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# Quantum Theory of Wave Fields in a Curved Space

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*Abstract.* Wave fields are studied in a space-time continuum whose curvature is independent of the physical phenomena. It is shown that a space of constant curvature is the most natural choice for this purpose, not only because of its geometrical simplicity, but because it guarantees the maximum number of certain constants of motion and leads to a propagation formula whose kernel is only a function of the geodesic distance. Therefore a de Sitter space is investigated in detail. A complete set of solutions is discussed for the scalar wave equation and for the first order wave equation of Dirac. Hadamard's propagation formula is written in a particularly symmetric form with the help of a propagation function which is similar to the well known $D$-function in flat space. A generalization for spinors and electromagnetic fields is given. In the latter case Huygens' principle is shown to hold even in this space of constant curvature.

A second general propagation formula is established whose kernel is again related to Hadamard's elementary solution. But it leads now to a propagation function which is similar to the $D_1$-function in flat space. Using this new propagation formula, every solution of the homogeneous wave equation (with a mass term) can be split into a sum of two such solutions which are shown to belong to two distinct classes. This separation is uniquely determined and invariant with respect to the group of motions. Moreover in the case of a spinor field these two classes are transformed into each other by the operation of charge conjugation. Therefore they are interpreted as states of "positive energy" resp. "negative energy", although there does not exist in this space an operator like the Hamiltonian in flat space. Finally the various propagation functions are represented as sums over the complete sets of solutions which were mentioned previously.

As an example a process of second quantization is applied to a spinor field which is coupled to a pseudoscalar field. The method imitates the old non-relativistic procedure for a particular space-like surface, but the result is invariant and compatible with the field equations. If the coupling between pseudoscalar and spinor field vanishes, all the field operators can be explicitly stated in terms of the complete sets of solutions. This leads at once to general commutation rules using Hadamard's propagation formula. Moreover the vacuum can be defined in accordance with Dirac's hole theory. Therefore all the necessary elements are assembled for studying the various radiation effects in this more general scheme.

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## Introduction.

The theory of quantized fields is usually restricted to the assumptions which are made by the theory of special relativity in its classical form. These assumptions can be slightly generalized, without however changing in any way the conceptual basis of special relativity. It is indeed possible to replace the flat space-time continuum of special relativity by a curved continuum, provided the curvature does not depend on the physical phenomena which take place in it. Thus it will be assumed in this investigation that the metric tensor of the space-time continuum is known *a priori* in a particular system of coordinates. Contrary to general relativity there shall be no interaction between the geometry and the physical events. Therefore no argument will be used which would properly belong into general relativity.

Among the many possible space-time continua the spaces of constant curvature have received special attention from several investigators[1]). Their reason for doing so is not always quite clear except for the fact that a space of constant curvature has many simple geometric properties. However it is more satisfactory to use an argument of the following type: Unless a space does not possess a number of geometric properties (which will be enumerated in section 1), it does not constitute a proper basis for the description of physical phenomena in the sense of special relativity. It will be shown that only a space of constant curvature has the required properties. Moreover the theory of quantized fields will be discussed in detail for the special case of a de Sitter space and all the essential elements for such a theory will be assembled.

It is difficult to predict the advantages and the disadvantages of this theory compared to the ordinary theory of quantized fields in flat space. For instance it may be of some help to have a denumerable set of independent solutions for the wave equation, as opposed to the non-denumerable set in flat space. The use of a space of constant curvature can indeed be interpreted as a "quantization" of the momentum space, and it does not have the drawbacks of the "big but finite" box which is usually invoked in order to make the process of second quantization in flat space easier to visualize. As all the important $D$- and $S$-functions are explicitly constructed in this investigation, and as their connection with the solutions of the first and second order wave equation is shown to be the same as in flat space, all the formulas for the various radiation effects in flat space can at least be written down also in this space of constant curvature. But the numerical evaluation is much more complicated than in flat space, although the labor can be reduced by using the geometrical properties of the space, such as the group of motions.

The flat space does not seem to be simply the limiting case for a space of constant curvature whose radius of curvature increases indefinitely. This might be true locally, but it can hardly be so *in globo,* because the space which is studied in this paper has the topology of a cylinder, not that of a plane as the flat space. Moreover there are ten independent constants of motion in this space of constant curvature, but none of them or no combination can be compared to the Hamiltonian which plays such an outstanding role in flat space. Therefore the limit of vanishing curvature will not be discussed, although it seems to the author that the Compton wavelength of the elementary particles should be considered as very small compared to the radius of curvature of the space.

## 1. Some characteristic properties of a space of constant curvature.

The distance between two neighboring points is given by the known quadratic form

$$ds^2 = g_{ij}\, dx^i\, dx^j \tag{1.1}$$

(latin indices run from 0 to 3, the summation is made over the indices which occur twice), which has the signature

$$- + + + . \tag{1.2}$$

All the notations are chosen in accordance with EISENHART[2]). (E.g. covariant differentiation is denoted by a comma, etc.)

The condition for constant curvature

$$R_{hijk} = K_0(g_{hj}\, g_{ik} - g_{hk}\, g_{ij}) \tag{1.3}$$

can be obtained as a consequence of certain requirements. Thus it results if the Riemannian curvature at each point is required to be the same for every orientation (Theorem of Schur). Similarly equation (1.3) follows from the existence of a group of motions such that each point and a quadruple of directions in it can be transformed into any other point and an arbitrary quadruple of directions in it (Theorem of Bianchi). Two other arguments will now be considered which also lead to a space of constant curvature.

Suppose that it has been possible to define a symmetric tensor $T_{ij}$ in terms of some field quantities in such a way that

$$g^{jk}\, T_{ij,\,k} = 0 \tag{1.4}$$

as a consequence of the field equations. If $\Sigma$ and $\Sigma'$ are two arbitrary space-like surfaces which enclose the volume $V$, it follows from Gauss' theorem that

$$\int_{\Sigma} g^{jk} T_{ij} \, d\Sigma_k - \int_{\Sigma'} g^{jk} T_{ij} \, d\Sigma'_k = \frac{1}{2} \int_V \frac{\partial g_{jk}}{\partial x^i} T^{jk} \, dV. \tag{1.5}$$

The signs of $d\Sigma_k$ and $d\Sigma_k{}'$ are determined such that

$$\delta x^i \, d\Sigma_i > 0, \qquad \delta x^{i'} \, d\Sigma_i{}' > 0 \tag{1.6}$$

for time-like displacements $\delta x^i$ resp. $\delta x^{i'}$ which point toward the future. $\Sigma'$ is assumed to be in the past of $\Sigma$. Equation (1.4) implies that the integral

$$\int_{\Sigma} g^{jk} T_{ij} \, d\Sigma_k \tag{1.7}$$

is a constant of motion for the field quantities under consideration, provided the coordinate system is such that

$$\partial/\partial x^i (g_{jk}) = 0 \tag{1.8}$$

for the particular coordinate $x^i$ and for all indices $j$ and $k$. Equation (1.8) implies that $x^i$ is the parameter of a group of motions. Therefore the existence of the maximum number of ten independent constants of motions (1.7) follows from (1.4) provided the space is of constant curvature.

The work of HADAMARD on Cauchy's problem[3]), i.e. on the inhomogeneous wave equation

$$g^{ij} \psi_{,ij} - \varkappa^2 \psi = f(x), \tag{1.9}$$

is of fundamental importance for any field theory of elementary particles. HADAMARD's main result can be written in some spaces with the help of a Green's function $D(x, \xi)$ in the form

$$\psi(x) = -\int_V D(x,\xi) f(\xi) \, dV + \int_{\Sigma} \left( \frac{\partial D(x,\xi)}{\partial \xi^i} \psi(\xi) - D(x,\xi) \frac{\partial \psi}{\partial \xi^i} \right) d\Sigma^i. \tag{1.10}$$

$\Sigma$ is an arbitrary space-like surface and $V$ is the volume between the point $x$ and the surface $\Sigma$. The sign of $d\Sigma$ is determined according to the convention (1.6). $D(x, \xi)$ vanishes outside the light cone of $x$. Its behavior inside the light cone of $x$ is intimately connected with HADAMARD's elementary solution of the homogeneous wave equation [i.e. (1.9) with $f(x) = 0$], and it is dictated only by the geometry of the space and the value of the mass constant $\varkappa^2$ in (1.9). It seems to the author that $D(x, \xi)$ is purely a function of the geodesic distance between the point $x$ and the point $\xi$, if and only if the space is of constant curvature.

