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A Hyperon Model

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Abstract. A study is made of a nucleonic particle having three elementary states of neutron, proton, and lambda character, coupled with a K-meson field by Yukawa interactions. The binding of K-mesons gives rise to a spectrum of states which contains all levels of Gell-Mann's hyperon scheme, but also other levels, all of them at higher energies with one exception.

The mass level scheme of hyperons, as known today, is sufficiently complex to warrant the question whether it is perhaps deducible from a field-theoretical model involving a smaller number of "elementary particles". One possible approach would be to follow M. Goldhaber's suggestion that the hyperons result from a binding of $\bar{K}$ mesons by nucleons[1]). Here we propose to study a slightly different model.

Assume a K-meson field to be coupled, by Yukawa interactions, with a "baryon" which possesses three elementary states, to be labelled as N, P, and Λ, having respectively the same physical characteristics as a real neutron, proton, and Λ^0-hyperon. The Yukawa interactions will give rise to the following first order transitions:

$$N \leftrightarrow \Lambda + K^0, \qquad N + \bar{K}^0 \leftrightarrow \Lambda,$$
$$P \leftrightarrow \Lambda + K^+ \qquad P + \bar{K}^- \leftrightarrow \Lambda.$$

Among the stationary states of this system, "isobaric states" will occur, and we are interested in their mass and charge spectrum and their isotopic spin assignment. It will be shown that there is, to some extent, a qualitative resemblence with the observed spectrum.

In this preliminary study, no sophisticated field-theoretical treatment is attempted. Attention will be focussed on the qualitative features of the model. It is then justified to simplify the model as much as possible. The K-mesons are taken to be scalar, and the

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[1]) M. Goldhaber, Phys. Rev. **101**, 433 (1956).

baryon spin will be ignored. A "strong cutoff" will be adopted so that no recoil problems arise. Introducing a lattice space[2]) this implies that the baryon interacts only with mesons located in the *same space cell,* and all other mesons can be disregarded as far as the bound states are concerned.

The Hamiltonian of the free K-meson field can then be written in terms of two complex variables ψ_1, ψ_2, for the neutral and the charged variety respectively, and the conjugate momenta π_1, π_2:

$$H = \pi_1^* \pi_1 + \mu^2 \psi_1^* \psi_1 + \pi_2^* \pi_2 + \mu^2 \psi_2^* \psi_2 .$$

This is equivalent to a four-dimensional oscillator. With the substitution

$$\left.\begin{array}{l} \psi_1 = r_1 e^{i\vartheta_1} = r \cos\varphi\, e^{i\vartheta_1} \\ \psi_2 = r_2 e^{i\vartheta_2} = r \sin\varphi\, e^{i\vartheta_2} \end{array}\right\} \tag{1}$$

$$(r \geqslant 0, \quad 0 \leqslant \varphi \leqslant \pi/2, \quad 0 \leqslant \vartheta_i \leqslant 2\pi)$$

one obtains:

$$H = -\frac{1}{4}\left(\frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r}\right) + \frac{1}{r^2} K^2 + \mu^2 r^2 \tag{2}$$

$$K^2 = -\frac{1}{4}\left(\frac{\partial^2}{\partial \varphi^2} + 2\cot 2\varphi \frac{\partial}{\partial\varphi} + \frac{1}{\cos^2\varphi}\frac{\partial^2}{\partial {\vartheta_1}^2} + \frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial {\vartheta_2}^2}\right). \tag{3}$$

Interpreting the mesons K^0 and K^+ as an isotopic spin dublet, we can (by an infinitesimal rotation in "charge space" applied to the Lagrangian) derive the isotopic spin vector operator (for the mesons alone):

$$K_x = -\frac{1}{2i}\left(\sin(\vartheta_2 - \vartheta_1)\frac{\partial}{\partial\varphi} + \cos(\vartheta_2 - \vartheta_1)\left(\tan\varphi \frac{\partial}{\partial \vartheta_1} + \cot\varphi\frac{\partial}{\partial\vartheta_2}\right)\right)$$

$$K_y = +\frac{1}{2i}\left(\cos(\vartheta_2 - \vartheta_1)\frac{\partial}{\partial\varphi} - \sin(\vartheta_2 - \vartheta_1)\left(\tan\varphi \frac{\partial}{\partial \vartheta_1} + \cot\varphi\frac{\partial}{\partial\vartheta_2}\right)\right)$$

$$K_z = \frac{1}{2i}\left(\frac{\partial}{\partial\vartheta_2} - \frac{\partial}{\partial\vartheta_1}\right)$$

