Zeitschrift: Helvetica Physica Acta
Band: 32 (1959)
Heft: V

Artikel: Scattering integral equations in Hilbert space
Autor: Tixaire, Alberto Galindo
DOI: https://doi.org/10.5169/seals-113011

Scattering Integral Equations in Hilbert Space

von **Alberto Galindo Tixaire**
Junta de Energía Nuclear, Madrid

(1. VI. 1959)

Summary. This paper is devoted to extend the mathematically and physically rigorous theory given by Jauch for the multichannel scattering systems. With this purpose, several integral representations of the Möller operators are obtained which are related to three integration methods: Riemann, Cauchy and Riemann-Stieltjes integrals of operator valued functions. Accordingly we derive as well three types of integral equations for the ingoing and outgoing waves within Hilbert space. We study their validity range, and give reasonable conditions on the channel interaction hamiltonians in order to justify them. Our results apply with a wide class of «switching factors». The Lippmann-Schwinger equations appear as formal solutions of our RS-integral equations outside Hilbert space.

1. Introduction

In spite of the importance that the mathematical theory of scattering presents in relation to physics, we can say that almost every paper dealing with it, is reduced to a formal and heuristic handling of some known topics, without a careful examination of their legitimacy and physical sense. The existence of some limiting properties was not warranted, and every operator was handled as if it might possess all the conditions required to legitimate the expressions involving it. But, of course, that is not a right way. A great part of the inconsistencies we meet in quantum field theory are due to an incorrect use of senseless functions and operators, as many authors have already emphasized[1][2][3]. In scattering theory we are faced with a similar situation. The time-independent formalism is not the most appropiate tool for its treatment because of two main reasons: the first one, of a physical nature, since such a formalism does not describe the evolution character of the scattering systems; the last one, of a mathematical nature, because no scattering state can have a well-defined energy and therefore, it cannot be a stationary state. Even the time- dependent formalism used so far, with the exception next quoted, needed a careful reviewing, owing to its mathematically incomplete exposure*). Not much time ago, JAUCH[4][5]) afforded a first

*) Although there are some other papers on scattering whose mathematical correctness is quite complete, they lack generality and are reduced to study some special points concerning this subject.

approach to this problem, by giving out a rigorous basis to handle the scattering systems. We go on this way, and we make his results more explicit for applications, overpassing as well the above difficulties.

Let us recall some questions needed to understand the following. $\mathfrak{H}$ will be the physical state space, and so, it will be considered as a Hilbert space*), with a definite metric. The motion of the physical system which we shall be concerned with is given by an unitary group V_t, whose infinitesimal generator is a self-adjoint[7][8]) time-independent hamiltonian operator H of $\mathfrak{H}$ (so we assume the system to be closed). The scattering states ψ_t always belong to the continuum subspace $C \subseteq \mathfrak{H}$, i.e., the subspace associated with the continuum spectrum of H. According to the results by JAUCH[4])[5])

$$\mathfrak{C} = \overline{\sum_\alpha \mathfrak{R}_+^{(\alpha)}} = \overline{\sum_\alpha \mathfrak{R}_-^{(\alpha)}} \tag{1.1}$$

where $\sum_\alpha$ denotes direct summation, α is the channel index, and $\mathfrak{R}_\pm^{(\alpha)}$ are orthogonal**) subspaces of $\mathfrak{H}$, ranges of partial isometries $\Omega_\pm^{(\alpha)}$ (Möller operators) with domains $\mathfrak{D}_\pm^{(\alpha)} \subseteq \mathfrak{H}$***). $\mathfrak{D}_\pm^{(\alpha)}$ need not being disjoint, since it is possible[9]) that one of them be the whole $\mathfrak{H}$ (that is the case when we consider the «free» channel, in wich all the particles are free either at $t = -\infty$, or at $t = +\infty$). The operators $\Omega_\pm^{(\alpha)}$ are given[4])[5]) by the limits

$$\Omega_+^{(\alpha)} = \lim_{t \to -\infty} V_t^\dagger U_t^{(\alpha)} \ ,, \ \Omega_-^{(\alpha)} = \lim_{t \to +\infty} V_t^\dagger U_t^{(\alpha)} \tag{1.2}$$

when they exist in the strong topology[7])[8])[10]); it can be shown that these limits exist on some subspaces $\mathfrak{D}_\pm^{(\alpha)} \subseteq \mathfrak{H}$, and they are next prolonged to be zero on the orthogonal complements of $\mathfrak{D}_\pm^{(\alpha)}$. $U_t^{(\alpha)}$ are the α channel unitary operators, generated by the self-adjoint hamiltonians $H_0^{(\alpha)}$.

The Section 2 of this paper is devoted to derive some integral representations of $\Omega_\pm^{(\alpha)}, \Omega_\pm^{(\alpha)\dagger}$. They are connected to three types of operator valued functions integrals[10]): the R (iemann) integration, which provides the R-integral representation with a large class of «weight» functions, enclosing the one given by JAUCH[4]); the C (auchy) integration, which give to us the C-integral representation, and finally, the R(iemann) – S(tieltjes) integration, by means of which we obtain the RS-integral representation,

*) We do not assume $\mathfrak{H}$ to be separable, and so, our results are also valid for 'myriotic fields' [1])[2])[3])[6]) in which the actual possibility of an infinite number of 'bare' particles makes the Hilbert space non separable.

