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Autor: Wigner, E.

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Relativistic Invariance of Quantum-Mechanical Equations

by E. WIGNER (Princeton)

Introduction

Relativity Theory, of which we are celebrating the 50$^{\text{th}}$ anniversary, and quantum theory, which is about equally old, originated and developed in very different ways. The theory of relativity owes its origin to a set of experimental facts which can be epitomised as the independence of light velocity from the state of motion of emitter and absorber. However, its guiding stars in the course of its development were conceptual problems, problems of the measurement of space and time and of observation. Experimental facts played a relatively subordinate role in the development of relativity theory at least for the last 25 years. Quantum theory, on the contrary, originated as the result of the discussion of a conceptual problem, the inconsistencies in the classical description of black body radiation. However, the guiding stars of quantum theory were experimental facts: the photoelectric effect, the STERN-GERLACH phenomenon, the BOTHE-GEIGER-COMPTON-SIMON experiments and, before all, the immense amount of detailed information which was accumulated before the war on atomic spectra and is being accumulated now on nuclear forces and 'elementary particles'. The discovery of much of this information is traceable to the stimulus provided by quantum theory, all of it exerted a profound influence on quantum theory's development.

Similarly, the objects on which relativity and quantum theories focus their prime attention are very different. Relativity theory deals principally with macroscopic objects. In particular, the coordinate systems for which the special theory postulates equivalence are macroscopic, not subject to quantum uncertainties. Furthermore, at least as far as the general theory is concerned, its great successes are all in the domain of macroscopic physics. Its principal subject is a phenomenon which modern experimental physics would surely have overlooked were it not for the rather extraneous fact that the experimenter and his apparatus are constantly pulled to the floor of the Laboratory. The phenomena of principal interest for the quantum theorist are of microscopic nature, particles so light that one can be almost sure that their individual gravitational effect is unobservable even in principle. Because of this difference in the development, and in the subject matter of the theories of relativity and of quanta, it is hardly surprising that the efforts at their unification met serious difficulties. In fact, it would be hardly surprising if the knowledge of a new set of phenomena were required before a complete unification will become possible. It is most gratifying therefore that the attempts at a partial unification met with such striking success, often in an unforeseeable way. As the principal ones I would quote DIRAC's demonstration that the simplest relativistically invariant single particle equation attributes a spin to the particle described [1], PAULI's demonstration that the simplest way to quantise the single particle equation naturally leads to his equivalence postulate for all particles and to the exclusion principle for particles with half integer spin [2], and the success of the relativistically invariant perturbation methods of TOMONAGA, SCHWINGER and FEYNMAN in describing the finest details in the electronic spectra of hydrogen and other elements [3]. However, notwithstanding their importance, I shall not devote the body of my address to an elaboration of these points but will try to give a bird's eye view of the situation. In particular I shall try to trace the effect of the transition from classical to relativity theory as it manifests itself in the quantum mechanical equations of elementary particles and the invariants of these equations. I shall begin with classical theory; the transition to the special theory of relativity will be described rather completely. As intermediate point for a future transition to the general theory of relativity, the quantum mechanical properties of the DE SITTER spaces will be discussed next. These also provide the mathematically neatest embodiments of relativity theory short of a full reformulation of our space-time concepts. It is well to emphasize, however, that the consideration of the DE SITTER spaces does not yet meet with the very difficult conceptual problems which a full incorporation of the deep physical ideas of the general theory of relativity demands. These problems will be touched upon at the end of the discussion.

I shall base most of the discussion on the equivalence between the quantum mechanical equations for single particles and the simplest (so called irreducible) representations (up to a factor) of the symmetry group of the world in which those equations apply. I had an occasion, a few weeks ago, to present some aspects of this point of view and I shall try to avoid repeating myself[1]). The basic idea of the point of view which I am recapi-

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[1]) At the meeting of the International Union for Pure and Applied Physics in Pisa, June 13 to 17, 1955; cf. Nuovo Cimento X 3, 517 (1956) It should be pointed out that the considerations referred to in the text apply not only to the possible

tulating is to consider the transformation of the state vector (or wave function) for all the elements of the symmetry group of the underlying 'world' on the same footing [4]. The progress of time, i.e. the displacement of the time axis is one such transformation. It happens to be the one in which the more usual formulation of the theory is principally interested. However, one gains a deeper insight into the significance of relativistic invariance by considering this transformation not separately but in its relation to the other relativistic transformations.