$$K_x^2 + K_y^2 + K_z^2 = K^2 .$$

Note that with the substitution

$$2\varphi = \chi, \quad \vartheta_1 + \vartheta_2 = \vartheta_+, \quad \vartheta_2 - \vartheta_1 = \vartheta_-:$$

$$K^2 = -\left(\frac{1}{\sin\chi}\frac{\partial}{\partial\chi}\sin\chi\frac{\partial}{\partial\chi} + \frac{1}{\sin^2\chi}\left(\frac{\partial^2}{\partial\vartheta_+^2} + \frac{\partial^2}{\partial\vartheta_-^2} + 2\cos\chi\frac{\partial^2}{\partial\vartheta_+\partial\vartheta_-}\right)\right).$$

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[2]) G. Wentzel, Helv. Phys. Acta **13**, 269 (1940), §§ 2, 3.

This is the Hamiltonian of a spherical top in terms of Eulerian angles, whose eigenfunctions are well known. They have the form

$$e^{im\vartheta_+ + in\vartheta_-} f(\chi) = e^{im_1\vartheta_1 + im_2\vartheta_2} f(2\varphi)$$

where

$$m, n = j, \quad j-1, \quad j-2, \cdots, \quad -j$$

if $j(j+1)$ is the eigenvalue of K^2. Here, j, m, n can be all integral or all half-integral, whereas $m_1 = m - n$ and $m_2 = m + n$ are integral. Note that m_2 is the charge of the meson field, whereas $m_1 + m_2 = 2\,m$ stands for the total number of mesons minus antimesons, i.e. the "strangeness" of the meson state:

$$s_m = 2\,m = m_1 + m_2 \qquad (|s_m| \leqslant 2\,j)\,. \tag{4}$$

Since $K_z = n = \frac{1}{2}(m_2 - m_1)$, the charge is $m_2 = K_z + s_m/2$.

To introduce the coupling with the baryon, we distinguish three components of the state vector, F_0, F_1, F_2, referring to the cases that the "bare" baryon is in the Λ, or N, or P state, and write the Schrödinger equation as follows:

$$\left.\begin{aligned} &(-E+H)\,F_1 - g\,\psi_1^* F_0 = 0\\ &(-E+H)\,F_2 - g\,\psi_2^* F_0 = 0\\ &(-E+H+M)\,F_0 - g(\psi_1 F_1 + \psi_2 F_2) = 0\,. \end{aligned}\right\} \tag{5}$$

g is the coupling constant, and M is the mass difference of the bare Λ and the bare nucleon. Regarding isotopic spin, Λ is a singlet and N, P form a dublet. The baryon contribution to the isotopic spin, $\vec{T}$, is then easily constructed as a 3×3 vector matrix, involving the Pauli 2×2 matrices $\vec{\sigma}/2$ as submatrices, other elements being zero. The total isotopic spin

$$\vec{I} = \vec{K} + \vec{T}$$

is conserved, and the eigenvalues of I^2 are $i(i+1)$, with i integral or half-integral.

We now proceed to construct such solutions F of the equations (5) which are also eigenfunctions of $I^2 = K^2 + 2\,\vec{K}\cdot\vec{T} + T^2$. First we observe that in the Λ component of $I^2 F$, the matrices $\vec{T}$ and T^2 give no contribution: $(I^2F)_0 = K^2F_0$. Hence F_0 must be an eigenfunction of K^2, belonging to the eigenvalue $j = i$. Therefore, let

$$F_0 = \alpha(r)\,u(\varphi\,\vartheta_1\,\vartheta_2) \tag{6}$$

where

$$K^2 u = i\,(i+1)\,u\,.$$

For a given i-value, there are $(2\,i+1)^2$ such eigenfunctions u. Substituting (6) and (1) into (5), we expand

$$\left.\begin{aligned}\cos\varphi\, e^{-i\vartheta_1} u &= \sum_j b_j v_j(\varphi\,\vartheta_1\,\vartheta_2)\\ \sin\varphi\, e^{-i\vartheta_2} u &= \sum_j c_j w_j(\varphi\,\vartheta_1\,\vartheta_2)\end{aligned}\right\} \tag{7}$$

$$K^2 v_j = j(j+1)\, v_j\,, \quad K^2 w_j = j(j+1)\, w_j\,.$$

Actually, the expansions have (at most) two terms each: $j = i \pm \frac{1}{2}$ (see Appendix). Now, letting

$$F_1 = \sum_j b_j \beta_j(r)\, v_j$$
$$F_2 = \sum_j c_j \gamma_j(r)\, w_j$$

the first two equations (5) yield

$$\begin{aligned}(-E+H_j)\,\beta_j(r) - g\,r\,\alpha(r) &= 0\\ (-E+H_j)\,\gamma_j(r) - g\,r\,\alpha(r) &= 0\end{aligned} \tag{8}$$

where (see (2))

$$H_j = -\frac{1}{4}\left(\frac{d^2}{d r^2} + \frac{3}{r}\frac{d}{d r}\right) + \frac{j(j+1)}{r^2} + \mu^2 r^2.$$