**) The proof of this orthogonality[5]), is based upon the mean ergodic theorem, which is valid also when $\mathfrak{H}$ is non separable[7]).

***) It is not necessary to suppose that $\mathfrak{D}_+^{(\alpha)} = \mathfrak{D}_-^{(\alpha)}$, as made by JAUCH. But it can be proved, as he remarked[4]), that such an equality comes from the time-reversal invariance.

that provides a rigorous sense to the formal results by GELL-MANN and GOLDBERGER[11]).

In Section 3 we apply the results of Section 2 to derive correspondingly three integral equations for the ingoing (outgoing) states. Some difficulties arise owing to the possible unboundness of the interaction hamiltonians $H_{\mathrm{I}}^{(\alpha)}$. To overpass them we state explicitly some very general conditions on the domains $\mathfrak{D}_H$, $\mathfrak{D}_{H_0^{(\alpha)}}$ and $\mathfrak{D}_{H_{\mathrm{I}}^{(\alpha)}}$ of $H, H_0^{(\alpha)}$ and $H_{\mathrm{I}}^{(\alpha)}$. Our R-integral equations generalizes the results by COOK[12]) and HACK[13]), and the RS-integral equations show a quite clear ressemblance to the LIPPMANN-SCHWINGER[14]) ones, which appear to be their formal solution outside the Hilbert space. Questions concerning the iteration solution method and the scattering operator will be treated in a subsequent paper.

2. Integral Representations of $\Omega_{\pm}^{(\alpha)}$, $\Omega_{\pm}^{(\alpha)\dagger}$.

On brevity sake all the proofs and supplementary requirements will be given for $\Omega_{+}^{(\alpha)}$, since for the other MÖLLER operators they are quite similar; so we shall merely write mutatis mutandis the corresponding results. In addition we shall only indicate the outlines of the proofs, the details being omitted whenever that does not get any trouble.

a) *R-integral Representation*

Let $g_+(t, \varepsilon)$ be a non-negative real function of $t \in J_- \equiv (-\infty, 0]$ and $\varepsilon \in J_{\varepsilon_0} \equiv (0, \varepsilon_0]$; we assume that $g_+(t, \varepsilon)$ is continuous in t, and that

$$\int_{-\infty}^{0} g_+(t,\varepsilon)\, dt = 1 \ , \quad \text{for every } \varepsilon \in J_{\varepsilon_0} \tag{2a.1}$$

The operator valued function $g_+(t, \varepsilon)\, V_t^\dagger\, U_t^{(\alpha)}$ is strongly continuous in t, and, in addition, its norm $g_+(t, \varepsilon)$ is integrable on J_- for every $\varepsilon \in J_{\varepsilon_0}$. Therefore, there exists the R-integral[10]) [15])

$$\Omega_+^{(\alpha)}[g_+(\,.\,, \varepsilon)] \equiv \int_{-\infty}^{0} g_+(t, \varepsilon)\, V_t^\dagger\, U_t^{(\alpha)}\, dt \tag{2a.2}$$

in the strong topology*). The operator $\Omega_+^{(\alpha)}[g_+(\cdot, \varepsilon)]$ is bounded, and $|| \Omega_+^{(\alpha)}[g_+ c\cdot, \varepsilon)] || \leqslant 1$. Let us next assume that

$$\lim_{\varepsilon \to +0} \int_{-\tau}^{0} \frac{g_+(t/\varepsilon, \varepsilon)}{\varepsilon}\, dt \leqslant K\,\tau \quad \text{for every} \quad \tau \in (0, \tau_0) \tag{2a.3}$$

*) To avoid repetitions in the following, whenever we speak on the existence of a limit or an integral, it must be understood in the strong convergence, unless we specify the contrary.

Then, by means of a similar procedure to that used by JAUCH[4]) in his Lemma 6, it can be inmediately shown that

$$\Omega_{+}^{(\alpha)} = \lim_{\varepsilon \to +0} \Omega_{+}^{(\alpha)}[g_{+}(\,.\,,\varepsilon)] \quad \text{on} \quad \mathfrak{D}_{+}^{(\alpha)} \tag{2a.4}$$

Analogously

$$\Omega_{-}^{(\alpha)} = \lim_{\varepsilon \to +0} \Omega_{-}^{(\alpha)}[g_{-}(\,.\,,\varepsilon)] \equiv \lim_{\varepsilon \to +0} \int_{0}^{\infty} g_{-}(t,\varepsilon)\, V_t^{\dagger}\, U_t^{(\alpha)}\, dt \quad \text{on} \quad \mathfrak{D}_{-}^{(\alpha)} \tag{2a.5}$$

and

$$\Omega_{\pm}^{(\alpha)\dagger} = \lim_{\varepsilon \to +0} \Omega_{\pm}^{(\alpha)\dagger}[g_{\pm}(\,.\,,\varepsilon)] \equiv \lim_{\varepsilon \to +0} \pm \int_{\mp\infty}^{0} g_{\pm}(t,\varepsilon)\, U_t^{(\alpha)\dagger} V_t\, dt \quad \text{on} \quad \mathfrak{R}_{\pm}^{(\alpha)} \tag{2a.6}$$