Before starting on the principal discussion, let me illustrate this point by means of an example based on the special theory of relativity. Let us begin with the operator of an infinitesimal time displacement – which is in fact the crucial operator in the more usual formulation of the theory. It follows from the general principles of quantum mechanics that this infinitesimal operator has the form H/i where H is a self-adjoint operator. This gives the well known equation for the state vector Φ

$$i \frac{\partial \Phi}{\partial t} = H \, \Phi \, . \tag{1}$$

If one denotes the characteristic functions and the characteristic values of H by ψ_k and ν_k the general solution of (1) can be written down at once

$$\Phi = \sum a_k \, e^{-i \nu_k t} \, \psi_k \tag{1a}$$

where the a_k are arbitrary constants. This is a complete solution of the equations of motion. It is, however, not a complete solution of the physical problem because the physical properties of the states ψ_k are not known. One can say that (1a) tells us *how* it moves but not *what* moves.

It is at this point that the relation of the time displacement operator and the other relativistic operators becomes crucial. It tells us, for instance, that the infinitesimal operators for displacements along the spatial axes iP_x, iP_y, iP_z commute with H-simply because one obtains the

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states of a single particle but to all sets of states which are as small as possible consistent with the requirement of the superposition principle and relativistic invariance. Thus, for example, they apply equally well to all states of motion of an oxygen atom (or almost any other atom) in its normal state. The requirement 'as small as possible' excludes, however, the states of motion of an oxygen atom in two or more states of excitation because one can select, from these states, a smaller set for which the superposition principle applies and which can be characterised in a relativistically invariant fashion: 'normal state' is such a relativistically invariant characterisation. Thus the expression 'single particle' is somewhat too exclusive for the description of the physical systems to which the considerations of the text apply and these systems have been termed 'elementary systems'. However, single particles are the most important physical systems of this nature. For a further elaboration of this point, see [5].

same result no matter in which order one carries out displacements in space and time. This shows that all those ν_k are equal the corresponding ψ_k of which can be obtained from each other by displacements. This is still a rather trivial result. However, the consideration of the proper LORENTZ transformations and of the rotations gives more significant results and permits one, in fact, to determine the physically important properties of the ψ_k as long as the set of operators which correspond to all relativity transformations is irreducible [5]. This is the essence of the point of view from which I wish to compare the classical and relativistic quantum theories.

Given two relativity transformations r and s (such as the displacements considered above) the corresponding unitary operators may be denoted by U_r and U_s. The operator U_{rs} which corresponds to the product $r\,s$ of the two relativity transformations would seem to satisfy the equation

$$U_{rs} = U_r\, U_s\,.$$

It is a very essential point that this equation is not a necessary consequence of the basic postulates of quantum theory and of the invariance of the equations. Instead, because of the indeterminate factor in the state vectors, only

$$U_{rs} = \omega(r, s)\; U_r\, U_s \tag{2}$$

can be inferred where $\omega(r, s)$ is a function (c-number) depending on r and s. The mathematical term for unitary operators which satisfy (2) is that they form a (unitary) representation up to a factor of the invariance group. The discussion will therefore be based on the representations, up to a factor, of the group of classical mechanics (GALILEI group), of the group of the special theory of relativity (inhomogeneous LORENTZ group or POINCARÉ group), of the DE SITTER space, etc. We begin with the classical theory.

Classical Theory

The symmetry group consists in this case of the symmetry of Euclidean space – that is rotations and displacements – coupled with GALILEI transformations, that is transformations to a moving coordinate system. The corresponding transformations can be represented by the matrices

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} & v_x & a_x \\ R_{21} & R_{22} & R_{23} & v_y & a_y \\ R_{31} & R_{32} & R_{33} & v_z & a_z \\ 0 & 0 & 0 & 1 & a_t \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \\ t' \\ 1 \end{bmatrix}. \tag{3}$$

Such a matrix when applied to a vector the components of which are x, y, z, t, 1 gives a new vector the last component of which is again 1. This component has no physical significance; it is introduced only for mathematical convenience. The first four components of the vector obtained by applying our matrix are the transformed coordinates, x', y', z', t'. The R are components of a rotation; they represent the rotation contained in the generalised GALILEI transformation; v_x, v_y, v_z are the components of the velocity of the second coordinate system with respect to the first and the a its displacement in space and time. The group of matrices of the form as given in (3) is the GALILEI group. It is the group of classical mechanics. An investigation of the representations up to a factor of the GALILEI group gives a rather surprising result [6]: there are two and only two types of representations. The first type is the simplest possible; its operators satisfy the simple equation $U_r U_s = U_{rs}$ in which r and s are any two GALILEI transformations. In particular U_r and U_s commute if r is a spatial displacement and s a transition to a moving coordinate system. In the second type of representations, spatial displacements and transitions to a moving coordinate system do not commute, taken in different orders they differ by a factor