Since obviously $\gamma_j(r) = \beta_j(r)$, the combination $\psi_1 F_1 + \psi_2 F_2$ occurring in the last equation (5) can be written

$$r\sum_j \beta_j(r)\left[b_j v_j \cos\varphi\, e^{i\vartheta_1} + c_j w_j \sin\varphi\, e^{i\vartheta_2}\right].$$

Here, the expression in square brackets reduces, for each j-value, to u times a constant a_j (for details see the Appendix). Thus with (6):

$$(-E + H_i + M)\,\alpha(r) - g\,r\sum_j a_j \beta_j(r) = 0\,. \tag{9}$$

It remains to solve the ordinary differential equations (8), (9) for the three functions $\beta_{i\pm\frac{1}{2}}(r)$ and $\alpha(r)$.

Each such stationary state is characterized essentially by the function u, or its quantum numbers i, m_1, m_2, which specify the meson field in the presence of a bare Λ. Since the Λ has no charge nor isotopic spin, and its strangeness is -1 (by definition), the total system has isotopic spin i, strangeness $s = s_m - 1 = m_1 + m_2 - 1$, and charge $m_2 = I_z + s/2 + \frac{1}{2}$. The energy E will depend on i and s, but not on m_2.

Table I lists, in the third and fourth columns, the values of the constants a_j appearing in (9) for $i = 0, \frac{1}{2}, 1$, and for the various allowed s-values ($/s + 1/ \leqslant 2\,i$, see (4)).

Table I.

i	s	$a_{i-\frac{1}{2}}$	$a_{i+\frac{1}{2}}$	ε_∞	ε_0
0	-1	0	1	0	M
$^1/_2$	0	$^1/_2$	$^1/_2$	1	0
	-2	0	1	2	$\mu+M$
1	$+1$	$^2/_3$	$^1/_3$	3	μ
	-1	$^1/_3$	$^2/_3$	4	μ
	-3	0	1	5	$2\mu+M$

In order to see qualitatively how the level scheme changes with varying coupling strength g, we have studied the limiting cases $g \to \infty$ and $g \to 0$. In the strong coupling case[2], for the low energy states, because the center of the wavefunction is shifted to large r-values, the "centrifugal" terms $j(j + 1)/r^2$ may be considered as small perturbations, and the same is true for the mass term M, provided that M is of the order μ or smaller. Neglecting these perturbations:

$$\beta_j(r) = \alpha(r), \qquad (-E + H_0 - g\,r)\,\alpha(r) = 0;$$

the effective potential $\mu^2 r^2 - g\,r$ has its minimum at $r = g/2\,\mu^2$, and the ground state energy is roughly $-\,g^2/4\,\mu^2$ (the strong coupling condition is $g^2 \gg \mu^3$). Then, adding the perturbations, their lowest order contributions to E are found to be

$$\delta E = \frac{M}{2} + \frac{2\,\mu^4}{g^2}\Big(\sum_j a_j\, j(j+1) + i\,(i+1) + \text{const.}\Big) + \cdots$$

In other words, the strong coupling mass spectrum is essentially given by the expression

$$\varepsilon_\infty = \sum_j a_j\, j\,(j+1) + i\,(i+1) - \frac{3}{4}\,. \tag{10}$$

This is listed in the fifth column of Table I (also see Appendix).

As the coupling strength decreases, the perturbations, which act as repulsive potentials, will favour the probability amplitude with the lowest j-value, i.e. $\beta_{i-\frac{1}{2}}$ provided that $b_{i-\frac{1}{2}} \neq 0$. Also, α is somewhat disfavoured (but less than $\beta_{i+\frac{1}{2}}$) owing to the mass term, if we assume $M > 0$ (but $M < \mu$ if the Λ is to be stable against $\bar{K}$-meson decay in the weak coupling case). It is then easily seen which amplitude survives as $g \to 0$, i.e. which free particle state results in this limit. This is indicated in the last column of Table I. The appearance of M means, of course, that the Λ-state survives; this is the case whenever $a_{i-\frac{1}{2}} = 0$. The zeropoint energy of the oscillator ($2\,\mu$) has been substracted, so μ indicates the presence of a free meson, for $g \to 0$. It should be noted that not all free particle states appear in the list because some of them do not give rise to bound states when coupled.