These characteristic properties of a space of constant curvature are just what we would postulate for a curved space whose geometry is independent of its physical content. It would indeed be hard to understand the lack of symmetry which is inherent to all spaces of non-constant curvature, without assuming some interaction between geometry and physical phenomena. On the other hand there seems to be no a priori reason which would exclude the spaces of nonvanishing constant curvature from further consideration. Therefore it was thought worthwhile to examine in more detail such a space.

## 2. The de Sitter space.

In assuming (1.3) a new "constant of nature", namely the radius of curvature of the space, is introduced into the theory. It is therefore convenient to chose such units as to make this radius equal to one unity of length. Moreover the time scale and the mass scale are determined by putting equal to one the velocity of light and Planck's constant divided by $2\pi$. Then all physical quantities are expressed in natural units.

A space of constant curvature can be imbedded in a five-dimensional flat space[4]). There are only two cases in accordance with the signature (1.2). In this investigation only the case is studied which leads to space-like geodesics of finite length and time-like geodesics of infinite length. This space can be most easily described by the Weierstrassian coordinates $z^\alpha$ with $\alpha^\alpha = 0, 1, \ldots, 4$. Equation (1.1) becomes

$$ds^2 = c_{\alpha\beta}\, dz^\alpha\, dz^\beta \tag{2.1}$$

with

$$c_{\alpha\beta}\, z^\alpha\, z^\beta = 1 \tag{2.2}$$

and

$$c_{00} = -1,\ c_{11} = c_{22} = c_{33} = c_{44} = 1,\ c_{\alpha\beta} = 0 \text{ for } \alpha \neq \beta. \tag{2.3}$$

(Greek indices run from 0 to 4.) Moreover the curvature $K_0$ in (1.3) is uniquely determined as

$$K_0 = 1. \tag{2.4}$$

The geodesic distance $s$ between a point $P = (z^0, \ldots, z^4)$ and a point $Q = (\zeta^0, \ldots, \zeta^4)$ is given by

$$c_{\alpha\beta}\, z^\alpha\, \zeta^\beta = \begin{matrix} \cos s & \text{for space-like connection,} \\ \cosh s & \text{for time-like connection.} \end{matrix} \tag{2.5}$$

The group of motions is the group of linear transformations

$$\bar{z}^\beta = a^\beta_\alpha z^\alpha, \tag{2.6}$$

such that

$$c_{\alpha\beta} \bar{z}^\alpha \bar{z}^\beta = c_{\alpha\beta} z^\alpha z^\beta. \tag{2.7}$$

However the invariance of the physical laws will be postulated only with respect to the subgroup which does not reverse the time axis.

Applying the method of COURANT and HILBERT or of RIESZ[5]), equation (1.10) can be derived with $D(x, \xi)$ given by

$$2\pi \cdot \varepsilon(z^0 - \zeta^0) \cdot D(x, \xi) = \delta(\sinh^2 s) + \\ + 1/2\,(1 - \varkappa^2/2)\, F(3/2 + i\alpha,\, 3/2 - i\alpha,\, 2;\, -\sinh^2 s/2), \tag{2.8}$$

where

$$i\alpha = (9/4 - \varkappa^2)^{1/2} \tag{2.9}$$

and

$$\varepsilon(z^0 - \zeta^0) = \begin{matrix} +1 \text{ for } z^0 > \zeta^0, \\ -1 \text{ for } z^0 < \zeta^0. \end{matrix} \tag{2.10}$$

The usual notation for the hypergeometric function is used. As $\varkappa$ is the reciprocal Compton wavelength of a particle, one has in most cases

$$|\varkappa| \gg 1. \tag{2.11}$$

Therefore if $\varkappa$ is real, $\alpha$ is real too and $D(x, \xi)$ is a real function of the geodesic distance $s$. But it will be seen in section 5 that $\alpha$ has a small imaginary addition in the case of a spinor particle. $D(x, \xi)$ is then a complex valued function of the geodesic distance.

The transition from Weierstrassian coordinates to an ordinary coordinate system can be made if the $z^\alpha$ are known functions of $x^0$, $x^1$, $x^2$, $x^3$ in accordance with condition (2.2). It follows then from (1.1) and (2.1) that

$$g_{ij} = c_{\alpha\beta} z^\alpha{}_{,i} z^\beta{}_{,j}, \tag{2.12}$$

where

$$z^\alpha{}_{,i} = \partial/\partial x^i (z^\alpha).$$

For instance insert into (2.12) the expressions

$$z^0 = \sinh x^0,\; z^\alpha = \cosh x^0 \cdot f^\alpha(x^1, x^2, x^3) \text{ for } \alpha \neq 0. \tag{2.13}$$

It follows that

$$g_{00} = -1,\; g_{i0} = 0,\; g_{ij} = \cosh^2 x^0\, \bar{g}_{ij} \text{ for } i, j \neq 0 \tag{2.14}$$

with

$$\bar{g}_{ij} = \sum_{\alpha=1}^{4} (\partial/\partial x^i f^\alpha)\, (\partial/\partial x^j f^\alpha). \tag{2.15}$$

The coordinates $x^1$, $x^2$ $x^3$ describe the four-dimensional unit sphere and $\bar{g}_{ij}$ is the metric tensor for the unit sphere.

### 3. The solutions of the scalar wave equation.

The homogeneous wave equation can be separated in the coordinate system (2.14). Writing

$$\psi(x) = y_m(x^0) \cdot Y^\mu_m(x^1, x^2, x^3), \tag{3.1}$$

$Y^\mu_m$ is found to be an eigenfunction of the Laplacian

$$\Delta = \bar{g}^{-1/2} \sum_{i,j=1}^{3} \partial/\partial x^i (\bar{g}^{1/2} \bar{g}^{ij} \partial/\partial x^j) \tag{3.2}$$

on the unit sphere. Therefore $Y^\mu_m$ is a generalized harmonic and can be treated as the ordinary spherical harmonics[6]). It follows in particular that

$$\Delta Y^\mu_m = -m(m+2)\, Y^\mu_m. \tag{3.3}$$

The $(m+1)^2$ eigenfunctions are orthonormalized by

$$\int Y^{\mu'}_m Y^{\mu''*}_m \bar{g}^{1/2} d^3x = \delta_{\mu' \mu''}, \tag{3.4}$$

where the star denotes the complex conjugate function. This leads to the addition theorem[7])

$$V_m(\cos \sigma) = \sum_\mu Y^\mu_m(x^1, x^2, x^3)\; Y^{\mu*}_m(\xi^1, \xi^2, \xi^3) \tag{3.5}$$

with

$$V_m(\cos \sigma) = (m+1) \sin(m+1)\sigma \cdot (2\pi^2 \sin \sigma)^{-1},$$

where the geodesic distance $\sigma$ on the sphere is given by

$$\cos \sigma = \sum_{\alpha=1}^{4} f^\alpha(x^1, x^2, x^3)\; f^\alpha(\xi^1, \xi^2, \xi^3). \tag{3.6}$$

The equation for $y_m(x^0)$ becomes with (3.1) and (3.3)

$$\frac{1}{\cosh^3 x^0} \frac{d}{dx^0}\left(\cosh^3 x^0 \frac{dy_m}{dx^0}\right) + \frac{m(m+2)}{\cosh^2 x^0} y_m + \varkappa^2 y_m = 0. \tag{3.7}$$

$(m+1)$ can be compared with the absolute value of the momentum in view of (3.3) and also because there are $(m+1)^2$ solutions in the range $(m - \frac{1}{2}, m + \frac{1}{2})$ of $m$. Moreover (3.7) shows that $y_m$ oscillates with circular frequency $(\varkappa^2 + (m+1)^2)^{1/2}$ for $|x^0| \ll 1$, provided (2.11) holds. An expansion in terms of increasing $m$ corresponds therefore to an expansion with respect to increasing momentum.