$$\frac{\omega(r, s)}{\omega(s, r)} = e^{i m \boldsymbol{a} . \boldsymbol{v}} \tag{4}$$

where m is an arbitrary real constant. The SCHRÖDINGER equation (for a single particle) is of the second type, the m in (4) plays the role of the mass of the particle. An investigation of the representations of the second type, along the lines sketched in the introduction, leads to the usual operators for momentum, velocity, energy, and position. In order to obtain, for instance, the position operators, one has to look for three commuting operators which transform as a vector under rotations, to which a spatial displacement $\boldsymbol{a}$ adds the components of $\boldsymbol{a}$ and which are invariant under transitions to a moving coordinate system. There is only one triad of such operators and these are the usual position operators. The momentum and velocity operators can be defined in similar ways and their ratio is given by the m in (4).

All this is quite satisfactory but the point which I wish to make is that the same postulates cannot be satisfied for representations of the first type. There is, in the HILBERT space of these representations, no triad of operators which satisfies, for instance, the conditions enumerated for the momentum operators [7]. The infinitesimal displacement operators p_x, p_y, p_z in particular transform under a transition to a moving coordinate system like

$$\boldsymbol{p} \rightarrow \boldsymbol{p} \qquad \varepsilon \rightarrow \varepsilon + \boldsymbol{p} \cdot \boldsymbol{v} \, .$$

That is, the spatial displacement operators remain unchanged under these transformations. One has to conclude that these representations have no physical significance, that there are no particles the state vectors of which would transform, under Galilei transformations, by means of a representation of the first type. For the time being, this seems to be an isolated result but it will appear in a new light when classical mechanics is viewed as a limiting case of the special theory of relativity.

Special Relativity Theory

The symmetry group consists in this case of a combination of spatial and time displacements with ordinary (homogeneous) Lorentz transformations. The latter are combinations of rotations and transitions to a moving coordinate system. The transformations of this group can be represented by matrices of the form

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & \Lambda_{10} & a_x \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} & \Lambda_{20} & a_y \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} & \Lambda_{30} & a_z \\ \Lambda_{01} & \Lambda_{02} & \Lambda_{03} & \Lambda_{00} & a_t \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \\ t' \\ 1 \end{bmatrix} . \tag{5}$$

The last coordinate is introduced again for mathematical convenience, the Λ are components of a homogeneous Lorentz transformation, that is the transformation leaves the quadratic form $x^2 + y^2 + z^2 - t^2$ invariant (however, we do not consider, at this point reflections or inversions). It is well known that the Galilei transformations (3) can be considered as limiting cases of the transformations (5) and it is interesting to see how the representations of the Galilei group can be obtained as limiting cases of the representations (5), i. e. how the quantum theory of classical physics is obtained as a limiting case of the quantum theory of special relativity.

No representation of the Poincaré group (5) has a factor similar to the factor (4) of the Galilei group. Instead, all of them can be normalised in such a way that

$$U_r\, U_s = \pm\, U_{rs} \tag{6}$$

holds for any two transformations r and s. The representations can be classified, on the other hand, by the value which the Lorentzian sum of the squares of the infinitesimal displacement operators

$$P_t^2 - P_x^2 - P_y^2 - P_z^2 \tag{7}$$

assumes. This quantity can be positive, 0, or negative.

Only the first two cases have been investigated thoroughly and they prove to be equivalent with equations for particles with positive and zero restmass respectively. If (7) is positive the momentum, velocity and position operators can be defined in the same fashion as was explained at the discussion of the GALILEI group; if (7) is zero it is, in some cases, not possible to define position operators but the postulates for the momentum and velocity operators have a unique solution in every case. When the transition from (5) to (3) is made, i. e. when the light velocity is assumed to be infinite, the representations just described go over into the representations of the GALILEI group.
The correspondence is as follows [8]

$$P_t^2 - P_x^2 - P_y^2 - P_z^2 = m^2 > 0 \qquad \omega = e^{im\,\boldsymbol{a}.\boldsymbol{v}}$$

$$P_t^2 - P_x^2 - P_y^2 - P_z^2 = 0,\ \text{finite spin} \qquad \omega = e^{iS\,\boldsymbol{a}.\boldsymbol{v}}$$

$$P_t^2 - P_x^2 - P_y^2 - P_z^2 = 0,\ \text{infinite spin} \qquad \omega = 1$$

$$P_t^2 - P_x^2 - P_y^2 - P_z^2 < 0 \qquad \omega = 1;$$

ω is the expression which appears in equation (4). The positive restmass and the ordinary zero restmass representations have a reasonable non relativistic limit; the other cases lead to representations in which neither momentum nor position coordinates can be defined. The case in which (7) is negative, i. e. for which the infinitesimal displacement operators form a space like vector are commonly assumed to violate the KRAMERS-KRONIG causality conditions. They can be seen also to have no reasonable non-relativistic. This is then the interpretation of the true representations (i. e. for which $U_r U_s = \pm\, U_{rs}$ for all r and s) of the GALILEI group: they form the non relativistic limit of relativistic particles with space-like momentum which violate the principle of causality.

