition of a complete set of commuting observables valid in the finite dimensional case:

*Definition:* A set of commuting operators $\mathfrak{S} = \{A_i\}, i \in I$ ($I$ an index set) is said to be *complete* if the algebra $\mathfrak{A} = \mathfrak{S}''$ generated by the set is maximal abelian:

$$\mathfrak{A}' = \mathfrak{A}.$$

Our next task is to show that this definition can be literally taken over to the infinite dimensional case. The main point is to show that the connection between oprators with simple spectrum (properly generalized to the infinite dimensional case) and maximal abelian algebras remain valid for operators in Hilbert space. We shall devote the next three sections to this task.

### 3. Von Neumann Algebras of Bounded Operators

We shall now consider the transition from the finite to the infinite dimensional space. First we must take into account that the self-adjoint operators $A$ are in general unbounded. An operator $A$ is said to be bounded with bound $M = \|A\| < \infty$ if $M$ is the least upper bound of $\|A\,\psi\|$ for all $\psi \in \mathfrak{H}$ on the unit sphere: $\|\psi\| = 1$

$$\|A\| = \underset{\|\psi\|=1}{l.u.b.} \|A\,\psi\| \tag{3.1}$$

If $M$ is infinite then the operator is said to be unbounded. Some of the most commonly used operators, such as the Cartesian coordinates, the momenta, or the total energy are unbounded operators.

The complication with the unbounded operators is that they can never be defined on the whole space $\mathfrak{H}$ but only on a certain dense linear manifold of $\mathfrak{H}$. Since two dense linear manifolds have as an intersection again a linear manifold which need be neither dense, nor even $\neq 0$ the unrestricted combination of such operators may lead to mathematical difficulties. It is therefore simpler to work with unbounded operators only.

Since we are working exclusively with self-adjoint or normal operators for which there always exists a unique spectral resolution we can always use the set of spectral projections to replace the operator. They are of course all bounded.

The spectral projections of $A$ are a uniquely determined family of projections $E_\lambda$ depending on a parameter $\lambda$ $(-\infty < \lambda < +\infty)$ and having the following properties

(i) $E_\lambda^2 = E_\lambda, \; E_\lambda^* = E_\lambda$

(ii) $\lambda_1 < \lambda_2$ implies $E(\lambda_1) < E(\lambda_2)$

(iii) $E_{\lambda+0} = E_\lambda$

(iv) $E_{-\infty} = 0, \; E_{+\infty} = I$

(v) $A = \int_{-\infty}^{+\infty} \lambda \, dE_\lambda$

The last equation is to be interpreted as the Stieltjes integral valid for any pair of elements $f, g$

$$(f, Ag) = \int_{-\infty}^{+\infty} \lambda \, d(f, E_\lambda g)$$

Let now $A$ be a self-adjoint operator. We say the bounded operator $T$ commutes with $A$ if it commutes with all spectral projections of $A$. The set $\mathfrak{N}$ of all elements $T$ which commute with a given self-adjoint operator $A$ are a family of bounded operators which have the following properties

(i) If $T \in \mathfrak{N}$ then $\alpha T \in \mathfrak{N}$ for any complex $\alpha$

(ii) If $T_1 \in \mathfrak{N}$ and $T_2 \in \mathfrak{N}$ then $T_1 + T_2 \in \mathfrak{N}$ and $T_1 T_2 \in \mathfrak{N}$

(iii) If $T \in \mathfrak{N}$ then $T^* \in \mathfrak{N}$

(iv) $\mathfrak{N}$ is weakly closed

With weakly closed we mean the following: An operator $A$ is weakly adherent to $\mathfrak{N}$ if for any pair of elements $f, g \in \mathfrak{H}$ and $\varepsilon > 0$ there exists an element $T \in \mathfrak{N}$ such that

$$|(f, Sg) - (f, Tg)| < \varepsilon$$

$\mathfrak{N}$ is weakly closed if every element $T$ weakly adherent to $\mathfrak{N}$ is also contained in $\mathfrak{N}$.