With $x = i \sinh x^0$ the function $y_m(x^0)$ can be written in terms of generalized Legendre functions, namely

$$y_m(x^0) = (x^2-1)^{-1/2} P^{m+1}_{i\alpha-1/2}(x) \quad \text{or} \quad (x^2-1)^{-1/2} Q^{m+1}_{i\alpha-1/2}(x) \tag{3.8}$$

*

in the notation of HOBSON[8]). Another pair of solutions can be written in terms of hypergeometric functions

$$\begin{aligned} g_m(x^0) &= \cosh^m x^0 \cdot F\big(1/2\,(m+3/2+i\alpha),\ 1/2(m+3/2-i\alpha),\ 1/2; \\ &\quad -\sinh^2 x^0\big)\,, \\ h_m(x^0) &= \cosh^m x^0 \cdot \sinh x^0 \cdot F\ \big(1/2(m+5/2+i\alpha),\ 1/2(m+5/2-i\alpha), \\ &\quad 3/2; -\sinh^2 x^0\big). \end{aligned} \tag{3.9}$$

$g_m$ and $h_m$ have simple initial values for $x^0 = 0$.

In order to study the asymptotic behavior of $y_m(x^0)$ for large values of $x^0$, define moreover

$$\begin{aligned} \Gamma(1+i\alpha)\, g_m^{(\pm)}(x^0) &= \Gamma(m+3/2+i\alpha) \cdot \cosh^m x^0 \cdot e^{\pm(m+3/2+i\alpha)\,x^0} \\ &\quad \cdot F(m+3/2,\ m+3/2+i\alpha,\ 1+i\alpha;\ -e^{\mp 2x^0}), \end{aligned} \tag{3.10}$$

$$\begin{aligned} \Gamma(-m-1/2+i\alpha)\, h_m^{(\pm)}(x^0) &= \Gamma(i\alpha) \cosh^m x^0\, e^{\mp(m+3/2-i\alpha)\,x^0} \,\cdot \\ &\quad \cdot F(m+3/2,\ m+3/2-i\alpha,\ 1-i\alpha;\ -e^{\mp 2x^0}), \end{aligned} \tag{3.10}$$

which are again solutions of (3.7). At the same time they yield asymptotic expansions for $x^0 \gg +1$ (upper sign) and for $x^0 \ll -1$ (lower sign). The linear transformation which transforms the pair $(g_m^{(+)}, h_m^{(+)})$ into the pair $(g_m^{(-)}, h_m^{(-)})$ is given by

$$(\sin i\alpha\pi)^{-1} \cdot \begin{pmatrix} (-1)^m & \sin i\alpha\pi \\ -\cos i\alpha\pi \cdot \operatorname{cotg} i\alpha\pi & (-1)^{m+1} \end{pmatrix}. \tag{3.11}$$

In view of the simple exponential behavior of the solutions (3.10) for $x^0 \gg +1$ resp. for $x^0 \gg -1$, we could have hoped to find a simpler connection between the remote future and the remote past. In particular this might have yielded a convenient way of defining solutions of positive resp. negative "frequency". But the matrix (3.11) shows that this is not feasible. A quite different method will therefore be used in section 8 to bring about such a distinction which is of prime importance in order to apply Dirac's hole theory.

### 4. The electromagnetic field.

Maxwell's equations for the skew symmetric tensor $F_{ij}$ of the field strengths are written as usual

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0\,, \tag{4.1}$$

$$g^{ij} F_{kj,i} = J_k\,, \tag{4.2}$$

where $J_k$ is the external electric current. (4.1) guarantees the existence of a vector field $A_i$, such that

$$F_{ij} = A_{i,j} - A_{j,i}\,, \tag{4.3}$$

where $A_i$ is determined by $F_{ij}$ up to the gradient of an arbitrary scalar field. Equation (4.2) demands the continuity equation

$$g^{jk} J_{j,k} = 0. \tag{4.4}$$

Inserting (4.3) into (4.2) it follows with (1.3) and (2.4) that

$$g^{ij} A_{k,ij} - 3 A_k - g^{ij} A_{i,jk} = J_k. \tag{4.5}$$

With the help of Ricci's identities and (4.4) it follows that

$$g^{hk}(g^{ij} A_{i,j}),_{hk} = 0. \tag{4.6}$$

Therefore the Lorentz condition

$$g^{ij} A_{i,j} = 0 \tag{4.7}$$

can be considered as an initial condition rather than an identity to be satisfied by the vector field $A_i$. With (4.7) the field equations (4.5) for $A_i$ are reduced to

$$g^{ij} A_{k,ij} - 3 A_k = J_k. \tag{4.8}$$

But these equations are not suitable for computations because there does not seem to exist a coordinate system in which each potential $A_k$ appears in exactly one equation.

Such a separation of the components of the vector field $A_k$ can be achieved as follows. Define

$$B^\alpha = z^\alpha,_i \, g^{ij} A_j \tag{4.9}$$

with the help of (2.12). This gives the identity

$$\Box B^\alpha = z^\alpha,_k (g^{ij} A^k,_{ij} - 3 A^k) - 2 z^\alpha g^{ij} A_{i,j}, \tag{4.10}$$

where the d'Alembertian $\Box$ is defined by

$$\Box = (-g)^{-1/2} \partial/\partial x^i (-g)^{1/2} g^{ij} \partial/\partial x^j - 2. \tag{4.11}$$

Moreover it follows from (4.9) and (2.2) that

$$c_{\alpha\beta} z^\alpha B^\beta = 0. \tag{4.12}$$

There exists a one to one correspondence between the four potentials $A_i$ which satisfy (4.7) and (4.8), and the five potentials $B^\alpha$ which satisfy (4.12) and

$$\Box B^\alpha = z^\alpha,_k \, g^{kj} J_j. \tag{4.13}$$

It is possible to find a complete set of solutions of (4.12) and (4.13) if $J_k = 0$. This set is similar to the solutions of the homogeneous wave equation in section 3.

Each one among the equations (4.13) has the form of equation (1.9) with $\varkappa^2 = 2$. Equation (2.8) shows that $D(x, \xi)$ has only the $\delta$-like singularity on the light cone in this case, and vanishes everywhere inside the light cone. This is exactly the behavior of the $D$-function for a wave field of vanishing mass in flat space, and it can be interpreted as the validity of Huygens' principle for the electromagnetic field in the space of constant curvature. The similarity of the operator (4.11) with the d'Alembertian in flat space can be recognized[9]), if the following coordinate system is used

$$g_{ij} = \nu^{-2} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ with } \nu = 1 + 1/4\,(-\,x_0^2 + x_1^2 + x_2^2 + x_3^2),$$

so that it follows according to (4.11) that

$$\square = \nu^3(-\,\partial^2/\partial x_0^2 + \partial^2/\partial x_1^2 + \partial^2/x_2^2 + \partial^2/\partial x_3^2)\,\nu^{-1}.$$

## 5. The spinor field.

According to Dirac's method for treating the spinor field in a de Sitter space[10]), the Weierstrassian coordinates are interpreted as cartesian coordinates in a five-dimensional flat space and the space of constant curvature is given by (2.2). In order to make the five coordinates $z^\alpha$ more symmetric, introduce

$$y_0 = i\,z^0,\; y_\alpha = z^\alpha \qquad \text{for} \qquad \alpha \neq 0. \tag{5.1}$$

