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Autor: Ruelle, D.
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Analyticity of Wightman functions at completely space-like points

D. Ruelle

Eidgenössische Technische Hochschule, Zürich*)

(14. II. 59)

Introduction

The Wightman function¹⁾

$$\mathfrak{W}(z_0, z_1, \dots, z_n) \equiv W(\zeta_1, \dots, \zeta_n) \text{ with} \quad \begin{aligned} \zeta_k &= z_k - z_{k-1} \\ z_k &= x_k + i y_k \end{aligned} \quad \begin{aligned} \zeta_k &= \xi_k + i \eta_k \end{aligned} \quad (1)$$

is defined by analytic continuation of $\langle A(x_0)A(x_1)\dots A(x_n) \rangle_0$ where A is a scalar field operator.

It is known to be analytic in UPR_n' , i.e. in the union of the extended tubes after permutation of the variables. The real points in R_n' are the J points²⁾ defined by the condition

$$(\sum_i \lambda_i \xi_i)^2 < 0 \quad \text{for} \quad \sum_i \lambda_i = 1, \lambda_i \geq 0 \quad (2)$$

We define the S points as the real points such that

$$1. \quad (x_i - x_k)^2 < 0 \quad \text{for} \quad i \neq k \quad (3)$$

2. For no permutation

$$(i_0, \dots, i_n) \text{ of } (0, \dots, n) \text{ is } (x_{i_1} - x_{i_0}, \dots, x_{i_n} - x_{i_{n-1}})$$

a J point (otherwise stated: $(\xi_1, \dots, \xi_n) \notin UPR_n'$).

For $n = 2$, S points do not exist, which means that a completely space-like point (3) is a permuted J point.

*) Permanent address: Université Libre de Bruxelles, Service de Physique théorique, Bruxelles, Belgium.

For $n = 3$, the following is a S point:

$$\begin{aligned}x_0 &= (1 - \varepsilon, 1, 1, 0), x_1 = (1 - \varepsilon, -1, -1, 0), x_2 = (\varepsilon - 1, 1, -1, 0), \\x_3 &= (\varepsilon - 1, -1, 1, 0)\end{aligned}$$

where ε is a small positive number**).

The purpose of this note is to prove the following

Theorem ***)

The real completely space-like points are contained in the envelope of holomorphy of UPR_n' .

In fact, the analyticity at completely space-like points follows from local commutativity expressed as a condition on the boundary values of \mathfrak{W}^3). It also follows from the existence of boundary values only⁴⁾. It is however interesting, in the frame of the Wightman Programme, that this existence already follows from the assumed analyticity of \mathfrak{W} in UPR_n' .

Proof

ζ_k is chosen of the following form

1. (ξ_k) belongs to a completely space-like point.
2. (η_k) is along the time axis: $\eta_k = (t_k, \vec{0})$.

Thus, we consider W as a function of $3n$ real (space) variables and of n complex (time) variables.

Let (t_k) have any fixed value, then (y_k) is determined up to a common additive constant. We order (y_k) in (y_{i_k}) according to increasing time coordinates. We have thus

$$W(\zeta_1, \dots, \zeta_n) = \mathfrak{W}(z_0, z_1, \dots, z_n) = \mathfrak{W}(z_{i_0}, z_{i_1}, \dots, z_{i_n}) = W(\zeta'_1, \dots, \zeta'_n)$$

with $t'_i \geq 0$. If for all i , $t'_i > 0$, then, W is analytic in (ζ_i) .

If, for exactly m coordinates t'_i , $t'_i = 0$, we say that W has an m -singularity at (ζ_i) . In the space (t_i) , the singularities appear thus on a finite number of linear subspaces, m -singularities being located on subspaces with dimension $n-m$. From this it follows that an m -singularity is never limiting point of m' -singularities for $m' > m$. We now use induction to remove the m -singularities for increasing m . When $m = 1$, there is no singularity because the ξ' corresponding to $t' = 0$

**) This result is due to O. STEINMANN. I thank him for his permission to quote it.

***) This theorem was proved originally by DYSON, but remained unpublished (F. J. DYSON, private communication).

is space-like and ζ' can therefore be sent into the imaginary upper half cone by an infinitesimal complex Lorentz transformation²⁾. For $m > 1$, let t_1'', \dots, t_m'' be the vanishing arguments in (t_i'') , then, W is analytic if (t_i') is held fixed except for t_1'' , and t_2'' which vary over some independent neighbourhoods of their original positions and

$$t_1'' \text{ and } t_2'' \text{ are not both zero} \quad (4)$$

We call x and y respectively the complex variables from which t_1'' and t_2'' originate. Applying then the Kantensatz⁵⁾ to the intersection of the analytic hypersurfaces:

$$\operatorname{Im} x = 0, \quad \operatorname{Im} y = 0$$

we see that there are no singularities on this intersection. This completes the proof of the theorem.

We finally remark that it simply follows from the proof that $\mathfrak{W}(z_0, z_1, \dots, z_n)$ is analytic when the vectors z_i have real space components and purely imaginary time components except when two of them coincide⁶⁾.

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