Only property (iv) is not immediately obvious. So we give a short proof of it here. What we must show is: if $S$ is weakly adherent to $\mathfrak{N}$ then $AS = SA$. We write for $T \in \mathfrak{N}$

$$AS - SA = A(S - T) - (S - T)A$$

since $T$ commutes with $A$. Taking the scalar product for any pair of elements $f, g \in \mathfrak{H}$ we have

$$|(f,(AS - SA)g)| \leqslant |(f, A(S - T)g)| + |(f, (S - T)Ag)|$$

Since $S$ is weakly adherent to $\mathfrak{N}$ we can choose $T$ such that each expression on the right is $< \varepsilon/2$. Since $\varepsilon$ is arbitrary we have for any pair of elements $f, g$

$$(f,(AS - SA)g) = 0$$

or

$$AS = SA \quad \text{q. e. d.}$$

The first three properties are unified in the statement that $\mathfrak{A}$ is a *-algebra.

A set of elements which satisfies the four properties listed above will be called a von Neumann algebra [5]). An important property of a von Neumann algebra (which may also be used as a defining property) is that it is identical with its double commutant

$$\mathfrak{N}'' = \mathfrak{N} \tag{3.2}$$

We shall call the algebra $\mathfrak{A} = \mathfrak{N}' = \{A\}''$ generated by the operator $A$. It is the smallest von Neumann algebra $\mathfrak{A}$ containing $A$. Because $A \in \{A\}' = \mathfrak{N}$ it follows that $\mathfrak{N}' = \mathfrak{A} \subseteq \mathfrak{N}$ and therefore

$$\mathfrak{A} \subseteq \mathfrak{A}' \tag{3.3}$$

that is, $\mathfrak{A}$ is abelian. If $\mathfrak{A}'$ is also abelian and therefore $\mathfrak{A} = \mathfrak{A}'$ we call it maximal abelian.

The generation of an abelian von Neumann algebra can be generalized to any set of commuting self-adjoint operators. Let $\mathfrak{S} = \{A_i\}$ be such set where $i$ is from from some (not necessarily countable) index set $I$. We define by

$$\mathfrak{A} = \mathfrak{S}'' \tag{3.4}$$

the abelian von Neumann algebra generated by the set $\mathfrak{S}$.

In the previous section we mentioned the easily verified fact that in a finite dimensional space any abelian algebra of linear operators generated by a set of commuting self-adjoint operators can also be generated by a single self-adjoint operator.

According to VON NEUMANN [6]) this theorem is still valid in the infinite dimensional case. *Every abelian von Neumann algebra can be generated with a single self adjoint operator.* Because of this theorem we can limit the discussion to the case of von Neumann algebras generated by one single self-adjoint operator. This is a great simplification and this we shall henceforth do.

The elements in the algebra $\mathfrak{A} = \{A\}''$ generated by the bounded self-adjoint operator can also be characterized in a different way. If $u(\lambda)$ is a complex valued function with the property that it is measurable with respect to the density function $(f, E_\lambda g)$ for any pair of elements $f$ and $g$ and essentially bounded then the operator $u(A)$ defined by

$$u(A) = \int_{-\infty}^{+\infty} u(\lambda)\, dE_\lambda$$

is a bounded operator contained in $\mathfrak{A}$. The converse of this is also true: Any operator $T \in \mathfrak{A}$ is of the form $u(A)$ for some function $u(\lambda)$ [7]).

### 4. Operators with Simple Spectrum

The main question which we shall be concerned with next is how the property of the generating element $A$ to have a simple spectrum is mirrored in the properties of the algebra $\mathfrak{A} = \{A\}''$ which it generates.

In the finite dimensional case we have seen that the corresponding property of $\mathfrak{A}$ is that it is maximal abelian. There is a similar connection in the infinite case, but so far we are not able to formulate this connection properly because we have not yet stated what we mean with '$A$ has a simple spectrum'.

In the finite case the spectrum of $A$ is called simple if every eigenvalue of $A$ is non degenerate, that is if the equation

$$A\,\psi = a\,\psi \tag{4.1}$$

has exactly one solution (apart from an arbitrary factor) for each eigenvalue $a$ of $A$. In the infinite-dimensional case this definition of a simple spectrum is unusable because if $a$ lies in the continuous spectrum then (4.1) has no solution. Thus we must proceed differently. We must find a definition of the simple spectrum which coincides in the finite case with the one just given but which makes no reference to eigenvalues and eigenfunctions. This can be done with the notion of the generating element.