Moreover the field quantities are written as homogeneous functions of $y_\alpha$. The differentiations with respect to $y_\alpha$ occur only in the combination

$$m_{\alpha\beta} = y_\alpha\,\partial/\partial y_\beta - y_\beta\,\partial/\partial y_\alpha, \tag{5.2}$$

which is compatible with the condition (2.2) or

$$y_\alpha\, y_\alpha = 1\,. \tag{5.3}$$

The wave operator (4.11) can be written as

$$\square + 2 = \sum_{\alpha<\beta} (m_{\alpha\beta})^2. \tag{5.4}$$

With a set of five Hermitian 4 by 4 matrices $\gamma_\alpha$ such that

$$\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha = 2\,\delta_{\alpha\beta}\,I \tag{5.5}$$

and

$$\gamma_0\,\gamma_1\,\gamma_2\,\gamma_3\,\gamma_4 = I \text{ (unit-matrix)}, \tag{5.6}$$

a first order differential operator $M$ is defined by

$$M = \sum_{\alpha<\beta} \gamma_\alpha \gamma_\beta m_{\alpha\beta}, \tag{5.7}$$

which has the property

$$(M - 3/2)^2 = - \Box + 1/4. \tag{5.8}$$

The 4 by 4 matrices act on a spinor $\psi$ (1-column matrix) and it is useful to define an adjoint spinor $\psi^+$ (1-row matrix) by

$$\psi^+ = i\psi^* \gamma_0, \tag{5.9}$$

where $\psi^*$ is the 1-row matrix whose elements are complex conjugate to the elements of $\psi$. A current vector $J_k$ can be derived from the five components

$$j_\alpha = \varphi^+ \gamma_\alpha \psi - y_\alpha y_\beta \varphi^+ \gamma_\beta \psi, \tag{5.10}$$

where $\varphi$ and $\psi$ are two arbitrary spinors. The relation between $J_k$ and $j_\alpha$ is the same as between $A_k$ and $iB^0$, $B^1$, ..., $B^4$ using (4.9). This leads to the identity

$$g^{mn} J_{m,n} = \varphi^+ \gamma (\vec{M} - 2) \psi - \varphi^+ (\overleftarrow{M} + 2) \gamma \psi, \tag{5.11}$$

where the arrow indicates the spinor upon which the differential operator $M$ acts, and where $\gamma$ is given by

$$\gamma = \gamma_\alpha y_\alpha . \tag{5.12}$$

With a linear transformation (2.6) of the coordinates, the spinors undergo a linear transformation $\Lambda$, namely

$$\bar{\psi} = \Lambda \psi, \ \bar{\psi}^+ = \psi^+ \Lambda^{-1}. \tag{5.13}$$

The five components $j_\alpha$ transform like the coordinates and the operator $M$ undergoes the transformation

$$\bar{M} = \Lambda M \Lambda^{-1}. \tag{5.14}$$

For the reflection of the spatial coordinate $y_\alpha$ ($\alpha \neq 0$) we have

$$\Lambda = \gamma_\alpha . \tag{5.15}$$

The first order equation of Dirac for the free spinor is given by

$$\begin{aligned} (M - 2 + ia)\psi &= 0, \\ \psi^+(M + 2 + ia) &= 0, \end{aligned} \tag{5.16}$$

where $a$ is real and can be chosen positive because

$$\gamma(M - 2) = -(M - 2)\gamma . \tag{5.17}$$

It follows from (5.8) and (5.16) that

$$\Box \psi - (a^2 + i a) \psi = 0 . \tag{5.18}$$

On the other hand if $\chi$ is a spinor whose components satisfy the second order wave equation (5.18), then the spinor

$$\psi = (M - 1 - i a) \chi \tag{5.19}$$

satisfies the first order wave equation (5.16) because of (5.8). Moreover the divergence (5.11) vanishes for two arbitrary solutions $\varphi$ and $\psi$ of (5.16). Therefore the integral

$$\int_{\Sigma} \varphi^+ \gamma_\alpha \psi \, y_{\alpha, i} \, d\Sigma^i \tag{5.20}$$

is independent of the particular space-like surface $\Sigma$. Thus (5.20) is an invariant scalar product for two solutions $\varphi$ and $\psi$ of (5.16) with a positive definite value if $\varphi = \psi$.

## 6. The solutions of Dirac's equation.

A complete set of solutions for Dirac's equation can be constructed with the help of an operator $N$ which is given by (6.1) and is related to the absolute value of the momentum (cf. the discussion after (3.7) concerning the index $m$ in the solutions of the scalar wave equation). It will be convenient to use such a set of solutions in order to represent various propagation functions. The operator

$$N = \gamma_0 \left( \sum_{\alpha < \beta}^{1 \dots 4} \gamma_\alpha \gamma_\beta m_{\alpha \beta} - 3/2 \right) \tag{6.1}$$

is Hermitian for spinors with the norm (5.20), where the space-like surface $\Sigma$ is given by $y_0 =$ const. or also $\varrho =$ const. with

$$\varrho = (y_1^2 + y_2^2 + y_3^2 + y_4^2)^{1/2}. \tag{6.2}$$

The same is true for the operators

$$M_{\alpha \beta} = i (m_{\alpha \beta} + 1/2 \, \gamma_\alpha \gamma_\beta) \quad \text{for} \quad \alpha, \beta \neq 0 . \tag{6.3}$$

The variables $y_0$ and $\varrho$ can be written as

$$y_0 = i \sinh x^0, \ \varrho = \cosh x^0 \tag{6.4}$$

according to (2.13). Finally define an operator $\eta$ by

$$\varrho \eta = i \gamma_0 \sum_{\alpha = 1}^{4} \gamma_\alpha y_\alpha . \tag{6.5}$$

The following relations are easily proved

$$\begin{aligned}[N, \gamma_0] &= [N, \eta] = 0, \\ [N, \varrho] &= [N, \partial/\partial \varrho] = 0;\end{aligned} \tag{6.6}$$

$$\begin{aligned}[M_{\varkappa\lambda}, \gamma_0] &= [M_{\varkappa\lambda}, \eta] = 0, \\ [M_{\varkappa\lambda}, \varrho] &= [M_{\varkappa\lambda}, \partial/\partial \varrho] = 0;\end{aligned} \tag{6.7}$$

$$i[M_{\varkappa\lambda}, M_{\mu\nu}] = \partial_{\varkappa\mu} M_{\lambda\nu} + \partial_{\lambda\nu} M_{\varkappa\mu} - \partial_{\varkappa\nu} M_{\lambda\mu} - \partial_{\lambda\mu} M_{\varkappa\nu}; \tag{6.8}$$

$$[M_{\varkappa\lambda}, N] = 0; \tag{6.9}$$

where always $\varkappa, \lambda, \mu, \nu \neq 0$ and [,] stands for the commutator. Moreover

$$\begin{aligned}[\gamma_0, \varrho] &= [\gamma_0, \partial/\partial \varrho] = 0, \\ [\eta, \varrho] &= [\eta, \partial/\partial \varrho] = 0;\end{aligned} \tag{6.10}$$

$$\eta\gamma_0 + \gamma_0\eta = 0, \; \eta^2 = \gamma_0^2 = I. \tag{6.11}$$

With (5.7) it follows that

$$M = \eta(\partial/\partial x^0 + 3/2 \tanh x^0) + \gamma_0(1 - \eta \tanh x^0) N + 3/2, \tag{6.12}$$

and equation (5.16) becomes therefore

$$\left\{\partial/\partial x^0 + 3/2 \tanh x^0 + (\eta + \tanh x^0)\gamma_0 N + i\left(a + \frac{i}{2}\right)\eta\right\}\psi = 0. \tag{6.13}$$

If the spinor $\psi$ is a solution of (5.16) and belongs to the eigenvalue $n$ of $N$ for a particular space-like surface $x^0 =$ const., i. e. if for a particular value of $x^0$ we have

$$N\psi = n\psi, \tag{6.14}$$

then (6.14) holds for all values of $x^0$ because of (6.13) and (6.6). The same is true for $M_{\varkappa\lambda}$ because of (6.7) and (6.9). In view of (6.8) it is therefore possible to find solutions of (5.16) which are simultaneous eigenfunctions of the operators $N$, $M_{12}$, and $M_{34}$. The eigenvalues of these operators can be derived by v. d. Waerden's method[11]) and with the help of the formula

$$1 + (M_{12} \pm M_{34})^2 + (M_{23} \pm M_{14})^2 + (M_{31} \pm M_{24})^2 = (N \pm 1/2)^2. \tag{6.15}$$

The eigenvalues of $N$ are then found to be the positive and negative halfintegers except $+ 1/2$ and $- 1/2$.