As is well known, in order to characterise a representation (or the equivalent equation) completely, one has to give in addition to the mass, the spin S of the particle. If the restmass is positive, it is permissible to consider the particle to be at rest; the number of its states at rest is then $2\,S + 1$. The same is the number of states with any given momentum. However, if the restmass is zero, there are for all $S \geq 1/2$, only two different states. A particle of restmass zero cannot be, of course, considered to be at rest. Nevertheless, it seems worthwhile to explain more in detail this difference in the behavior of the particles with zero and with finite restmass.

If a particle is at rest, and its spin has a definite value in a given direction, the $2\,S + 1$ states of the particle can be obtained by considering the particle from coordinate systems which arise by a rotation from the original coordinate system. However, if the particle is in rapid motion, i. e. if the spatial component and the time component of its momentum are nearly equal, and if its spin has a definite value in the direction of its motion, this state of affairs will be invariant under rotations. In order to change the spin component in the direction of motion substantially, one has to consider the particle in a coordinate system in which it is nearly at rest, that is in which its velocity is well below the light velocity. This cannot be done if the restmass of the particle is zero and hence the spin of such particles in the direction of their motion is a relativistically invariant characteristic. The fact that such particles have two directions of polarization, instead of only one, is a result of the reflection symmetry. A reflection transforms a spin component S in the direction of propagation, into a spin component $-S$ in the same direction. In the case of finite restmass, the $-S$ state (as well as all spin directions) can be obtained also by a rotation; the parity is the ratio of the state vectors obtained by rotation and by reflection. Since, for zero restmass, the $-S$ state cannot be obtained by rotation from the $+S$ state, these particles have no parity or rather have both even and odd states with respect to reflection at the same energy and momentum. Only the KLEIN-GORDON particle ($S = 0$) is an exception: in this case the reflection does not produce a new state vector but reproduces the original one with positive or negative sign.

The fact that a fast moving particle's polarization does hardly change under a not too drastic LORENTZ transformation can be seen most easily on the example of the DIRAC electron. If one decomposes a state which has positive polarization in the direction of motion into characteristic functions of $\gamma = i\,\gamma_t\,\gamma_x\,\gamma_y\,\gamma_z$, the absolute values of the coefficients are

$$\frac{P_t + m + P}{(4\,P_t\,(P_t + m))^{1/2}} \qquad \text{and} \qquad \frac{P_t + m - P}{(4\,P_t\,(P_t + m))^{1/2}} \tag{8}$$

The state vector for which $\gamma = 1$ remains such a vector under LORENTZ transformations and the same holds for the $\gamma = -1$ vector. If the coefficient of the former is practically 1, this condition will remain unchanged under LORENTZ transformations which do not change the length of the $\gamma = 1$ vector too drastically. It follows that the polarization will hardly change under such LORENTZ transformations.

It is possible to express this in another way which shows, at the same time, that the property in question is a property of the LORENTZ group and not of a particular representation thereof, i. e., that it is true for all values of the spin. Consider a particle at rest and polarized in the z direction. Impart to it a velocity in the z direction by subjecting it to a LORENTZ transformation with the hyperbolic angle α. Later, this angle will be assumed to be very large so as to make the particle highly relativistic. At any rate, we now have a particle which is polarized in the direction of its motion – which is in the z direction. In order to obtain a particle which is polarized in the direction of its motion, but is moving in another direction, one would first subject the particle to a rotation to bring the polarization into the direction of its projected motion and then accelerate it in the desired direction. In order to test whether the statement, that the polarization has the direction of the motion of the particle, is relativistically invariant we subject the particle which moves in the direction z and is properly polarized, to a second acceleration, in the x direction, by the hyperbolic angle ε. This angle is arbitrary but will be assumed, at the end, to be much smaller than α. The particle could have achieved the same state of motion by being accelerated by the hyperbolic angle α' in the direction which includes an angle ϑ with the z axis where

$$\operatorname{Cosh} \alpha' = \operatorname{Cosh} \alpha \operatorname{Cosh} \varepsilon\,, \quad \sin \vartheta = \operatorname{Cosh} \alpha \operatorname{Sinh} \varepsilon / \operatorname{Sinh} \alpha'\,. \tag{8a}$$