Let $A$ be an operator in the $n$-dimensional space $\mathfrak{H}$ ($n < \infty$) and consider a polynomial $p(x)$ of the independent variable $x$. To every such polynomial we can associate an operator called $p(A)$ by substituting in the expression for $p(x)$ the operator $A$ at the place of $x$. Let us now consider an element $g \in \mathfrak{H}$ then the set of elements $p(A)g$ obtained by letting $p(x)$ run through all the polynomials is a linear subspace $\mathfrak{G}$ of $\mathfrak{H}$. (Note that only polynomials of degree $< n$ will occur in $p(A)$ since $A$ satisfies an algebraic equation of degree $n$). If this subspace $\mathfrak{G} = \mathfrak{H}$ we call $g$ a generating element for $A$.

We verify that such a generating element exists if and only if the spectrum of $A$ is simple. Indeed let $A$ have simple spectrum and let $\varphi_r$ be the eigen vector of $A$ to the eigenvalue $a_r$. Then

$$g = \sum_{r=1}^{n} \varphi_r$$

is a generating element. In order to obtain the element

$$f = \sum_{r=1}^{n} x_r\,\varphi_r$$

with arbitrary coefficients $x_r$ in the form $f = p(A)g$ we construct a polynomial $p(x)$ with the property $p(a_r) = x_r$ ($r = 1, \ldots, n$). This can be done with a polynomial of degree $< n$. It follows that

$$p(A)g = \sum_{r=1}^{n} p(A)\,\varphi_r = \sum_{r=1}^{n} x_r\,\varphi_r = f$$

In order to see conversely that a generating element implies a simple spectrum, we assume that the eigenvalue $a = a_1$ is degenerate. Let $M$ be the subspace spanned by the proper elements associated with $a$. Let $g$ be the assumed generating element and $g_1$ its component in $M$. Since $M$ reduces $A$, any transformation $p(A)$ transforms $g_1$ according to

$$p(A)\,g_1 = p(a)\,g_1$$

Let $f_1 \in M$, and $(f_1, g_1) = 0$, $f_1 \neq 0$. Such an element exists because $a_1$ is degenerate. For every $p(A)$ we have then

$$(f_1, p(A)g) = (f_1, p(a)g_1) = p(a)\,(f_1, g_1) = 0$$

Hence $f_1$ is orthogonal to all elements of the form $p(A)g$. Thus $f_1 = p(A)g$ is impossible and $g$ is not a generating element. This contradicts the assumption that it is.

Thus the following definition is an equivalent characterization of an operator with simple spectrums.

*Definition:* The self-adjoint operator in the finite dimensional space $\mathfrak{H}$ is said to have a simple spectrum if there exists a generating element $g$ such that every element $f$ in $\mathfrak{H}$ can be represented in the form $f = p(A)g$ with some polynomial $p(A)$.

The transcription of this definition to the infinite case is now fairly obvious. Instead of the algebra generated by polynomials of $A$ we substitute the von Neumann algebra generated by $A$. We denote by $\{\mathfrak{A}g\}$ the set of all elements $Tg$ with $T \in \mathfrak{A}$. This set is obviously a linear manifold but in general we cannot assert that it is closed in the topology induced by the norm. The best we can expect is that it is dense in $\mathfrak{H}$. Thus we define

*A self-adjoint operator $A$ is said to have a simple spectrum if the linear manifold $\{\mathfrak{A}g\}$ is dense in $\mathfrak{H}$.*

## 5. Algebras Generated by Operators with Simple Spectrum

We are now prepared for answering the question posed at the beginning of the last section. It is contained in the following

*Theorem:* The necessary and sufficient condition that the abelian von Neumann algebra $\mathfrak{A}$ is maximal abelian is that it is generated by a self-adjoint operator $A$ with simple spectrum.

Although this theorem occurs in various contexts in the literature we shall give here a short proof so that the reader may be spared the labour of searching in the literature for it. Moreover the proof employs some of the typical reasonings involved for some proofs needed in the remaining part of the paper which will not be given.