Equation (6.14) can easily be discussed in the special representation of the $\gamma$-matrices given by

$$\gamma_0 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, \quad \gamma_k = \begin{pmatrix} 0 & i\sigma_k \\ -i\sigma_k & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{6.16}$$

where $I$, $\sigma_1$, $\sigma_2$, $\sigma_3$ are the usual 2 by 2 spin matrices. With

$$\begin{aligned} \varrho\,\sigma &= \sigma_1 y_1 + \sigma_2 y_2 + \sigma_3 y_3 + i\,y_4 I, \\ \varrho\,\sigma^+ &= \sigma_1 y_1 + \sigma_2 y_2 + \sigma_3 y_3 - i\,y_4 I, \end{aligned} \tag{6.17}$$

it follows that

$$\eta = \begin{pmatrix} 0 & \sigma^+ \\ \sigma & 0 \end{pmatrix}. \tag{6.18}$$

Equation (6.14) splits into two independent couples of first order equations in the coordinates of the four-dimensional sphere $\varrho =$ const. Therefore (6.14) has two types of solutions: $\psi'$ has vanishing third and fourth components, $\psi''$ has vanishing first and second components. A correspondence

$$\psi'' = \eta\,\psi', \quad \psi' = \eta\,\psi'' \tag{6.19}$$

can be established between $\psi'$ and $\psi''$, which preserves the normalization and the eigenvalues of $N$ and $M_{\varkappa\lambda}$ because of (6.6) and (6.7). Only $\psi'$ has to be discussed therefore. The 2 non-vanishing components of $\psi'$ satisfy the equation

$$\Delta\,\psi' = -\left((n - 1/2)^2 - 1\right)\psi', \tag{6.20}$$

which is identical with (3.3) if $(m+1)^2 = (n - 1/2)^2$ with $m \geqq 0$. Moreover it can be shown that there is exactly one eigenfunction $\psi'$ for every possible set of simultaneous eigenvalues $N$, $M_{12}$, and $M_{34}$, and these eigenfunctions form a complete set for the spinors of the type $\psi'$ for a particular value $x^0 =$ const.

An arbitrary spinor $\psi$ which belongs to the eigenvalue $n$ of $N$, can now be written as

$$\psi = \varphi'(x^0)\cdot\psi'(x^1, x^2, x^3) + \varphi''(x^0)\cdot\psi''(x^1, x^2, x^3), \tag{6.21}$$

where the functions $\varphi'(x^0)$ and $\varphi''(x^0)$ are determined by

$$\left.\begin{aligned} \left(\partial/\partial x^0 - (n - 3/2)\tanh x^0\right)\varphi' + \left(n + i(a + i/2)\right)\varphi'' &= 0, \\ \left(\partial/\partial x^0 + (n + 3/2)\tanh x^0\right)\varphi'' + \left(-n + i(a + i/2)\right)\varphi' &= 0. \end{aligned}\right\} \tag{6.22}$$

After eliminating $\varphi''$, it is found that $\varphi'$ satisfies (3.7) with $(m+1)^2 = (n - 1/2)^2$. $\varphi''$ satisfies (3.7) with $(m+1)^2 = (n + 1/2)^2$. In both cases $\varkappa^2$ is replaced by $(a + i/2)^2 + 9/4$. The initial values of $\varphi'$ and $\varphi''$ can be chosen arbitrarily.

### 7. The spinor field in arbitrary coordinates.

For some applications it is more convenient to use the spinor formalism in arbitrary coordinates which has been developped by several authors[12]). It is sufficient to list the results for the space of constant curvature. Define in terms of the matrices $\gamma_\mu$ and $\gamma$ in (5.5) and (5.12)

$$\begin{aligned} \bar{\alpha}_\mu &= (i\gamma_\mu - i\gamma\, y_\mu - y_\mu)\gamma, \\ \bar{\Lambda}_\mu &= 1/2(y_\mu - \gamma_\mu \gamma), \\ A &= \gamma_0 \gamma; \end{aligned} \tag{7.1}$$

$$\begin{aligned} \alpha_j &= y_{\mu,j}\, \bar{\alpha}_\mu, \\ \alpha &= y_\mu\, \bar{\alpha}_\mu = -\gamma, \\ \Lambda_j &= y_{\mu,j}, \bar{\Lambda}_\mu, \end{aligned} \tag{7.2}$$

from which follows that

$$A\,\alpha_j \text{ and } A\,\alpha \text{ are Hermitian,} \tag{7.3}$$

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\, g_{ij}\, I, \quad \alpha_i \alpha + \alpha \alpha_i = 0, \quad \alpha^2 = I. \tag{7.4}$$

The covariant derivatives of a spinor $\Psi$ (1-column matrix) and for a spinor $\Phi$ (1-row matrix) are given by

$$\Psi,_i = \partial/\partial_i \Psi + \Lambda_i \Psi; \; \Phi,_i = \partial/\partial x^i\, \Phi - \Phi \Lambda_i. \tag{7.5}$$

A spinor $\Psi^+$ of the type $\Phi$ can be defined from $\Psi$ by

$$\Psi^+ = \Psi^* A, \tag{7.6}$$

where $\Psi^*$ is the 1-row matrix whose elements are complex conjugate to $\Psi$. The covariant derivatives of $\alpha_i$, $\alpha$, and $A$ are defined by

$$\begin{aligned} \alpha_{i,j} &\equiv \partial/\partial x^j\, \alpha_i - \Gamma^h_{ij}\alpha_h + \Lambda_j \alpha_i - \alpha_i \Lambda_j = 0, \\ \alpha,_j &\equiv \partial/\partial x^j\, \alpha + \Lambda_j \alpha - \alpha \Lambda_j = 0, \\ A,_j &\equiv \partial/\partial x^j A - A\Lambda_j - \Lambda_j^+ A = 0, \end{aligned} \tag{7.7}$$

where $\Lambda_j^+$ is the Hermitian conjugate matrix of $\Lambda_j$. The interchange of covariant differentiation on a spinor gives the following simple result with (1.3) and (2.4)

$$\Psi,_{ij} - \Psi,_{ji} = P_{ij}\Psi; \; \Phi,_{ij} - \Phi,_{ji} = -\Phi P_{ij} \tag{7.8}$$

with

$$P_{ij} = 1/4\,(\alpha_j \alpha_i - \alpha_i \alpha_j).$$

The $\overline{\alpha}_\mu$ in (7.1) has been chosen such as to change equation (5.16) into

$$\alpha^j \Psi_{,j} + a\Psi = 0;\ \Psi^+{}_{,j}\alpha^j - a\Psi^+ = 0 \qquad (7.9)$$

with $\alpha^j = g^{ji}\alpha_i$. This gives at once (4.4) with

$$J_k = \Psi^+ \alpha_k \Psi. \qquad (7.10)$$

Equation (7.9) follows from the Lagrangian

$$L = \Psi^+ \alpha^k \Psi_{,k} + a\Psi^+\Psi. \qquad (7.11)$$

The energy-momentum tensor

$$T_{ij} = i/4(\Psi^+\alpha_i\Psi_{,j} + \Psi^+\alpha_j\Psi_{,i} - \Psi^+{}_{,i}\alpha_j\Psi - \Psi^+{}_{,j}\alpha_i\Psi) \qquad (7.12)$$

satisfies (1.4) by virtue of (7.8) and (7.9).

The propagation formula (1.10) can be applied to each component of a spinor with the help of (5.18). The term with $\Psi_{,j}\,d\Sigma^j$ can be transformed because of (7.9). It follows that

$$\Psi(x) = -\int K(x,\xi)\,\alpha_j(\xi)\,\Psi(\xi)\,d\Sigma^j, \qquad (7.13)$$

with

$$K(x,\xi) = -\alpha^k(\xi)\,\partial/\partial\xi^k\,D(x,\xi) - (a-i)\,D(x,\xi), \qquad (7.14)$$

or

$$K(x,\xi) = -i(M_y - 1 - i\,a)\,D(x,\xi), \qquad (7.15)$$

if the coordinates $x^i$ and $\xi^i$ are replaced by $y_\alpha$ and $\eta_\alpha$ according to (5.1). $D(x,\xi)$ here is given by (2.8) with a complex mass term

$$\alpha = a + i/2 \qquad (7.16)$$

according to (2.9), (5.18), (4.11), and (1.9).