Although for our purposes we need no more than (2a.4), (2a.5), (2a.6), it is interesting, from a theoretical point of view, to investigate if the restrictions on the validity domains of the above expressions can be dropped or weakened. We know that to be true for simple scattering systems[4]) whenever we take $g_{\pm}(t,\varepsilon) = \varepsilon\, e^{\pm \varepsilon t}$. However we have failed to prove it in the most general multichannel case*).

We have then been able to write $\Omega_{\pm}^{(\alpha)}$, $\Omega_{\pm}^{(\alpha)\dagger}$, as a limit of an integral average of the unitary operators $V_t^{\dagger}\, U_t^{(\alpha)}$, $U_t^{(\alpha)\dagger}\, V_t$, with a suitable class of weight functions $g_{\pm}(\mathrm{t},\varepsilon)$, enclosing the special case treated by JAUCH[4]). It is worth to remark that the «adiabatic hypothesis» with a general class of «switching factors» is not but a simple consequence of treating with scattering systems[4])[5]).

b) *C-integral Representation*

Let σ_α be any compact subset of the spectral set of $H_0^{(\alpha)}$, and $\mathfrak{M}(\sigma_\alpha)$ its corresponding subspace. One has

$$U_t^{(\alpha)} = \frac{1}{2\pi i} \int_{C[\sigma_\alpha]} e^{-izt} R(z, H_0^{(\alpha)})\, dz\,, \quad \text{on} \quad \mathfrak{M}(\sigma_\alpha) \tag{2b.1}$$

where $C[\sigma_\alpha]$ is an oriented envelope[10]) of an arbitrary bounded open subset**) of the complex plain containing σ_α, and where $R(z, H_0^{(\alpha)}) \equiv (z - H_0^{(\alpha)})^{-1}$ is the resolvent of $H_0^{(\alpha)}$ at z.

If we assume that for a fixed $g_{+}(t,\varepsilon)$, and for every compact subset σ_α, there exists some $C[g_{+}(\cdot,\varepsilon), \sigma_\alpha]$ such that

$$\int_{-\infty}^{0} g_{+}(t,\varepsilon)\, e^{-izt}\, V_t^{\dagger}\, dt \equiv N_{+}[g_{+}(\,.\,,\varepsilon), z] \ \text{ exists} \tag{2b.2}$$

*) We are indebted to Professor JAUCH for a private communication on this subject.

**) We suppose that this oriented envelope consists of a finite number of closed simple Jordan curves.

for every $z \in C[g_+(\cdot, \varepsilon), \sigma_\alpha]$ and $\varepsilon \in J_{\varepsilon_0}$, $N_+[g_+(\cdot, \varepsilon), z]$ being strongly continuous on $C[g_+(\cdot, \varepsilon), \sigma_\alpha]$ as a function of z, we can sustitute (2b.1), (2b.2) in (2a.2) to obtain, by interchanging the integration order (whose legitimacy comes from the above conditions), the following expression

$$\Omega_+^{(\alpha)}[g_+(\,.\,,\varepsilon)] = \frac{1}{2\pi i} \int\limits_{C[g_+(\,.\,,\varepsilon)\,\sigma_\alpha]} N_+[g_+(\,.\,,\varepsilon), z]\, R(z, \mathrm{H}_0^{(\alpha)})\, dz \quad \text{on } \mathfrak{M}(\sigma_\alpha) \tag{2b.3}$$

Therefore

$$\Omega_\pm^{(\alpha)} = \lim_{\varepsilon \to +0} \int\limits_{C[g_\pm(\,.\,,\varepsilon),\sigma_\alpha]} N_\pm[g_\pm(\,.\,,\varepsilon), z]\, R(z, H_0^{(\alpha)})\, dz \quad \text{on } \mathfrak{M}(\sigma_\alpha) \cap \mathfrak{D}_\pm^{(\alpha)} \tag{2b.4}$$

$$\Omega_\pm^{(\alpha)\dagger} = \lim_{\varepsilon \to +0} \int\limits_{C[g_\pm(\,.\,,\varepsilon),\,\sigma]} N_\pm^{(\alpha)}[g_\pm(\,.\,,\varepsilon), z]\, R(z, H)\, dz \quad \text{on } \mathfrak{M}(\sigma) \cap \mathfrak{R}_\pm^{(\alpha)} \tag{2b.5}$$

which are the C-integral representations we were looking for, and whose applicability is restricted to the elements of $\mathfrak{D}_\pm^{(\alpha)}$, $\mathfrak{R}_\pm^{(\alpha)}$, which contain only bounded energies in relation to $H_0^{(\alpha)}$, H, respectively*). In the simple case $g_\pm(t, \varepsilon) = \varepsilon\, e^{\pm \varepsilon t}$, we get