*Proof:* Let $\mathfrak{A} = \{A\}''$ and let $A$ have simple spectrum. We desire to show that $\mathfrak{A} = \mathfrak{A}'$. Since always $\mathfrak{A} \subseteq \mathfrak{A}'$ we need only verify $\mathfrak{A}' \subseteq \mathfrak{A}$. Thus let $T \in \mathfrak{A}'$. This $T$ commutes with $A$. Since the spectrum of $A$ is simple $T$ is a function of $A$[8]), consequently it is contained in $\mathfrak{A}$. Thus we have verified $\mathfrak{A}' \subseteq \mathfrak{A}$. Therefore $\mathfrak{A}' = \mathfrak{A}$ and $\mathfrak{A}$ is maximal abelian.

Next let us assume that $\mathfrak{A}$ is maximal abelian and $A$ one of its generating operators. We desire to show that the spectrum of $A$ is simple, that is that there exists a generating element $g$ such that the linear manifold is everywhere dense in $\mathfrak{H}$. We note by $[\mathfrak{A}g]$ the closure of the linear manifold generated by $g$ and our aim is to show $[\mathfrak{A}g] = \mathfrak{H}$.

Let us begin with an arbitrary $g_1 \in \mathfrak{H}$, $\| g_1 \| = 1$. Denote by $M_1 = [\mathfrak{A}g_1]$ the closure of the linear manifold generated by $g_1$. If $M_1 = \mathfrak{H}$ we are finished with $g = g_1$. If $M_1 \subset \mathfrak{H}$, denote by $P_1$ the projection with range $M_1$. Since $\mathfrak{A}$ contains the unit element $0 < P_1$.

The projection $P_1$ reduces $\mathfrak{A}$, this means it belongs to $\mathfrak{A}'$. In order to show this we verify that $M_1$ and $M_1{}^{\perp}$ are separately left invariant under the operators in $\mathfrak{A}$. For $M_1$ this is an immediate consequence of the definition. For $M_1{}^{\perp}$ we assume $f \in M_1{}^{\perp}$, $g \in M_1$ and $T \in \mathfrak{A}$ arbitrary. By definition of $g$ we have $g = Sg_1$ with some $S \in \mathfrak{A}$, and so we find

$$(g, Tf) = (Sg_1, Tf) = (T^* Sg_1, f)$$

Since $\mathfrak{A}$ is a *-algebra $T^{\perp}S \in \mathfrak{A}$ and therefore

$$T^* Sg_1 \in M_1. \quad \text{Hence}$$

$$(T^* Sg_1, f) = 0$$

and thus for arbitrary $g \in M_1$

$$(g, Tf) = 0$$

Therefore $Tf \in M_1{}^{\perp}$. This means $P_1$ commutes with $T$ and since $T$ is arbitrary $\in \mathfrak{A}$ we have $P_1 \in \mathfrak{A}'$. Since $\mathfrak{A}$ is maximal abelian $\mathfrak{A} = \mathfrak{A}'$ and it follows that $P_1 \in \mathfrak{A}$.

Now let $g_r$ be a normalized maximal set of generating elements*) for the orthogonal subspaces. $M_r = [\mathfrak{A}g_r]$. The corresponding projections $P_r$ are then mutually orthogonal. Let $P = \underset{r}{\subset} P_r$ be the least upper bound of these projections. Because of the maximal properties of $g_r$ $P = I$.

If now the set of elements $g_r$ is finite, we have a generating element with $g = \Sigma g_r$. If their number is infinite it is countably infinite because

*) This is the non-trivial part of the proof. The existence of the required maximal set of $g_r$ in the stated properties is guaranteed only by virtue of ZORN's lemma, which in turn is equivalent to the axiom of choice. This appeal to the famous transfinite axiom at this point is essential and unavoidable.

the Hilbert space $\mathfrak{H}$ is separable. We choose an arbitrary sequence of positive numbers $a_r > 0$ such that

$$\sum_r a_r^2 < \infty$$

and define

$$g = \sum_r a_r g_r$$

with the finite norm

$$\| g \|^2 = \sum_r a_r^2$$

We want to show, this $g$ is a generating element for $\mathfrak{A}$. Let $f \in \mathfrak{H}$ be an arbitrary element and decompose

$$f = \sum_r f_r \qquad f_r = P_r f.$$

Given $\varepsilon > 0$ we must find $T \in \mathfrak{A}$ such that

$$\| f - Tg \| < \varepsilon$$

Choose first an $N$ such that

$$\sum_{r=N+l}^{\infty} \| f_r \|^2 < \frac{\varepsilon}{2}$$

Since $f_r \in M_r$ we can choose for every $r \leq N a$ $T_r \in \mathfrak{A}$ such that

$$\| f_r - T_r g_r \| < \frac{\varepsilon}{2N}$$

The operator

$$T = \sum_{r=l}^{N} T_r$$

is contained in $\mathfrak{A}$ and for it we have

$$\| f - Tg \|^2 = \sum_{r=l}^{N} \| f_r - T_r g_r \|^2 + \sum_{r=N+l}^{\infty} \| f_r \|^2 \leqslant N \frac{\varepsilon}{2N} + \frac{\varepsilon}{2} = \varepsilon$$