However, the direction of polarization would not be the same in the second case as in the first case. In order to make it the same, one has to rotate the system, before accelerating it in the ϑ direction, by an angle $\vartheta - \delta$ where δ is given by

$$\sin \delta = \operatorname{Sinh} \varepsilon / \operatorname{Sinh} \alpha' = \operatorname{Sinh} \varepsilon\, (\operatorname{Cosh}^2 \alpha \operatorname{Cosh}^2 \varepsilon - 1)^{-1/2}\,. \tag{8b}$$

This follows, simply, from the identity for LORENTZ transformations

$$A\left(\frac{1}{2}\pi, \varepsilon\right) A(0, \alpha) = A(\vartheta, \alpha')\, R\, (\vartheta - \delta). \tag{8}$$

where α, ε are arbitrary while α', ϑ and δ are defined by the last two equations. $A(\vartheta, \alpha)$ is the acceleration by a hyperbolic angle α in that direction in the $x\,z$ plane which includes an angle ϑ with the z axis; $R(\varphi)$ is a rotation by φ in the $x\,z$ plane. If δ were zero, the particle which was polarized in the direction of its motion after the acceleration α, would have remained polarized in the direction of its new motion (i. e., the ϑ direction) after the second acceleration, by ε. This is not the case, as δ is finite. However, δ is very small if $\varepsilon \ll \alpha$, i. e., if the second acceleration is by a much smaller hyperbolic angle than the first, and if $\alpha \gg 1$.

De Sitter Spaces

The symmetry group of the ordinary DE SITTER space consists of the transformations which leave the DE SITTER world

$$x^2 + y^2 + z^2 + w^2 - t^2 = R^2$$

invariant. The group of special relativity can be considered as a limiting case of this group in exactly the same sense in which the GALILEI group is a limiting case of the inhomogeneous LORENTZ group. The quadratic forms which these groups leave invariant are

$$x^2 + y^2 + z^2; \quad x^2 + y^2 + z^2 - t^2; \quad x^2 + y^2 + z^2 + w^2 - t^2 .$$

The equations which are invariant in DE SITTER space, or the representations of the group of DE SITTER space [9], have many interesting features. The distinction between particles with finite and zero restmass becomes unsharp – the finite restmass particles have to be characterized by the statement that their COMPTON wave length is very small as compared with the size of the universe. Even more remarkable are, however, the properties of the particles with respect to the discrete operations of the group such as space and time inversion. In particular, it is difficult to maintain the positive definite nature of the energy, or of its substitute, in DE SITTER space. This is hardly surprising because the same transformation which advances time in one part of space, retards it in another. Again, the physical significance of this circumstance will become clearer when the next stage of the theory, the gerneral relativity is considered.

General Relativity

I am approaching this subject with a great deal of hesitation because even the general outlines of a quantum mechanics which conforms with the ideas of the general relativity theory are quite unclear. Much of what has been said and written on this subject is more nearly a special relativistic theory of a particle with spin 2 rather than an adaptation of quantum theory to the thinking and principles which EINSTEIN has put forward.

Most of us would consider two observations to form the basis of the general theory of relativity. EINSTEIN's first observation is that coordinates have no independent meaning; that only coincidences in space-time can be observed directly and only these should be the subject of physical theory. This observation is so stringent that, properly considered, every physical theory conforms with it and I shall show that this is true also of present day quantum mechanics. However, the realisation that this is the case will show us, at the same time, that it presupposes rather special circumstances. Although these special circumstances prevail for ordinary laboratory experiments, they are special circumstances nevertheless. The necessity of observing these circumstances renders ordinary quantum theory quite artificial from the point of view of EINSTEIN's first axiom.

EINSTEIN's second axiom is the equivalence principle. This gives a preferred role to gravitation over all other types of interactions. The justification for this preference comes from the particular simplicity of gravitational interaction, from the equality of gravitational and inertial mass. The validity of this principle in the microscopic domain is not as evident as the validity of the coincidence axiom. We know of many rules which apply with great rigor to electromagnetic and other types of interaction and it is conceivable that the special role of the gravitational interaction may dissolve in a higher harmony. For this reason, I shall pay prime attention to EINSTEIN's first observation, that only coincidences have a direct physical meaning, values of coordinates do not.