$$\Omega_\pm^{(\alpha)} = \lim_{\varepsilon \to +0} \frac{\pm \varepsilon}{2\pi} \int\limits_{C(\varepsilon,\sigma_\alpha)} R(z \pm i\,\varepsilon\,, H)\, R(z, H_0^{(\alpha)})\, dz \quad \text{on } \mathfrak{M}(\sigma_\alpha) \cap \mathfrak{D}_\pm^{(\alpha)} \tag{2b.6}$$

$$\Omega_\pm^{(\alpha)\dagger} = \lim_{\varepsilon \to +0} \frac{\pm \varepsilon}{2\pi} \int\limits_{C(\varepsilon,\sigma)} R(z \pm i\varepsilon, H_0^{(\alpha)})\, R(z, H)\, dz \quad \text{on } \mathfrak{M}(\sigma) \cap \mathfrak{R}_\pm^{(\alpha)} \tag{2b.7}$$

where $C(\varepsilon, \sigma_\alpha)$, $C(\varepsilon, \sigma)$ can be chosen inside the strip $|\, Imz \,| \leqslant \varepsilon/2$.

c) *RS-integral Representation*

Let us consider again (2a.2). If $E_\lambda^{(\alpha)}$ is the identity resolution corresponding to $H_0^{(\alpha)}$, and if $P_J^{(\alpha)}$ is the spectral projector on a closed interval $J \equiv [\lambda_0, \lambda_1]$, one has[7])

$$U_t^{(\alpha)} P_J^{(\alpha)} = \int\limits_J e^{-i\lambda t}\, dE_\lambda^{(\alpha)} \quad \text{in the norm topology,} \tag{2c.1}$$

*) Although it is not easy to check mathematically if these elements exist, it is very likely to happen that on physical grounds. In the worst case, however, we can perform a double limit in (2b.3), by putting $P(\sigma_\alpha)$ (spectral projector on σ_α) on the right of $R(z, H_0^{(\alpha)})$ and making $\sigma_\alpha \to (-\infty, +\infty)$, $\varepsilon \to +0$.

Let ϱ be a partition of J, and $U^{(\alpha)}_{t,\varrho}\,P^{(\alpha)}_J$ a corresponding sum in (2c.1). We form the operator

$$\Omega^{(\alpha)}_{+}[g_+(\,.\,,\varepsilon),\varrho]\,P^{(\alpha)}_J \equiv \int\limits_{-\infty}^{0} g_+(t,\varepsilon)\,V^{\dagger}_t\,U^{(\alpha)}_{t,\varrho}\,P^{(\alpha)}_J\,dt \qquad (2\,c.2)$$

If $\varphi \in \mathfrak{H}$, one gets

$$\|\,\Omega^{(\alpha)}_{+}[g_+(\,.\,,\varepsilon)]\,P^{(\alpha)}_J\,\varphi - \Omega^{(\alpha)}_{+}[g_+(\,.\,,\varepsilon),\varrho]\,P^{(\alpha)}_J\,\varphi\,\| \leqslant$$

$$\leqslant \int\limits_{-T}^{0} g_+(t,\varepsilon)\,\|\,U^{(\alpha)}_t P^{(\alpha)}_J \varphi - U^{(\alpha)}_{t,\varrho} P^{(\alpha)}_J \varphi\,\|\,dt\, +$$

$$\int\limits_{-\infty}^{-T} g_+(t,\varepsilon)\,\|\,U^{(\alpha)}_t P^{(\alpha)}_J \varphi - U^{(\alpha)}_{t,\varrho} P^{(\alpha)}_J \varphi\,\|\,dt \qquad (2\,c.3)$$

We can choose T to be large enough such that, for an arbitrary $\delta > 0$,

$$\int\limits_{-\infty}^{-T} g_+(t,\varepsilon)\,dt < \delta/4\,\|\,\varphi\,\| \qquad (2\,c.4)$$

and since $e^{-i\lambda t}$ in uniformly continuous in $\lambda \in J$, $t \in [-\,T, 0]$,

$$\int\limits_{-T}^{0} g_+(t,\varepsilon)\,\|\,U^{(\alpha)}_t P^{(\alpha)}_J \varphi - U^{(\alpha)}_{t,\varrho} P^{(\alpha)}_J \varphi\,\|\,dt < \delta/2 \qquad (2\,c.5)$$

for a partition norm $|\,\varrho\,|$ sufficiently small. Therefore, we have

$$\Omega^{(\alpha)}_{+}[g_+(\,.\,,\varepsilon)]\,P^{(\alpha)}_J = \lim_{|\varrho|\to 0} \Omega^{(\alpha)}_{+}[g_+(\,.\,,\varepsilon),\varrho]\,P^{(\alpha)}_J \qquad (2\,c.6)$$