Since $\varepsilon$ is arbitrary, we have shown $\mathfrak{H} = [Ag]$ and this shows that $g$ is a generating element and consequently $A$ has simple spectrum.

## 6. The Structure of the Algebra Generated by Observables

Let $A_i$ $(i \in I)$ be the set of all observables of a given physical system and $\mathfrak{N} = \{A_i\}''$ the von Neumann algebra which it generates. Much of the structure of the physical system is already contained in the structure of the algebra $\mathfrak{N}$. For instance it is an essential feature of quantum mechanics that $\mathfrak{N}$ is not an abelian algebra.

In the structure theory of such algebras the centre $\mathfrak{Z}$ plays an important role. It is defined as the set of elements in $\mathfrak{N}$ which commute with all elements of $\mathfrak{N}$.

$$\mathfrak{Z} = \mathfrak{N} \cap \mathfrak{N}' \tag{6.1}$$

The centre is always an abelian subalgebra of $\mathfrak{N}$, but it is not necessarily maximal abelian. As explained in the previous section the requirement of an unambiguous representation of the physical states in terms of state vectors involves the existence of a maximal abelian algebra $\mathfrak{A}$ contained in $\mathfrak{N}$. From $\mathfrak{A} \subset \mathfrak{N}$ follows

$$\mathfrak{N}' \subset \mathfrak{A}'$$

and since $\mathfrak{A}' = \mathfrak{A}$ we have $\mathfrak{N}' \subset \mathfrak{A} \subset \mathfrak{N}$.

This says first of all that the commutant of $\mathfrak{N}$ is abelian and secondly that every operator in the commutant of $\mathfrak{N}$ is a function of observables and hence, if it is self-adjoint, itself an observable. A consequence of this is that the centre $\mathfrak{Z}$ is identical with $\mathfrak{N}'$

$$\mathfrak{Z} = \mathfrak{N}' \tag{6. 2.}$$

In order to progress further with the structure theory of $\mathfrak{N}$ we make use of the theory of the direct integral of Hilbert spaces. This is a generalization of the concept of the direct sum and will be very briefly outlined here[9]). No proofs will be given, they are found in the references of footnote 9.

Let $\sigma(\lambda)$ $(-\infty < \lambda < +\infty)$ be a function with the following properties

(i) $\sigma(\lambda)$ is real $(-\infty < \lambda < +\infty)$

(ii) $\lambda_1 \leqq \lambda_2$ implies $\sigma(\lambda_1) \leqq \sigma(\lambda_2)$

(iii) $\sigma(\lambda + 0) = \sigma(\lambda)$

(iv) $\sigma(-\infty) = 0,\ \sigma(+\infty) = 1.$

Such a function determines a certain class of sets $K$ on the real $\lambda$-axis which are called $\sigma$-measurable sets. For them a measure is defined $\mu(K)$ with the property

$$\mu(\mathrm{K}_1) + \mu(\mathrm{K}_2) = \mu(\mathrm{K}_1 \cup \mathrm{K}_2) + \mu(\mathrm{K}_1 \cap \mathrm{K}_2)$$

Every half open interval $[\alpha\, \beta]$ is measurable and its measure is given by any set which is built up from such intervals by the formation of unions, intersections and complements in a countable sequence of steps. Thus all Borel sets are measurable.

With this measure one can develop an integration theory in a standard fashion. The integral thus defined is called the Lebesgues-Stieltjes integral.