One should observe, in this connection, first, that if one deals only with a finite number of particles, EINSTEIN's first observation cannot be incorporated even into classical theory. The fundamental question to which such a theory would provide an answer would have a form such as 'We have ten particles, so far collisions have occurred between particles 1—2, 5—6 and 3—6. Will particles 1—5 collide?' No theory has yet been attempted to answer questions of this sort and this statement remains true even if some further structure is permitted, such as a time order for the collisions of every particle. The transition from the enumeration of coincidences to a continuous Riemannian space implies the existence of an infinite number of small particles which constantly fly around between the principal particles, collide with them and through an infinity of collisions provide a metric. Actually, the assumption of such an infinite substrate of very small particles is not very far from reality: the light emitted by the stars forms such a substrate. We know of the other stars because the light which was in coincidence with other stars comes to coincide with us. Neither is there a fundamental inconsistency in the assumption of the infinite substrate in classical theory because there is no limitation in classical theory on the size of a particle with given energy.

All this is quite different in quantum theory. First of all, the event of a collision is not an absolute one but is subject to observation. The most natural criterion for a collision to have taken place is that the momentum of the colliding particles has changed. However, the measurement of the momentum requires a finite volume and it is hard to claim, therefore, that the collision is the basic entity of physics in terms of which everything should be described.

Second, there are difficulties with the assumption of an infinite substrate of very small particles. In order to fix the position of a principal particle, such as a star, very accurately, one has to assume that the substrate consists of particles of very short wave length. The short wave length gives, however, a lower limit to the energy and hence also to the gravitational mass of the substrate. This difficulty would be a very real and very practical one were it not for the very small value of the gravitational constant. The smallness of this constant is quite essential for the formulation of present day quantum theory if one keeps in mind that only coincidences have physical significance. From the point of view of these ideas, a quantum mechanical experiment, undertaken on an isolated system, and the use of LORENTZ metric, would have to be described as follows. The isolated system is surrounded, in space, by a framework, containing clocks at the junctions, which can be used to ascertain, first, that space-time is approximately flat on the surface surrounding the isolated system and which enables one, then, to define a coordinate system with LORENTZ-metric to impart impulses to the system and to register its response. Such a framework-clock system could be used, for instance, to measure collision cross sections or even the whole S-matrix.

Before trying to simplify this scheme, it seems worth while to remark that the necessity of the framework-clock system raises doubts whether it is meaningful to consider a simple system, such as a particle, to be alone in the universe and to obey certain equations. In order to ascertain the behavior of the particle, it would have to be surrounded by our framework-clock system and clearly, the whole universe cannot be so surrounded. For this reason it now seems doubtful to me whether particle equations in DE SITTER space have much significance and in particular whether those properties of the equations are meaningful which follow from the symmetry of this space at large. I refer by this term to symmetry planes and to the symmetry center of such a space.

If one analyses the way in which measurements on so called isolated systems are interpreted, one does in fact find the framework-clock system described above surrounding the isolated system. The motion of the stars etc. guarantees the approximate flatness of the space and provides a coordinate system with LORENTZ metric; the apparatus used for the measurement provides and registers the patricles the properties of which are to be measured. A simple consideration shows also that the gravitational forces emanating from the 'isolated system' do not interfere with the possibility of measuring cross sections with arbitrary accuracy if one is allowed to make the volume surrounded by the framework arbitrarily large.

The state of affairs just described is, nevertheless, unsatisfactory from the point of view of the principle of general relativity because the physical entity which provides the metric is distinct from the entity the properties of which one investigates. The idealisation of a framework which provides a metric but does not influence the 'isolated system' by its gravitational field is possible only because the gravitational constant is so small. In a theory which fully accepts EINSTEIN'S basic observation it would not be necessary to divide the physical system into two parts: one which gives the metric and the other which is to be described.

The simplest way to accomplish this is to renounce the use of coordinates entirely. The questions to which physical theory would give an answer would be of the sort: I have a system in which there is a three dimensional manifold on which the probability of finding particle 5 is the product of finding particles 1, 2, 3 and 4. Is there another threedimensional manifold on which the probability of finding particle 5 is some other given function of the probabilities of finding particles 1, 2, 3 and 4. Such a theory would not deal with the coordinate derivatives of field quantities but with the derivatives of probability amplitudes with respect to each other.

If one assumes no interaction, the equations of such a theory should be obtainable from the usual quantum mechanical equations through elimination of the coordinates. It must be possible to accomplish this although I have so far succeeded to do so only in a most provisional and incomplete fashion. The equations obtained by such an elimination differ in fact less from usual equations than one might first think. In particular, quantities similar to the g_{ik} appear again. The content of the equations obtained is identical with the content of the equations from which one started and I mention a set of such equations only to illustrate what I have in mind, not because I believe that they solve some problem.