Writing

$$\int\limits_{-\infty}^{0} g_+(t,\varepsilon)\,V^{\dagger}_t\,e^{-i\lambda t}\,dt \equiv N_+[g_+(\,.\,,\varepsilon),\lambda]$$

we deduce

$$\Omega^{(\alpha)}_{+}[g_+(\,.\,,\varepsilon)]\,P^{(\alpha)}_J = \int\limits_{J} N_+[g_+(\,.\,,\varepsilon),\lambda]\,dE^{(\alpha)}_\lambda \qquad (2\,c.7)$$

Since $\Omega^{(\alpha)}_{+}[g_+(\cdot,\varepsilon)]$ is a bounded operator, we can pass to the limit when $J \to (-\,\infty, +\,\infty)$, and so

$$\Omega^{(\alpha)}_{+}[g_+(\,.\,,\varepsilon)] = \int\limits_{-\infty}^{+\infty} N_+[g_+(\,.\,,\varepsilon),\lambda]\,dE^{(\alpha)}_\lambda \qquad (2\,c.8)$$

Finally, by tending $\varepsilon \to +0$, we can write in general

$$\Omega_{\pm}^{(\alpha)} = \lim_{\varepsilon \to +0} \int_{-\infty}^{+\infty} N_{\pm}[g_{\pm}(\,.\,,\varepsilon), \lambda]\, dE_{\lambda}^{(\alpha)} \quad \text{on} \quad \mathfrak{D}_{\pm}^{(\alpha)} \tag{2c.9}$$

$$\Omega_{\pm}^{(\alpha)\dagger} = \lim_{\varepsilon \to +0} \int_{-\infty}^{+\infty} N_{\pm}^{(\alpha)}[g_{\pm}(\,.\,,\varepsilon), \lambda]\, dE_{\lambda} \quad \text{on} \quad \mathfrak{R}_{\pm}^{(\alpha)} \tag{2c.10}$$

If one takes $g_{\pm}(t, \varepsilon) = \varepsilon\, e^{\pm \varepsilon t}$, (2c.9) and (2c.10) bring over the following expressions

$$\Omega_{\pm}^{(\alpha)} = \lim_{\varepsilon \to +0} \pm i\, \varepsilon \int_{-\infty}^{+\infty} R(\lambda \pm i\, \varepsilon, H) \quad dE_{\lambda}^{(\alpha)} \text{ on } \mathfrak{D}_{\pm}^{(\alpha)} \tag{2c.11}$$

$$\Omega_{\pm}^{(\alpha)\dagger} = \lim_{\varepsilon \to +0} \pm i\, \varepsilon \int_{-\infty}^{+\infty} R(\lambda \pm i\, \varepsilon, H_0^{(\alpha)})\, dE_{\lambda} \quad \text{on} \quad \mathfrak{R}_{\pm}^{(\alpha)} \tag{2c.12}$$

which gives a precise meaning to previous results[11]).

3. Scattering integral equations

The integral expressions obtained in Section 2, although rigorous, are not suitable enough for practical purposes, owing to the non explicit appearance of the interaction hamiltonians $H_I^{(\alpha)} \equiv H - H_0^{(\alpha)}$, which in the most physical cases are treated as small perturbations whose effects one studies in different approximations. Nevertheless, we are faced with the difficulties arising from the possible unboundness of $H_I^{(\alpha)}$. We cannot avoid them because they often appear, as, for instance, the Coulomb potential with an appropiate cut-off and some velocity-dependent interactions. Not much time ago, it has been proved[12])[13]) that even with unbounded singular potentials satisfying some general conditions, it is possible to get simple scattering systems, whose Möller operators admit integral representations in which H_I appear explicitly, but whose validity is restrained to a suitable domain everywhere dense in $\mathfrak{H}$. Guided by their results and trying to generalize them, we shall be able to perform the same steps for each one of the integral representations that we have just found out.

a) *R-integral equations*

Let us suppose that for each $H_I^{(\alpha)}$ there exists a domain $\tilde{\mathfrak{D}}_{+}^{(\alpha)}$ everywhere dense in $\mathfrak{D}_{+}^{(\alpha)}$ such that

a′) $\tilde{\mathfrak{D}}_{+}^{(\alpha)} \subseteq \mathfrak{D}_{H^0}$., $U_t^{(\alpha)}\, \tilde{\mathfrak{D}}_{+}^{(\alpha)} \subseteq \mathfrak{D}_H, \mathfrak{D}_{H_I^{(\alpha)}}$ for $t \in J_-$

b′) $H_I^{(\alpha)} U_t^{(\alpha)}$ is strongly continuous on $\tilde{\mathfrak{D}}_{+}^{(\alpha)}$ as a function of $t \in J_-$.