Let $n(\lambda)$ be a measurable function from the real to the positive integers $\leq \infty$ and let there be given for each $\lambda$ a Hilbert space $\mathfrak{H}(\lambda)$ of dimension $n(\lambda)$. From every space $\mathfrak{H}(\lambda)$ select an element $f(\lambda)$ in such a way that $\| f(\lambda) \|^2$ is $\sigma$-measurable and

$$\int_{-\infty}^{+\infty} \| f(\lambda) \|^2 \, d\sigma(\lambda) < \infty$$

A family of such sets of elements, with the property that for any two of them $f = \{f(\lambda)\}$ and $g = \{g(\lambda)\}$ the function $(f(\lambda), g(\lambda))$ is $\sigma$-measurable, form a Hilbert space $\mathfrak{H}$ called the *direct integral* of the $\mathfrak{H}(\lambda)$ and denoted by

$$\mathfrak{H} = \int_{\oplus} \mathfrak{H}(\lambda) \sqrt{d\sigma(\lambda)} \tag{6.3}$$

The linear operations which make this a vector space are defined as

$$\left.\begin{aligned} \alpha f &= \{\alpha f(\lambda)\} \quad \alpha \text{ complex} \\ f + g &= \{f(\lambda) + g(\lambda)\} \end{aligned}\right\} \tag{6.4}$$

The scalar product of two elements $f$ and $g$ is

$$(f, g) = \int_{-\infty}^{+\infty} (f(\lambda), g(\lambda)) \, d\sigma(\lambda) \tag{6.5}$$

Two decompositions (6.3) with component spaces $\mathfrak{H}(\lambda)$ and $\mathfrak{H}'(\lambda)$ and density functions $\sigma(\lambda)$ and $\sigma'(\lambda)$ are considered equivalent if the spaces $\mathfrak{H}(\lambda)$ differ from $\mathfrak{H}'(\lambda)$ on at most a set of $\sigma$-measure zero, and if $\sigma(\lambda)$ and $\sigma'(\lambda)$ are absolutely continuous with respect to one another. This latter condition means that every $\sigma$-measurable set of $\sigma$-measure zero is also $\sigma'$-measurable of $\sigma'$-measure zero and vice versa.

As usual the words 'almost' or 'essentially' are used to express a property which holds with the exception of a set of $\sigma$-measure zero.

To every essentially bounded $\sigma$-measurable function $\phi(\lambda)$ there is associated a linear operator $L_\phi$ in $\mathfrak{H}$ such that

$$L_\Phi f = \{\phi(\lambda) f(\lambda)\} \tag{6.6}$$

The set of operators of this kind for all $\sigma$-measurable essentially bounded functions $\phi(\lambda)$ form an algebra and one can show that it is a weakly closed abelian *-algebra. This is just the kind of algebra which we have been considering in connection with the sets of commuting observables.

The full significance of the direct integral for our purpose is brought out in the converse of this connection contained in the following theorem[10])

Every weakly closed abelian *-algebra $\mathfrak{A}$ of bounded operators in $\mathfrak{H}$ determines almost uniquely a decomposition of $\mathfrak{H}$ into a direct integral (6.3) in such a way that $\mathfrak{A}$ consists exactly of the operators of the form $L_\phi$ where $\phi(\lambda)$ is a $\sigma$-measurable function of $\lambda$.

Using this theorem we can give a decomposition of the algebra into an almost uniquely determined direct integral.

Let $\mathfrak{Z} = \mathfrak{N} \cap \mathfrak{N}'$ be the centre of $\mathfrak{N}$. It is abelian and determines a direct integral (6.3) of Hilbert spaces $\mathfrak{H}(\lambda)$. Let us examine the structure of the operators in $\mathfrak{N}$ in this decomposition. The relevant theorem is the following:

Every bounded operator $A$ which commutes with all the operators in $\mathfrak{Z}$ can be represented in the form

$$A f = \{A(\lambda)\, f(\lambda)\}$$

where $A(\lambda)$ is a family of operators with the following two properties.

(a) $A(\lambda)$ is measurable: This means $A(\lambda)$ is defined almost everywhere with respect to $\sigma$ and for all $f = \{f(\lambda)\}$ and $\{A(\lambda)\, f(\lambda)\} \in \mathfrak{H}$. $A(\lambda)\, f(\lambda)$ is defined for almost all $\lambda$.

(b) $A(\lambda)$ is essentially bounded, meaning, there exists a number $M < \infty$ such that the sets of $\lambda$ for which $\| A(\lambda) \| > M$ is of $\sigma$-measure zero.