## 8. The second propagation formula.

A solution $\varphi(s)$ of the homogeneous wave equation which depends only on the geodesic distance $s$ to a fixed point, has to satisfy

$$\ddot\varphi + 3\coth s\,\dot\varphi + \varkappa^2\varphi = 0 \qquad (8.1\,\mathrm{a})$$

for time-like connection,

$$\ddot\varphi + 3\,\mathrm{cotg}\,s\,\dot\varphi - \varkappa^2\varphi = 0 \qquad (8.1\,\mathrm{b})$$

for space-like connection,

where the dot indicates the differentiation with respect to $s$. In terms of Legendre functions the solutions are

$$(\sinh s)^{-1} P^1_{i\alpha-\frac{1}{2}}(\cosh s),\ (\sinh s)^{-1} Q^1_{i\alpha-\frac{1}{2}}(\cosh s), \tag{8.2a}$$

resp.

$$(\sin s)^{-1} P^1_{i\alpha-\frac{1}{2}}(\cos s),\ (\sin s)^{-1} Q^1_{i\alpha-\frac{1}{2}}(\cos s)\,. \tag{8.2b}$$

A propagation function $D_1(x, \xi)$ can be constructed from (8.2) which is regular for fixed $x$ and varying $\xi$ except on the light cone of $x$, and the leading term at the light cone is $(2\pi^2 s^2)^{-1}$ for space-like connection and $-(2\pi^2 s^2)^{-1}$ for time-like connection. $D_1(x, \xi)$ is given by

$$(2\pi^2 \sinh s)^{-1}\{Q^1_{i\alpha-\frac{1}{2}}(\cosh s) + \pi/2 \operatorname{tang} i\alpha\pi \cdot P^1_{i\alpha-\frac{1}{2}}(\cosh s)\}, \tag{8.3a}$$

resp.

$$-(2\pi^2 \sin s)^{-1}\{Q^1_{i\alpha-\frac{1}{2}}(\cos s) + \pi/2 \operatorname{tang} i\alpha\pi \cdot P^1_{i\alpha-\frac{1}{2}}(\cos s)\}. \tag{8.3b}$$

For points $\xi$ which cannot be connected with the point $x$ by a geodesic, $D_1(x, \xi)$ can be continued without singularities and still be a solution of the homogeneous wave equation.

Consider now a volume $V$ which is contained between two space-like surfaces $\Sigma'$ and $\Sigma''$ in the past of the point $P = (x^0, x^1, x^2, x^3)$. Outside the light cone $H$ of $P$ a cone $H'$ is chosen which is generated by geodesics through $P$. A similar cone $H''$ is chosen inside $H$. $V$ is defined by the space between $\Sigma'$ and $\Sigma''$ except for the space between $H'$ and $H''$. Let $S$ be the surface of $V$; $S$ consists of parts which belong to $\Sigma'$, $\Sigma''$, $H'$, and $H''$. With an arbitrary function $\psi$ and with $f$ defined by (1.9) it follows from Green's formula that

$$\int_V D_1(x, \xi)\, f(\xi)\, dV = \int_S \left\{D_1(x, \xi) \frac{\partial \psi}{\partial \xi^j} - \frac{\partial D_1(x, \xi)}{\partial \xi^j} \psi(\xi)\right\} dS^j. \tag{8.4}$$

The sign of $dS^j$ is determined such that

$$\delta x^j\, dS_j > 0 \tag{8.5}$$

for an arbitrary displacement $\delta x^j$ pointing out of $V$.

The left hand side of (8.4) has a well defined limit as $H'$ and $H''$ approach $H$, provided that for the intersections $Q$, $Q'$, and $Q''$ of a space-like curve with $H$, $H'$, and $H''$ we have

$$\lim \frac{\text{distance } QQ'}{\text{distance } QQ''} = 1\,. \tag{8.6}$$

The contributions to the right-hand side of (8.4) which come from $\Sigma'$, $\Sigma''$, $H'$, and $H''$ do not tend to a finite limit separately. However the contributions from $H'$ and $H''$ can be integrated exactly between $\Sigma'$ and $\Sigma''$, and the result of this integration just cancels the terms in the contributions from $\Sigma'$ and $\Sigma''$ which do not tend to a finite limit. Therefore equation (8.4) can now be written with the convention (1.6) as

$$\mathfrak{F}\int\limits_{\dot{\Sigma}'}\left\{D_1(x,\xi)\,\frac{\partial\psi}{\partial\xi^j}-\frac{\partial D_1(x,\xi)}{\partial\xi^j}\,\psi(\xi)\right\}d\Sigma^{j\prime}=$$

$$=\mathfrak{F}\int\limits_{\dot{\Sigma}''}\left\{D_1(x,\xi)\,\frac{\partial\psi}{\partial\xi^j}-\frac{\partial D_1(x,\xi)}{\partial\xi^j}\,\psi(\xi)\right\}d\Sigma^{j\prime\prime}+\mathfrak{P}\int\limits_{\dot{V}}D_1(x,\xi)\,f(\xi)\,dV. \quad (8.7)$$

The symbols $\mathfrak{F}$ and $\mathfrak{P}$ indicate the limiting process which has to be used in order to make each term well defined. $\mathfrak{P}$ is a Cauchy principal value connected to the condition (8.6). In each of the two surface integrals $\mathfrak{F}$ means that the integrand has to be expanded about the intersection of $\Sigma$ with $H$ in powers of the distance perpendicular to this intersection and only those terms have to be retained which give a finite contribution to the surface integral in the sense of a Cauchy principal value with condition (8.6). With these definitions for $\mathfrak{P}$ and $\mathfrak{F}$ equation (8.7) holds even if $P$ has an arbitrary position with respect to $\Sigma'$ and $\Sigma''$, provided $P$ does not lie in $\Sigma'$ or $\Sigma''$.

As a consequence of (8.7) an arbitrary solution $\psi$ of the homogeneous wave equation has a unique adjoint function $\bar{\psi}$ which is given by

$$\bar{\psi}(x)=\mathfrak{F}\int\limits_{\dot{\Sigma}}\left\{\frac{\partial D_1(x,\xi)}{\partial\xi^j}\,\psi(\xi)-D_1(x,\xi)\,\frac{\partial\psi}{\partial\xi^j}\right\}d\Sigma^j. \quad (8.8)$$

The transition from $\psi$ to $\bar{\psi}$ is invariant with respect to the group of motions which was defined in section 2. Moreover $\bar{\psi}(x)$ satisfies the homogeneous wave equation. If the correspondence (8.8) is symbolically represented by $T$, it will be shown that

$$T^2=-E, \quad (8.9)$$

where $E$ is the identity. Therefore each solution $\psi(x)$ of the homogeneous wave equation can be uniquely written as the sum of two solutions $\psi^{(+)}(x)$ and $\psi^{(-)}(x)$, i. e.

$$\psi(x)=\psi^{(+)}(x)+\psi^{(-)}(x), \quad (8.10)$$

where

$$\psi^{(+)}=1/2(E+iT)\,\psi,\ \psi^{(-)}=1/2(E-iT)\,\psi.$$

In the case of a spinor $\psi$ which satisfies equation (5.16), the integral

(8.8) can be transformed in a way similar to (7.13) with the help of (7.9). This gives

$$\overline{\psi}(x) = -\int_{\Sigma} K_1(x,\xi)\,\alpha_j(\xi)\,\psi(\xi)\,d\Sigma^j \tag{8.11}$$

with

$$K_1(x,\xi) = -\,i(M_y - 1 - i\,a)\,D_1(x,\xi)\,.$$

Because of (5.19), $\overline{\psi}(x)$ is again a solution of Dirac's equation (5.16).

To prove (8.9) it is sufficient to show its validity for the complete set of solutions of the homogeneous wave equation which was discussed in section 3. The transforms $\overline{\psi}$ for this set follow from writing (8.3b) as

$$D_1(x,\xi) = \sum_{m \geqslant 0} c_m\,V_m(\cos s) \tag{8.12}$$

with

$$c_m\,d_m = -1$$

and

$$d_m = -\pi^{-2}\cos i\alpha\pi\,\Gamma\big(1/2(m+5/2+i\alpha)\big)\,\Gamma\big(1/2(m+5/2-i\alpha)\big)\times$$
$$\times\,\Gamma\big(1/2(-m+1/2+i\alpha)\big)\,\Gamma\big(1/2(-m+1/2-i\alpha)\big).$$

Formula (8.12) is a consequence of the addition theorem for generalized Legendre functions[13]). The transforms $\overline{\psi}$ are

$$\begin{array}{lll} d_m\,h_m(x^0)\,Y_m^\mu & \text{for} & g_m(x^0)\,Y_m^\mu\,, \\ c_m\,g_m(x^0)\,Y_m^\mu & \text{for} & h_m(x^0)\,Y_m^\mu\,, \end{array} \tag{8.13}$$

and the relation (8.9) follows immediately.