Conditions a') and b') able us to provide a mathematical sense to the forecoming expressions. In particular b') indicates that $H_I^{(\alpha)}$ is «well-behaved» enough on $\tilde{\mathfrak{D}}_+^{(\alpha)}$. Both conditions are reasonable on mathematical grounds. We suppose, in addition, that for some $g_+(t, \varepsilon)$,

$$\text{c}')\quad \int_{-\infty}^{0} g_+^1(t,\varepsilon)\, \| H_I^{(\alpha)} U_t^{(\alpha)} \varphi \|\, dt < +\infty \quad \text{for every} \quad \varphi \in \tilde{\mathfrak{D}}_+^{(\alpha)},$$

$$\text{where} \quad g_+^1(t,\varepsilon) \equiv \int_{-\infty}^{t} g_+(t,\varepsilon)\, dt$$

With these conditions in mind*), we obtain inmediately

$$\Omega_+^{(\alpha)}[g_+(\,.\,,\varepsilon)] = 1 - i \int_{-\infty}^{0} g_+^1(t,\varepsilon)\, V_t^\dagger H_I^{(\alpha)} U_t^{(\alpha)}\, dt \quad \text{on} \quad \tilde{\mathfrak{D}}_\pm^{(\alpha)} \qquad (3\text{a}.1)$$

Therefore, and with similar considerations for $\Omega_-^{(\alpha)}$, $\Omega_\pm^{(\alpha)\dagger}$,

$$\Omega_\pm^{(\alpha)} = 1 - i \lim_{\varepsilon \to +0} \int_{\mp\infty}^{0} g_\pm^1(t,\varepsilon)\, V_t^\dagger H_I^{(\alpha)} U_t^{(\alpha)}\, dt \quad \text{on} \quad \tilde{\mathfrak{D}}_\pm^{(\alpha)} \qquad (3\text{a}.2)$$

$$\Omega_\pm^{(\alpha)\dagger} = 1 + i \lim_{\varepsilon \to +0} \int_{\mp\infty}^{0} g_\pm^1(t,\varepsilon)\, U_t^{(\alpha)\dagger} H_I^{(\alpha)} V_t\, dt \quad \text{on} \quad \tilde{\mathfrak{R}}_\pm^{(\alpha)} \qquad (3\text{a}.3)$$

By applying (3a.3) to a $\psi_\alpha^\pm \in \tilde{\mathfrak{R}}_\pm^{(\alpha)}$, we get

$$\psi_\alpha^\pm = \varphi_\alpha - i \lim_{\varepsilon \to +0} \int_{\mp\infty}^{0} g_\pm^1(t,\varepsilon)\, U_t^{(\alpha)\dagger} H_I^{(\alpha)} V_t \psi_\alpha^\pm\, dt \qquad (3\text{a}.4)$$

which are the scattering R-integral equations. For $g_\pm(t,\varepsilon) = \varepsilon\, e^{\pm\varepsilon t}$, they adopt the following form:

$$\psi_\alpha^\pm = \varphi_\alpha^\pm - i \lim_{\varepsilon \to +0} \int_{\mp\infty}^{0} e^{\pm\varepsilon t}\, U_t^{(\alpha)\dagger} H_I^{(\alpha)} V_t \psi_\alpha^\pm\, dt \qquad (3\text{a}.5)$$

In (3a.4) as well as in (3a.5), $\varphi_\alpha^\pm$ belongs to $\Omega_\pm^{(\alpha)}\tilde{\mathfrak{R}}_\pm^{(\alpha)}$. In the particular case where $H_I^{(\alpha)}$ be bounded, the conditions a'), b') and c') can be partially removed and (3a.4) are valid for any $\psi_\alpha^\pm \in \mathfrak{R}_\pm^{(\alpha)}$. Indeed we need only assume the existence of domains $\tilde{\mathfrak{D}}_\pm^{(\alpha)}$, ($\tilde{\mathfrak{R}}_\pm^{(\alpha)}$), everywhere dense in $\mathfrak{D}_\pm^{(\alpha)}$, ($\mathfrak{R}_\pm^{(\alpha)}$), such that $U_t^{(\alpha)}\tilde{\mathfrak{D}}_\pm^{(\alpha)} \subset \mathfrak{D}_H$, ($V_t\tilde{\mathfrak{R}}_\pm^{(\alpha)} \subseteq \mathfrak{D}_{H_0^{(\alpha)}}$), to legitimate (3a.2), ((3a.3))

*) These three conditions are satisfied in [12]) [13]).

on $\mathfrak{D}_{\pm}^{(\alpha)}$, $(\mathfrak{R}_{\pm}^{(\alpha)})$. We must note that we can only assert (3a.4) to be the scattering R-integral equation in that case where the initial data $\varphi_{\alpha}^{\pm}$ belongs to $\Omega_{\pm}^{(\alpha)}\tilde{\mathfrak{R}}_{\pm}^{(\alpha)}$, which is everywhere dense in $\mathfrak{D}_{\pm}^{(\alpha)}$; therefore in practical problems, that must be kept well in mind*).

