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On Equations of Motion in General Relativity Theory

by L. Infeld (Warschau)

I would like to speak briefly on the history of the problem of motion since I had the great privilege of working on this problem with Einstein. For the first year I worked on the problem with Einstein and Hoffmann, later with Einstein only. At the end of my lecture, I shall say a few words about simplifications of the solution which I have recently developed.

When I went to Princeton in 1936, Einstein told me that he had been working on this problem for 15 years. Indeed, its history dates back to the paper by Einstein and Grommer in which they showed that the equations of the geodesic line can be deduced from the field equations of Relativity Theory. Although today this can be shown more simply and more correctly mathematically, the idea of deducing the equations of motion from the field equations is a very important one.

Let us take as an example the Newtonian theory of gravitation. Here we have the Laplace or Poisson equations as the field equations. In addition we have the Newtonian equations of motion connecting the acceleration with the potential gradient. In electrodynamics there is a similar situation. Here we have Maxwell's equations which are the field equations, and Lorentz's equations which are the equations of motion. Finally there is a similar situation in Einstein theory of gravitation. Here we have the field equations for the gravitational field, and the equations of a geodesic line, that is the equations of motion for the test particle.

But the difference between Poisson's equations of classical mechanics and Maxwell's equations on the one hand, and the field equations of relativity theory on the other hand, is this: whereas the former are linear equations, the latter are non-linear. From linear equations we cannot deduce the equations of motion. But there is a possibility of deducing them from non-linear equations. Indeed, Einstein and Grommer showed that the equations for a test particle – that is, the equations of a geodesic line – can be deduced from the field equations.

What about the motion of two bodies – say a double star – if the masses of both are of the same order? To this problem the solution was suggested by Einstein in 1936.

Let us assume the field equations for empty space, and that matter is represented by singularities of such a field. EINSTEIN found the general form of the equations of motion for the s-th singularity by considering four vanishing surface integrals around this s-th singularity. These surface integrals were independent of the two-dimensional surface und they vanish because of the BIANCHI identities.

In 1938, by using these surface integrals and the proper approximation method, EINSTEIN, HOFFMANN and I obtained the equations of motion to an approximation one step beyond the Newtonian. (Here I should like to mention one of EINSTEIN'S mistakes, because EINSTEIN'S mistakes are more important and interesting than the virtues of many other men. Since we have four equations for the motion of each singularity to determine three space coordinates as functions of time, EINSTEIN thought that the fourth equation would restrict the motion and perhaps give a quantum condition. This proved to be wrong and the fourth equation followed, roughly speaking from the other three equations).

A year later, V. I. FOCK'S paper appeared, in which he found, independently, the Newtonian equations of motion from the field equations. Professor FOCK characterized the difference between his approach and ours in two ways: first he took a continuous distribution of matter and no singularities; second, he used the harmonic coordinate system.

As to the first difference: EINSTEIN very much disliked the illegal marriage between the artificial energy-momentum tensor of matter and the curvature tensor. This was why we preferred to consider empty space with matter as its singularities. Yet the mathematical theory would remain exactly the same if we assumed continuous distribution of matter. The surface integral would merely have to enclose the continuously distributed particle at a given moment.

As to the second difference: In our 1938 paper we used a different coordinate condition than FOCK'S, but from PETROVA'S calculations published eleven years later, it followed that in spite of the different coordinate conditions, the post-Newtonian approximation is exactly the same for coordinate conditions!

In 1938, I received a letter from EINSTEIN about the objections of the mathematician LEVINSON to the rigor of our general theory of motion. EINSTEIN found the objections valid and suggested that we remove them. Thus our work by correspondence began. While removing LEVINSON'S objections, we revised the whole theory. There was one new result which I should like to mention here. We found that the equations of motion up to the post-Newtonian approximation are independent of coordinate conditions. In other words: our approximation method sufficiently determines the coordinate conditions. Therefore, it is absolutely unessential whether our coordinate conditions used in 1938, or FOCK'S, or none at all are used, as long as the proper approximation method is used.

Now I should like to say a few words about a simplification, I found recently (1954). The method has something in common with the work of PAPAPETROU, although he, like FOCK, used continuous distribution and harmonic coordinate conditions. In our work of 1938 (with EINSTEIN and HOFFMANN) and 1949 (with EINSTEIN) we represented matter by singularities. This was equivalent, mathematically, to considering the energy momentum tensor $T^{\mu\nu}$ as linearly and homogeneously dependent on DIRAC'S δ function, vanishing everywhere outside the singularities. Thus we had for the field equations

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -8 \pi T_{\alpha\beta} .$$

Now instead of forming surface integrals from the left-hand expressions, let us take the right-hand side. Because of BIANCHI'S identities we have

$$T^{\alpha\beta}{}_{;\beta} = 0 \quad \text{(; means covariant differentiation).}$$

Thus

$$\int T^{\alpha\beta}{}_{;\beta} \, d_{(3)}x = 0$$

is the equation of motion of the s-th singularity if the region of integration surrounds the s-th singularity.

To indicate the simplification, let us write

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \underset{1}{h}_{\alpha\beta} + \underset{2}{h}_{\alpha\beta} \ldots$$

where $\eta_{\alpha\beta}$ are the MINKOWSKI values for the metric tensor and the numbers under the h's denote their order. To calculate the equations of motion by the new method requires knowledge of

$$\underset{1}{h}_{\alpha\beta} \quad \text{and} \quad \underset{2}{h}_{00} ,$$

whereas to calculate the equations of motion by the old method requires knowledge of

$$\underset{1}{h}_{\alpha\beta} \quad \text{and} \quad \underset{2}{h}_{\alpha\beta}$$

which means more than ten times as much work. The result in all cases is of course exactly the same.

Diskussion – Discussion

W. HEITLER: This is the only case in physics where the equations of motion follow from the field equations. Can one find a general criterion characterizing the type of field theory from which the equations of motions follow or do not follow?

L. INFELD: There is no such general criterion.

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