$D(x,\xi)$ and $D_1(x,\xi)$ can be written as

$$D(x,\xi) = \sum_m \big(h_m(x^0)\,g_m(\xi^0) - g_m(x^0)\,h_m(\xi^0)\big)\,V_m(\cos\sigma)\,, \tag{8.14}$$

$$D_1(x,\xi) = \sum_m \big(c_m\,g_m(x^0)\,g_m(\xi^0) - d_m\,h_m(x^0)\,h_m(\xi^0)\big)\,V_m(\cos\sigma). \tag{8.15}$$

These formulas can be proved by inserting them into (1.10) and (8.8) with (3.5) for a space-like surface $x^0 = \text{const}$. Another $D$-function is defined by

$$2\,D_2(x,\xi) = \varepsilon(x^0 - \xi^0)\,D(x,\xi) + iD_1(x,\xi)\,. \tag{8.16}$$

It has the representation

$$2\,D_2(x,\xi) =$$
$$= \sum_m i\,c_m\big(g_m(x^0) \pm i\,d_m\,h_m(x^0)\big)\big(g_m(\xi^0) \mp i\,d_m\,h_m(\xi^0)\big)\,V_m(\cos s)\,, \tag{8.17}$$

with the upper sign for $x^0 > \xi^0$ and the lower sign for $x^0 < \xi^0$.

### 9. The positive and negative energy states of a spinor.

Let ~ indicate the operation of taking the complex conjugate (not the Hermitian conjugate). A matrix $C$ can be defined[14]) (with $C\tilde{C}$ = unit matrix) such that the spinor $\hat{\psi}(x)$ which is given by

$$\hat{\psi}(x) = \tilde{C}\,\tilde{\psi}(x)\,, \tag{9.1}$$

is a solution of (5.16), provided $\psi(x)$ is a solution of (5.16). In the set (6.16) of $\gamma$-matrices $C$ is given by

$$C = \gamma_2 \gamma_4 \gamma\,. \tag{9.2}$$

The correspondence (9.1) between $\psi$ and $\hat{\psi}$ is invariant under the group of motions which was defined in section 2, and it does not change the current vector (5.10) or (7.10). Let this correspondence be represented symbolically by $S$.

The correspondence $T$, i.e. (8.8), and the correspondence $S$, i.e. (9.1), are connected for an arbitrary solution $\psi$ of Dirac's equation (5.16) by the relations

$$\begin{aligned} (E + i\,T)\,S(E + i\,T)\,\psi &= 0\,, \\ (E - i\,T)\,S(E - i\,T)\,\psi &= 0\,, \end{aligned} \tag{9.3}$$

or $S\psi^{(+)}$ resp. $S\psi^{(-)}$ are of the type $\psi^{(-)}$ resp. $\psi^{(+)}$.

The relations (9.3) are easily reduced to

$$T\,S\,\psi = S\,T\,\psi. \tag{9.4}$$

This last equation can be proved separately for each solution of the complete set in section 6. Moreover it is sufficient to show (9.4) for a particular space-like surface, e.g. the surface $x^0 = 0$, because both spinors $TS\psi$ and $ST\psi$ are solutions of the first order wave equation (5.16). These two spinors are easily computed for each eigenvalue $n$ in the set (6.16) of $\gamma$-matrices with the help of (6.13) and (8.13). They are found to be equal, provided the following recursion formula is true

$$c_{m+1}\,(i\,a - m - 2) = \bar{c}_m(i\,a - m - 1) \tag{9.5}$$

for the coefficient $c_m$ in (8.13). Equation (9.5) follows indeed from (8.12), if $c_m$ is defined with the complex mass term $\alpha = a + i/2$.

Equation (8.10) shows explicitly how to split an arbitrary spinor into $\psi^{(+)}$ and $\psi^{(-)}$, and it is now legitimate to interpret $\psi^{(+)}$ as a "positive energy" state and $\psi^{(-)}$ as a "negative energy" state. $\psi^{(+)}$ is orthogonal to $\psi^{(-)}$ in the normalization (5.20). The spinors in the complete set of section 6 can therefore be uniquely determined by four labels, namely the eigenvalues of the four operators $t = iT$, $N$, $M_{12}$, and $M_{34}$. The functions $\varphi'(x^0)$ and $\varphi''(x^0)$ in (6.21) can indeed be worked out for $t = +1$ and $t = -1$.

The matrices $K(x, \xi)$ and $K_1(x, \xi)$ in (7.13) and (8.11) can be represented as sums over the complete set of spinors $\psi_\omega$, where $\omega$ stands for a set of simultaneous eigenvalues of $t$, $N$, $M_{12}$, and $M_{34}$. It is found that

$$K(x,\xi) = \sum_\omega \psi_\omega(x)\, \psi_\omega^+(\xi)\,,$$

$$K_1(x,\xi) = -i \sum_\omega t\, \psi_\omega(x)\, \psi_\omega^+(\xi),$$

$$K_2(x,\xi) = \begin{cases} \sum\limits_{t>0} \psi_\omega(x)\, \psi_\omega^+(\xi) & \text{for} \quad x^0 > \xi^0, \\ -\sum\limits_{t<0} \psi_\omega(x)\, \psi_\omega^+(\xi) & \text{for} \quad x^0 < \xi^0. \end{cases} \tag{9.6}$$

The proof follows from inserting these formulas into (7.13) and (8.11) with $\Sigma$ given by $x^0 = \text{const}$. The similarity with the $S$-functions in flat space is obvious.

## 10. Example of second quantization.

Consider a spinor field $\Psi$ which is coupled to a real pseudoscalar field $\varphi$ by a pseudoscalar coupling. The Lagrangian of the system is given by

$$L = i(\Psi^+ \alpha^k \Psi_{,k} + \alpha\, \Psi^+ \Psi) + 1/2(g^{jk} \varphi_{,j} \varphi_{,k} + \varkappa^2 \varphi^2) + k \varphi\, \Psi^+ \alpha\, \Psi. \tag{10.1}$$

The field equations are

$$\begin{aligned} \alpha^k \Psi_{,k} + a\,\Psi - i k \varphi\, \alpha\, \Psi &= 0\,, \\ \Psi^+{}_{,k}\, \alpha^k - a\,\Psi + i k \varphi\, \Psi^+ \alpha &= 0\,, \end{aligned} \tag{10.2}$$

$$g^{jk} \varphi_{,jk} - \varkappa^2 \varphi - k\, \Psi^+ \alpha\, \Psi = 0\,. \tag{10.3}$$

From these field equations follow the equation of continuity (4.4) with (7.10) and the conservation law (1.4) with the energy-momentum tensor

$$\begin{aligned} T_{jk} = {} & i/4(\Psi^+ \alpha_j \Psi_{,k} + \Psi^+ \alpha_k \Psi_{,j} - \Psi^+{}_{,k} \alpha_j \Psi - \Psi^+{}_{,j} \alpha_k \Psi) + \\ & + \varphi_{,j} \varphi_{,k} - 1/2\, g_{jk} (g^{mn} \varphi_{,m} \varphi_{,n} + \varkappa^2 \varphi^2). \end{aligned} \tag{10.4}$$

Equations (10.2) and (10.3) can be written as integral equations with the help of Hadamard's formula (1.10), namely

$$\Psi(x) = -i k \int\limits_V K(x, \xi)\, \alpha(\xi)\, \Psi(\xi)\, \varphi(\xi)\, dV - \\ - \int\limits_\Sigma K(x, \xi)\, \alpha^j(\xi)\, \Psi(\xi)\, d\Sigma_j, \tag{10.5}$$

$$\varphi(x) = -k \int\limits_V D(x, \xi)\, \Psi^+(\xi)\, \alpha(\xi)\, \Psi(\xi)\, dV + \\ + \int\limits_\Sigma \left\{ \frac{\partial D(x,\xi)}{\partial \xi^j}\, \varphi(\xi) - D(x, \xi)\, \frac{\partial \varphi}{\partial \xi^j} \right\} d\Sigma^j. \tag{10.6}$$

$K(x, \xi)$ is the same as for the free spinors, whereas $D(x, \xi)$ is given by (2.8) with the real mass term $\varkappa^2$.