The result is qualitatively summarized in Figure 1 which pictures the mass and charge spectrum for the two extreme cases (large g

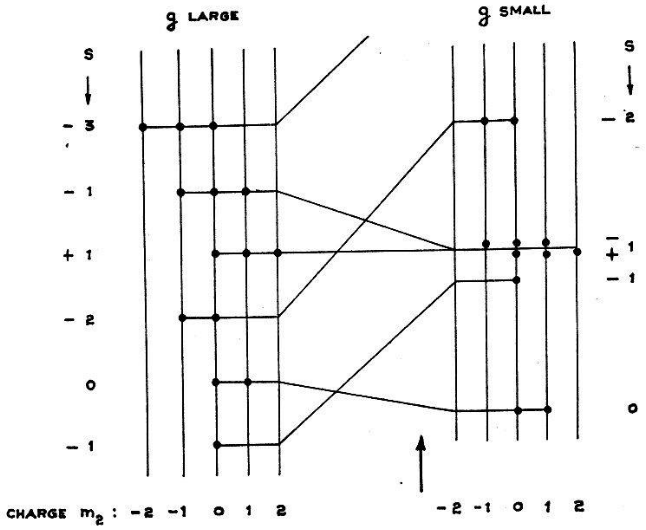

Fig. 1.

to the left, small g to the right). The units and zero levels on either side are unrelated. M has been arbitrarily chosen as $0.8\,\mu$. The straight lines connecting the two sides are meant only to illustrate the transition qualitatively, in the sense of an assignment. Nevertheless, the general character of the spectrum for intermediate g-values can be inferred.

The similarity with the Gell-Mann scheme is best noticeable at g-values close to the weak coupling side (arrow in the figure). The proton-neutron, Λ, $\Sigma(i = 1,\ s = -1)$, and $\Xi(i = \frac{1}{2},\ s = -2)$ are immediately recognizable. However, below the Σ there appears another charge triplet of strangeness $+1$ which is not observed*). For the sake of discussion, let us call this hypothetical hyperon (with $i = 1,\ s = +1$) Σ'. According to (8) and (9), its mass is certainly below the Σ-mass (because this is so for both large and small g-values). It is conceivable that its mass is even below the Λ and below the nucleon plus pion mass making it stable against pion

*) *Note added in proof:* The same difficulty arises in a "K-meson pair theory" corresponding to Goldhaber's model[1]).

decay. A very long lifetime might have prevented the detection of such a particle. Nonetheless, the associated production of Σ' plus $\overline{K}$, conserving strangeness, should then be expected to occur with a threshold much lower than for K plus $\overline{K}$ production. On the other hand, the asymmetry in K and $\overline{K}$ would still pertain since only K can be created in association with Λ, and this is by far the most frequent process. Indeed, if we apply perturbation theory since we found g rather small, $N \to \Lambda + K$ is a first order process whereas $N \to \Sigma + K$ and $N \to \Sigma' + \overline{K}$ are of higher order (going through the steps $N \to \Lambda + K \to N + \overline{K} + K$ where either $N + \overline{K}$ can bind to form a Σ, or $N + K$ can form a Σ'). We tend to believe, therefore, that the existence of a hyperon Σ' is not entirely ruled out by the present experimental information. The same can be said of the higher states (e. g. $i = 3/2, 2, \ldots$) predicted by our model whose excitation would require processes of still higher order, with the exception of states with $s = 0$, whose lifetimes are presumably very short.

If no Σ' exists below the Σ, this may mean that mesons of the same strangeness have a strong repulsive interaction. To give an example, let the interaction energy (of mesons in the same space cell) depend on

$$s_m^2 = (m_1 + m_2)^2 = \left[\frac{1}{i}\left(\frac{\partial}{\partial \vartheta_1} + \frac{\partial}{\partial \vartheta_2}\right)\right]^2$$

in such a way that it vanishes for $|s_m| = 0$ and 1 (there should be no "interaction" in a single-particle state) but rises very sharply as $|s_m|$ becomes 2, 3, ... (Since s_m commutes with the vector $\vec{K}$, isotropy in charge space is maintained). This interaction will appear as an additional term in the meson energies H_j in (8) and H_i in (9), and will depend on the s_m-values of the respective meson states. Comparing now the stationary states Σ and Σ', all component functions u, v_j, w_j of Σ belong to either $s_m = 0$ or $s_m = -1$, whereas in Σ' the component u belongs to $s_m = 2$. Consequently, the energy of Σ will be unaffected (the same is true for the nucleon and the Λ), whereas the Σ' level is *raised*, due to the meson repulsion, maybe substantially so, though not beyond the $N + K$ level. Hence, this Σ' would still be stable against K-decay, but the threshold for $\Sigma' + \overline{K}$ production might be almost as high as that for $K + \overline{K}$ production. Incidentally, this meson repulsion would also lift the level of the $\Xi(i = \frac{1}{2},\ s = -2)$, but Ξ would remain stable against the decay into $\Lambda + \overline{K}$.