Let me take, for simplicity, a world of only one spatial dimension and consider fields which obey the KLEIN-GORDON equation. Let us use two such fields φ_1 and φ_2 to eliminate the coordinates. The other fields shall be denoted by ψ_a. We then have

$$\frac{\partial \psi_a}{\partial x_\varkappa} = \sum_{i=1}^{2} \frac{\partial \varphi_i}{\partial x_\varkappa} \frac{\partial \psi_a}{\partial \varphi_i}$$

$$\frac{\partial^2 \psi_a}{\partial x_\varkappa^2} = \sum_{i,j=1}^{2} \frac{\partial \varphi_i}{\partial x_\varkappa} \frac{\partial \varphi_j}{\partial x_\varkappa} \frac{\partial^2 \psi_a}{\partial \varphi_i \, \partial \varphi_j} + \sum_{i=1}^{2} \frac{\partial^2 \varphi_i}{\partial x_\varkappa^2} \frac{\partial \psi_a}{\partial \varphi_i}. \tag{9}$$

The left side summed over $\varkappa$ is equal to $m_a^2 \psi_a$ and a similar substitution can be made on the right side

$$m_a^2 \psi_a = \sum_{ij} g_{ij} \frac{\partial^2 \psi_a}{\partial \varphi_i \, \partial \varphi_j} + \sum_i m_i^2 \varphi_i \frac{\partial \psi_a}{\partial \varphi_i} \tag{10}$$

where

$$g_{ij} = \sum_k \frac{\partial \varphi_i}{\partial x_\varkappa} \frac{\partial \varphi_j}{\partial x_\varkappa} \,. \tag{10a}$$

The first set of equations can be considered to determine the g_{ij}; the requirement of their consistency is the vanishing of a determinant of order 4 ($a = 1, 2, 3, 4$)

$$\left| m_a^2 \psi_a - \sum_i m_i^2 \varphi_i \frac{\partial \psi_a}{\partial \varphi_i} \quad \frac{\partial^2 \psi_a}{\partial \varphi_1^2} \quad \frac{\partial^2 \psi_a}{\partial \varphi_1 \partial \varphi_2} \quad \frac{\partial^2 \psi_a}{\partial \varphi_2^2} \right| = 0 \,. \tag{11}$$

This equation does not contain coordinates any more. As I said, it is mentioned only as an illustration and not because it has any status.

Diskussion – Discussion

D. van Dantzig: I first want to make a remark concerning the first part of your talk about special relativity in connection with relativistic invariance. It seems to me that the term 'relativistic invariance' is not completely unambiguous. Usually one requires invariance under all transformations which leave ds^2 unchanged. Half of these leave each half light cone, past as well as future, unaltered; the other ones interchange these.

Now, I am not quite sure that there is any *observational* evidence which makes it necessary to include in relativity theory the latter transformations also.

It seems to me rather that the only thing which is really guaranteed by experimental evidence is the group of transformations which leave the time direction unaltered.

There are several points where, without noticing it, one usually introduces a choice of time-direction. For instance in the expression of the energy-momentum of a charged particle, $p_j = m\, i_j - e\, \varphi_j$ ($c = 1$), i_j is a unit vector along the world line, which can be oriented in two directions. It *must* be oriented, however, so that the time component is *positive*; otherwise p_j is *not* the energy-momentum. In $i^h = dx^h/ds$ one can therefore not always take ds as the *positive* root $|ds|$ of ds^2, but one *must* have $ds = |ds| \operatorname{sgn} dt$, where $\operatorname{sgn} x$ is the sign of x ($= +1$, 0 or -1 according to $x > 0, = 0$ or > 0). This disturbes the invariance under the complete group. The equations of motion are only invariant under $t \to -t$ if also the sign of e is changed.

Just this fact seems to be a cause of many difficulties. For instance the necessity of introducing so many phantom particles in relativistic quantum theory (anti-proton, etc.) might disappear by considering only those quantities as relativistically invariant, which leave ds^2 as well as sgn dt invariant.

My second question concerns the second half of your talk. It is concerned with the possibility of identifying space-time points.

It seems to me that your remark is in complete agreement with what I said last monday, namely that it is impossible to define in empirical terms what a space-time-point is, and that the only thing we have are events.

I agree with you that it is not sufficient to take only observed events; we have to add to these also possibly observable, hence fictitious events. But I do not think that this leads to the necessity of introducing the concept of space and time, i. e. of a (four dimensional) *continuum* of such fictitious events. It seems to me rather that thereby something essentially unobservable is introduced, and that some kind of 'flash model', i. e. a discrete set of possibly observable events, comparable with a four dimensional crystal, the structure of which is determined (at least partially) by 'transition probabilities' instead of forces, might be more realistic.