As $A$ runs through the algebra $\mathfrak{N}$ the operators $A(\lambda)$ run through an algebra $\mathfrak{N}(\lambda)$. What can we say about the structure of the algebras $\mathfrak{N}(\lambda)$?

The following theorem gives the answer to this question: The family $\mathfrak{N}(\lambda)$ is irreducible for almost all $\lambda$ if and only if $\mathfrak{Z}$ is a maximal abelian subalgebra of $\mathfrak{N}'$.

Thus in order to verify that $\mathfrak{N}(\lambda)$ is irreducible for almost all $\lambda$ it sufficies to verify that $\mathfrak{Z}$ is maximal abelian in $\mathfrak{N}'$. This is very easy in our case since we have previously shown that $\mathfrak{Z} = \mathfrak{N}'$ (cf. equation (6.2)).

With this result we have obtained the complete structure theory of the observables in a physical system.

In order to clarify the physical significance of this result we consider an arbitrary normalized element $\psi \in \mathfrak{H}$ representing a state of the system. Let $\mathfrak{N}$ be the von Neumann algebra of bounded operators generated by the observables of the system, $\mathfrak{H}(\lambda)$ the component space in the essentially unique direct integral with respect to the centre $\mathfrak{Z}$ of $\mathfrak{N}$ and $\psi(\lambda) \in \mathfrak{H}(\lambda)$ the component vector of $\psi$ in the space $\mathfrak{H}(\lambda)$.

Let $U$ be a unitary operator in $\mathfrak{N}'$. Since every operator in $\mathfrak{N}'$ is also in the centre of $\mathfrak{N}$, $U$ must be of the form $U = L_\phi$

$$U \psi = L_\Phi \psi = \{\phi(\lambda)\, \psi(\lambda)\}$$

with some $\sigma$-measurable function $\phi(\lambda)$ satisfying $|\phi(\lambda)| = 1$.

Let $A \in \mathfrak{N}$ be any observable and consider the expectation value of $A$ in the state $U\psi$

$$(U\psi, AU\psi) = (\psi, U^*AU\,\psi) = (\psi, U^*UA\psi) = (\psi, A\psi)$$

Thus we see the expectation value of any observable is invariant under the transformation $U$. This means that it is impossible to distinguish any physical properties on the states represented by $\psi$ and $U\psi$. Another way of saying this is:

*The relative phases of the vectors in the different component spaces cannot be measured.*

As was shown in $W^3$ this is the characteristic property of a *superselection rule.*

We have thus obtained the most general structure of the space of state vectors in a physical system with superselection rules.

In the special case that $\sigma(\lambda)$ is a discontinuous function with a finite or countable number of discontinuities then the measure induced by $\sigma$ is discrete and the direct integral (6.3) becomes a direct sum of a finite or countably infinite set of orthogonal subspaces. This is the case discussed in $W^3$.

## 7. Concluding Summary

For convenience we summarize the results obtained in the preceding sections.

1. The observables of a physical system are self adjoint linear operators in a Hilbert space which generate a von Neumann algebra $\mathfrak{N}$ of bounded operators. $\mathfrak{N}$ is irreducible if and only if there are no superselection rules operating on the system.

2. A set of commuting observables generates an abelian subalgebra $\mathfrak{A} \subset \mathfrak{N}$. $\mathfrak{A}$ is maximal abelian if and only if the operators in $\mathfrak{A}$ are generated by a single observable with *simple* spectrum. In that case we say is generated by a *complete* set of commuting observables.

3. Assuming the existence of at least one complete set of commuting observables one can show that there exists an essentially unique decomposition of the Hilbert space $\mathfrak{H}$ into a direct integral of component spaces $\mathfrak{H}(\lambda)$ in such a manner that the operators of $\mathfrak{N}$ induce on $\mathfrak{H}(\lambda)$ an *irreducible* operator algebra.

4. The component vectors of $\psi(\lambda)$ of a state vector $\psi \in \mathfrak{H}$ correspond to the separation of $\mathfrak{H}$ into superselection subspaces in the sense that there exists no physical measurement which will give any information on the relative phase of the different components.

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8) Cf. N. I. ACHIESER and I. M. GLASMANN, loc. cit. section 75, theorem 1, p. 220.

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