b) *C-integral equations*

We adopt the same notations as in 2.b). We assume that

$$\int_{-\infty}^{0} g_{+}^{1}(t,\varepsilon)\, e^{-i z t} V_t^{\dagger}\, dt \equiv N_{+}^{1}[g_{+}(\,.\,,\varepsilon), z] \tag{3b.1}$$

and $H_I^{(\alpha)} R(z, H_0^{(\alpha)})$ exist, and that they are strongly continuous as functions of $z \in C[g_{+}(\cdot,\varepsilon), \sigma_\alpha]$ on a subset $'\tilde{\mathfrak{D}}_{\pm}^{(\alpha)}(\sigma_\alpha)$ everywhere dense in $\tilde{\mathfrak{D}}_{+}^{(\alpha)} \cap \mathfrak{M}(\sigma_\alpha)$, for every $\varepsilon \in J_{\varepsilon_0}$. Then, we can proceed with (3a.1) in the same fashion as with (2a.2), to obtain

$$\Omega_{+}^{(\alpha)}[g_{+}(\,.\,,\varepsilon)] = 1 - \frac{1}{2\pi} \int_{C[g_{+}(\,.\,,\varepsilon),\sigma_\alpha]} N_{+}^{1}[g_{+}(\,.\,,\varepsilon), z]\, H_I^{(\alpha)} R(z, H_0^{(\alpha)})\, dz \quad \text{on} \quad '\tilde{\mathfrak{D}}_{+}^{(\alpha)}(\sigma_\alpha) \tag{3b.2}$$

and similarly for $\Omega_{-}^{(\alpha)}[g_{-}(.,\varepsilon)]$, $\Omega_{\pm}^{(\alpha)\dagger}[g_{\pm}(.,\varepsilon)]$. Next we can take limits when $\varepsilon \to +0$ to derive the corresponding expressions for $\Omega_{\pm}^{(\alpha)}$, $(\Omega_{\pm}^{(\alpha)\dagger})$ on $'\tilde{\mathfrak{D}}_{\pm}^{(\alpha)}(\sigma_\alpha)$, $('\tilde{\mathfrak{R}}_{\pm}^{(\alpha)}(\sigma))$, and from these, the C-integral equations.

If $g_{+}(t,\varepsilon) = \varepsilon\, e^{\pm \varepsilon t}$, (3b.1) is authomatically satisfied whenever $|\, Im z \,| \leqslant \varepsilon/2$; the C-integral equations are in this case

$$\psi_\alpha^{\pm} = \varphi_\alpha^{\pm} + \frac{1}{2\pi i} \lim_{\varepsilon \to +0} \int_{C(\varepsilon,\sigma)} R(z \pm i\varepsilon, H_0^{(\alpha)})\, H_I^{(\alpha)} R(z, H)\, dz\, \psi_\alpha^{\pm} \tag{3b.3}$$

for every $\psi_\alpha^{\pm} \in {}'\tilde{\mathfrak{R}}_{\pm}^{(\alpha)}(\sigma)$.

c) *RS-integral equations*

Let us suppose that for every bounded $J \equiv [\lambda_0, \lambda_1]$ and any partition ϱ of J, $H_I^{(\alpha)} U_{t,\varrho}^{(\alpha)} P_J^{(\alpha)}$ exists on some $''\tilde{\mathfrak{D}}_{+}^{(\alpha)}$ everywhere dense in $\tilde{\mathfrak{D}}_{+}^{(\alpha)}$, and that

$$\int_{-\infty}^{0} g_{+}^{1}(\mathrm{t},\varepsilon)\, \| H_I^{(\alpha)} U_{t,\varrho}^{(\alpha)} P_J^{(\alpha)} \varphi \|\, dt < +\infty \quad \text{for every} \quad \varphi \in {}''\tilde{\mathfrak{D}}_{+}^{(\alpha)} \tag{3c.1}$$

*) It is clear that the requirement of $\tilde{\mathfrak{D}}_{\pm}^{(\alpha)}$, $(\tilde{\mathfrak{R}}_{\pm}^{(\alpha)})$, being everywhere dense in $\mathfrak{D}_{\pm}^{(\alpha)}$, $(\mathfrak{R}_{\pm}^{(\alpha)})$ is not essential to derive (3a.2), ((3a.3)). But we have given it in order to provide some importance to them, since if so, we can approach any element of $\mathfrak{D}_{\pm}^{(\alpha)}$, $(\mathfrak{R}_{\pm}^{(\alpha)})$ as much as needed by means of solution elements of the corresponding R-integral equations.

If in addition $H_I^{(\alpha)} U_{t,\varrho}^{(\alpha)} P_J^{(\alpha)} \xrightarrow[|\varrho| \to 0]{} H_I^{(\alpha)} U_t^{(\alpha)} P_J^{(\alpha)}$ on $''\tilde{\mathfrak{D}}_+^{(\alpha)}$, uniformly with respect to $t \in [-T, 0]$ (T any finite positive number), we can proceed as in (2.a) to derive from (3a.1)

$$\Omega_+^{(\alpha)}[g_+(\,.\,,\varepsilon)] = 1 - i \int_{-\infty}^{+\infty} N_+^1[g_+(\,.\,,\varepsilon), \lambda]\, H_I^{(\alpha)} dE_\lambda^{(\alpha)} \quad \text{on} \quad ''\tilde{\mathfrak{D}}_+^{(\alpha)} \qquad (3\text{c}.2)$$

and similar expressions for $\Omega_-^{(\alpha)}[g_-(.,\varepsilon)]$, $\Omega_\pm^{(\alpha)\dagger}[g_\pm(.,\varepsilon)]$, in each one of them we would take limits when $\varepsilon \to +0$ to obtain $\Omega_\pm^{(\alpha)}$, $\Omega_\pm^{(\alpha)\dagger}$.