F. Hoyle: I should like to express my agreement with Prof. Wigner's remarks on coördinates. It has seemed to me for some time that our present use of coördinates may be a psychological survival from the Newtonian era. Now that we realize that coördinates are nonmore than parameters that must be eliminated in determining relations between observables, it becomes natural to ask whether we are using the most advantageous parameters, or even whether any such parameters are necessary.

E. Wigner: I have learned since the Conference that some of the ideas which were expressed by me are similar to those presented earlier by D. van Dantzig, J. L. Synge and J. Géhéniau.

A. D. Fokker: If theory is concerned with coincidences only, taking such coincidences as consisting in collisions e. g. of particles 1 and 2, of particles 3 and 5, of particles 2 and 6, and so on, how can one know that the particle called 2 in the first event and the particle called 2 in the third event is the same? Does not theory require something more than coincidences only?

E. Wigner: The problem of identification of the particles is greatly affected by the equivalence principle. Naturally, one can distinguish between different types of coincidences, such as an electron-electron coincidence and a proton-electron coincidence. However it is not only difficult, but in principle impossible to identify the electrons (by name or number) between which a coincidence has taken place. However any theory which takes the equivalence theorem into account automatically conforms with this principle and this applies also to the type of equations which was described by me.

M. Born: Professor Wigner has put his finger exactly on the spot where the difficulty of reconciling general relativity and quantum mechanics lies, namely that relativity is based on the concepts of coincidences as the only observable things. Atomic physics has to do with collisions of particles; but quantum mechanics treats a collision as a spatially extended process of which only the asymptotic limits are observed.

Though I agree with this analysis, I am rather doubtful about the remedy suggested by Wigner. He tries to eliminate the coördinates and the time altogether and to reformulate the laws of quantum mechanics in terms of field components or wave functions alone. I prefer to adhere to Bohr's standpoint according to which all actual observations are made with the help of macroscopic systems to which the notions and laws of classical physics can be applied; the interpretation of the results then compels us to assume different laws for the underlying atomic processes. General relativity, in my opinion, has to do only with the macroscopic superstructure. In fact, of the three observable consequences of the theory only the purely macroscopic one, the anomaly of the perihelion of the orbit of Mercury, is explained by Einstein's theory without doubt; the other two effects, the deflection of light rays by the sun and the red-shift of spectral lines (which are micro-phenomena), are still controversial, at least in regard to magnitude. I think that general relativity as we know it may be invalid in this domain. Therefore I do not agree with the general negative attitude towards Freundlich's attempt to base the red-shift on new foundations. He has suggested a formula of the form $\Delta \nu/\nu = A\, l\, T^4$, where T is the absolute temperature on the surface of a star and l a certain 'length of penetration' of the surface layer. He has observed that this means probably an effect proportional to the radiation density u on the surface. If one writes his expression in the form $\Delta \nu/\nu = C\, l\, u/l_0 u_0$, and take for l_0 the atomic length $\hbar/m\, c = \lambda_0/2\,\pi$ ($m =$ atomic mass, $\lambda_0 =$ Compton wave length) and for u_0 the energy of one electron at rest per volume $l_0{}^3$, one finds for C the order of magnitude 1. As Freundlich's formula represents almost all known facts (including Wolf-Rayet stars where the relativity effect is about 100 times too small) I think this cannot be an accidental agreement and should not be dismissed without careful study.

P. G. Bergmann: (à propos Born's remark) Though I agree that the world point may lose its significance and invariant identity in the very small, I believe formly that there must remain an invariance group (perhaps larger than the invariance group of general relativity) which, correspondence-wise, goes over into that of general relativity. After all, the invariance group of general relativity is based on a sound physical principle, to which no exceptions have as yet been observed: the principle of equivalence. Personally I know of two different logical possibilties for constructing groups that 'emasculate' the meaning of the world point. One retains world points in any frame of reference but does not preserve their identities under the invariance group, and that is the group of canonical transformations under which a given covariant theory (such as general relativity) is invariant. There are (admittedly rather trivial) examples of transformations of this type that do not carry world point into world point. The other approach would consist of constructing 'spaces' that have certain topological properties similar to those of point spaces in the large but do not possess 'points' as elementary constituents; I am thinking of such structures as skew lattices, in which the skewness guarantees the non-existence of points. Whether either of these two approaches leads to anything physically promising I do not pretend to know.

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