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Appendix.

In regard to the computation of the constants b_j and c_j in (7), we observe that the matrix elements of

$$\xi_1 = \cos\varphi\, e^{i\vartheta_1}, \qquad \xi_2 = \sin\varphi\, e^{i\vartheta_2},$$

can be derived from the commutation relations

$$\begin{aligned}
[\xi_1, K_z] &= \frac{1}{2}\,\xi_1, & [\xi_2, K_z] &= -\frac{1}{2}\,\xi_2,\\
[\xi_1, K_-] &= 0\quad, & [\xi_2, K_-] &= \xi_1,\\
[\xi_1, K_+] &= \xi_2\quad, & [\xi_2, K_+] &= 0.
\end{aligned}$$

$(K_\mp = K_x \mp iK_y)$. Selection rules follow immediately. In particular, the identity

$$[[\xi, K^2], K^2] = \frac{1}{2}\,(\xi K^2 + K^2 \xi) + \frac{3}{16}\,\xi,$$

valid for both ξ_1 and ξ_2, yields the selection rule $\Delta j = \pm\frac{1}{2}$.

Characterizing the eigenstates of K^2 by the quantum numbers j, m, n, as defined in the text (viz. $2m = m_1 + m_2$, $2n = m_2 - m_1$; note the symmetry under the substitution $\vartheta_+ \leftrightarrow \vartheta_-$, $m \leftrightarrow n$), one finds the following non-vanishing matrix elements of ξ_1 and ξ_2:

$$\begin{aligned}
\left(j, m, n \,\middle|\, \xi_1 \,\middle|\, j+\frac{1}{2},\ m-\frac{1}{2},\ n+\frac{1}{2}\right) &= \left[\frac{(j-m+1)\,(j+n+1)}{2\,(2j+1)\,(j+1)}\right]^{\frac{1}{2}}\\
\left(j, m, n \,\middle|\, \xi_1 \,\middle|\, j-\frac{1}{2},\ m-\frac{1}{2},\ n+\frac{1}{2}\right) &= \left[\frac{(j+m)\,(j-n)}{2\,(2j+1)\,j}\right]^{\frac{1}{2}}\\
\left(j, m, n \,\middle|\, \xi_2 \,\middle|\, j+\frac{1}{2},\ m-\frac{1}{2},\ n-\frac{1}{2}\right) &= \left[\frac{(j-m+1)\,(j-n+1)}{2\,(2j+1)\,(j+1)}\right]^{\frac{1}{2}}\\
\left(j, m, n \,\middle|\, \xi_2 \,\middle|\, j-\frac{1}{2},\ m-\frac{1}{2},\ n-\frac{1}{2}\right) &= -\left[\frac{(j+m)\,(j+n)}{2\,(2j+1)\,j}\right]^{\frac{1}{2}}.
\end{aligned}$$

If the function u, in (6) and (7), is the state function $|\, i, m, n)$, then $v_j = |\, j, m-\frac{1}{2}, n+\frac{1}{2})$ and $w_j = |\, j, m-\frac{1}{2}, n-\frac{1}{2})$, with $j = i \pm \frac{1}{2}$. The coefficients b_j and c_j in (7) are matrix elements of ξ_1^* and ξ_2^*, e.g.

$$\begin{aligned}
b_j &= \left(j, m-\frac{1}{2},\ n+\frac{1}{2} \,\middle|\, \xi_1^* \,\middle|\, i, m, n\right)\\
&= \left(i, m, n \,\middle|\, \xi_1 \,\middle|\, j, m-\frac{1}{2},\ n+\frac{1}{2}\right),
\end{aligned}$$

and are immediately given by the four matrix elements listed above, with j replaced by i. The coefficients a_j appearing in (9) are then given by $a_j = b_j^2 + c_j^2$; hence

$$a_{i+\frac{1}{2}} = \frac{i-m+1}{2\,i+1}\ ; \qquad a_{i-\frac{1}{2}} = \frac{i+m}{2\,i+1}.$$

Inserting this into (10), the strong coupling energy spectrum is found to be

$$\varepsilon_\infty = 2\,i\,(i+1) - m.$$