In the special case where $g_\pm(t,\varepsilon) = \varepsilon\, e^{\pm \varepsilon t}$, they become

$$\Omega_\pm^{(\alpha)} = 1 + \lim_{\varepsilon \to +0} \int_{-\infty}^{+\infty} \frac{1}{\lambda \pm i\varepsilon - H} H_I^{(\alpha)} dE_\lambda^{(\alpha)} \quad \text{on} \quad ''\tilde{\mathfrak{D}}_\pm^{(\alpha)} \qquad (3\text{c}.3)$$

$$\Omega_\pm^{(\alpha)\dagger} = 1 - \lim_{\varepsilon \to +0} \int_{-\infty}^{+\infty} \frac{1}{\lambda \pm i\varepsilon - H_0^{(\alpha)}} H_I^{(\alpha)} dE_\lambda \quad \text{on} \quad ''\tilde{\mathfrak{R}}_\pm^{(\alpha)} \qquad (3\text{c}.4)$$

From (3c.4), we get

$$\psi_\alpha^\pm = \varphi_\alpha^\pm + \lim_{\varepsilon \to +0} \int_{-\infty}^{+\infty} \frac{1}{\lambda \pm i\varepsilon - H_0^{(\alpha)}} H_I^{(\alpha)} dE_\lambda\, \psi_\alpha^\pm \quad \text{for every} \quad \psi_\alpha^\pm \in {''\tilde{\mathfrak{R}}_\pm^{(\alpha)}} \qquad (3\text{c}.5)$$

whose analogy with those of LIPPMANN-SCHWINGER[14]) is quite evident. However (3c.5) hold inside the Hilbert space, while these other ones are their formal solutions outside $\mathfrak{H}$. We shall come back to this point elsewhere.

4. Conclusion

The results obtained in this paper show clearly the complications inherent to the problem which we have been concerned with. One of the most striking facts is the appearance of E_λ, in the RS-integral representations of $\Omega_\pm^{(\alpha)\dagger}$, and the RS-integral equations. This is due to the fact that to approach their direct expressions or solutions we need to know the very spectral structure of the hamiltonian operator H, a matter that is almost entirely unknown. A method to derive it from the $E_\lambda^{(\alpha)}$, would be of great interest for specific applications.

Finally, we want to emphasize that, in spite of the restrictions imposed to justify the derived expressions, it is very likely that it may be possible to fulfill them in the majority of the practical situations, as stressed by the fact that the potentials studied by HACK[13]), although very general, satisfy them.

Acknowledgments

The author would like to express its gratitude to Dr. GARRIDO and Mr. PASCUAL for their interesting and helpful discussions, as well as to the Junta de Energía Nuclear for its financial support.

Note added in proof. After this paper was written, Prof. JAUCH kindly pointed out to me that condition a' is very similar to what he calls 'admissible interaction operator' (J. M. JAUCH and I. I. ZINNES, Nuovo Cim. *11*, 553 (1959)).

References

1) FRIEDRICHS, K. O., Mathematical aspects of the quantum theory of fields, Interscience, New York (1953).
2) WAKITA, H., Prog. Theor. Phys. *20*, 35 (1958).
3) WAKITA, H., Prog. Theor. Phys. *21*, 299 (1959).
4) JAUCH, J. M., Helv. Phys. Acta *31*, 127 (1958).
5) JAUCH, J. M., Helv. Phys. Acta *31*, 661 (1958).
6) ALBERTONI, S., and DUIMIO, F., Nuovo Cim. *6*, 1193 (1957).
7) RIESZ, F., and SZ-NAGY, B., Leçons d'analyse fonctionnelle, Gauthier-Villars, Paris 1955.
8) STONE, M. H., Linear Transformations in Hilbert space and their applications to analysis, Amer. Math. Soc. Colloquium Pub. Vol. *15*, New York 1932.
9) EKSTEIN, H., Phys. Rev. *101*, 880 (1956).
10) HILLE, E., Functional analysis and semigroups, Am. Math. Soc. Colloquium Pub. Vol. 31, New York 1957.
11) GELL-MANN, M., and GOLDBERGER, M., Phys. Rev. *91*, 398 (1953).
12) COOK, J. M., Journ. of Math. and Phys. *36*, 81 (1957).
13) HACK, M. N., Nuovo Cim. *9*, 731 (1958).
14) LIPPMANN, B. A., and SCHWINGER, J., Phys. Rev. *79*, 469 (1950).
15) DUNFORD, N., and SCHWARTZ, J. T., Linear operators, part I, Interscience, New York 1958.