The transition from a $c$-number theory to a $q$-number theory will first be made on a particular space-like surface $\Sigma$ which is described by three parameters $v^1$, $v^2$, $v^3$. On $\Sigma$ a vector field $\tau_i(v^1, v^2, v^3)$ can be defined by

$$d\Sigma_i = \tau_i(v^1, v^2, v^3)\, dv^1\, dv^2\, dv^3 \tag{10.7}$$

with the convention (1.6). It simplifies the writing in the forthcoming derivation if this $\Sigma$ is assumed to be imbedded in a continuous sequence of space-like surfaces. Each surface in this sequence is labeled by a parameter $u^0$, and the points in each surface are labeled by parameters $u^1$, $u^2$, $u^3$ in such a way, that the curves $u^1 =$ const., $u^2 =$ const., $u^3 =$ const. are orthogonal to the surfaces $u^0 =$ const. The parameters $u^0$, $u^1$, $u^2$, $u^3$ are used as new coordinates and the new metric tensor has the property

$$g_{j0} = g_{0j} = 0 \quad \text{for} \quad j = 1, 2, 3. \tag{10.8}$$

Two auxiliary fields are defined by

$$\Phi(u) = (-g)^{1/2} \frac{\partial L}{\partial \left( \dfrac{\partial \Psi}{\partial u^0} \right)} = i (-g)^{1/2}\, \Psi^+(u)\, \alpha^0, \tag{10.9}$$

$$\pi(u) = (-g)^{1/2} \frac{\partial L}{\partial \left( \dfrac{\partial \varphi}{\partial u^0} \right)} = (-g)^{1/2}\, g^{00}\, \frac{\partial \varphi}{\partial u^0}. \tag{10.10}$$

The field equations can now be written as

$$\partial/\partial u^0 \Phi = -\sum_1^3 \partial/\partial u^k (\Phi \alpha_0 \alpha^k) + \Phi \alpha_0 \Big(\sum_0^3 \alpha^j \Lambda_j - ik\alpha\varphi + a\Big), \quad (10.11)$$

$$\partial/\partial u^0 \pi = -\sum_1^3 \partial/\partial u^j \big((-g)^{1/2} g^{jk} \cdot \partial/\partial u^k \varphi\big) + + (-g)^{1/2} (\varkappa^2 \varphi + k \Psi^+ \alpha \Psi). \quad (10.12)$$

Moreover a sort of Hamiltonian $H_\Sigma$ can be defined on $\Sigma$ by

$$H_\Sigma = \int_\Sigma \mathfrak{H}(u)\, du^1\, du^2\, du^3 \quad (10.13)$$

with

$$\mathfrak{H}(u) = (-g)^{1/2} L - \Phi \cdot \partial/\partial u^0 \Psi - \pi \cdot \partial/\partial u^0 \varphi.$$

This definition applies to each surface in an arbitrary sequence of space-like surfaces. This is important because $H_\Sigma$ will be used later to show the compatibility of the commutation rules with the field equations. On the other hand $H_\Sigma$ is not in general a constant of motion, except e.g. in the following special case. Let the sequence of space-like surfaces be generated by a 1-parameter group of motions in such a way that the curves perpendicular to the surfaces are the trajectories of the motion. The metric tensor does not depend on the coordinate $u^0$. Moreover after a proper choice of the coordinates in the spin space, covariant differentiation and ordinary differentiation with respect to $u^0$ become identical. Then $H_\Sigma$ turns out to be the same as the constant of motion (1.7) with $i = 0$ and (10.4). However compared to the cartesian coordinates in flat space this special coordinate system has the disadvantage that it is not regular everywhere and the surface $\Sigma$ does not sweep over all points in the space. A similar situation arises in flat space if one choses a coordinate system whose time-like coordinate is the parameter of a hyperbolic rotation (restricted Lorentz transformation). Therefore a general coordinate system will be used henceforth.

The components of the spinors $\Phi(u)$ and $\Psi(u)$, and the pseudoscalar fields $\varphi(u)$ and $\pi(u)$ are now considered as operators which satisfy on a fixed space-like surface $u^0 = \text{const.}$ the (anti)commutation rules

$$\begin{aligned} \{\Phi_a(u), \Psi_b(u')\} &= -i\delta_{ab}\, \delta(u-u'), \\ \{\Phi_a(u), \Phi_b(u')\} &= \{\Psi_a(u), \Psi_b(u')\} = 0; \end{aligned} \quad (10.14)$$

$$\begin{aligned} [\pi(u), \varphi(u')] &= i\,\delta(u-u'), \\ [\pi(u), \pi(u')] &= [\varphi(u), \varphi(u')] = 0; \end{aligned} \quad (10.15)$$

$\Phi_a(u)$ and $\Psi_b(u)$ commute with $\varphi(u)$ and $\pi(u)$ on $u^0 = \text{const.}$ As

*

usual $\{,\}$ stands for the anticommutator and $[,]$ for the commutator. $\delta(u - u')$ ist the triple $\delta$-function for the coordinates $u^1$, $u^2$, $u^3$ on the space-like surface $u^0 = \text{const}$.

The question arises whether the commutation rules (10.14) and (10.15) are compatible with the field equations. The propagation formulas (10.5) and (10.6) show indeed how to compute the field operators for the whole space, if they are given on a particular space-like surface. On the other hand the commutation rules (10.14) and (10.15) can be postulated equally well on any of the surfaces $u^0 = \text{const}$. The two procedures are consistent with each other, if it follows from the field equations and the commutation rules on a particular surface $u^0 = \text{const.}$, that the derivatives with respect to $u^0$ of the (anti)commutators (10.14) and (10.15) vanish. It is then indeed legitimate to put these (anti)commutators equal to a $c$-number independent of $u^0$. Therefore consider for instance the derivative

$$\partial/\partial u^0 \{\Phi_a(u), \Psi_b(u')\}. \tag{10.16}$$

It follows from (10.9), (10.10), (10.11), and (10.12) in the usual manner with the help of (10.14) and (10.15) that

$$\begin{aligned} \partial/\partial u^0 \Phi &= i[H_\Sigma, \Phi(u)],\ \partial/\partial u^0 \Psi = i[H_\Sigma, \Psi(u)];\\ \partial/\partial u^0 \pi &= i[H_\Sigma, \pi(u)],\ \partial/\partial u^0 \varphi = i[H_\Sigma, \varphi(u)]. \end{aligned} \tag{10.17}$$

Thus the expression (10.16) becomes

$$i\{[H_\Sigma, \Phi_a(u)], \Psi_b(u')\} + i\{\Phi_a(u), [H_\Sigma, \Psi_b(u')]\},$$

and this is written using Jacobi's identity as

$$i[H_\Sigma, \{\Phi_a(u), \Psi_b(u')\}].$$

But this last commutator vanishes, because the anticommutator $\{\Phi_a(u), \Psi_b(u')\}$ is a $c$-number.

The (anti)commutators in (10.14) and (10.15) can be written without the help of the special coordinate system (10.8). It follows from (10.7) that for instance

$$\{\Psi_a(v), \Phi_b(v')\} = -i\,\delta(v - v')\,\delta_{ab}, \tag{10.18}$$

$$[\pi(v), \varphi(v')] = i\,\delta(v - v'), \tag{10.19}$$

with

$$\Phi = i\,\Psi^+(\alpha^i \tau_i) \qquad \text{and} \qquad \pi = \varphi_{,j}\, g^{jk}\, \tau_k,$$

which is obviously independent of the particular coordinate system.

It is easy to construct operators which satisfy the postulated commutation rules on a particular space-like surface, e.g. $x^0 = 0$ in the coordinate system (2.13). For this purpose the complete sets of solutions which were discussed in section 3 and section 6 can be used exactly as complete sets of plane waves are used in flat space. But it is not necessary here to make an assumption such as the "big but finite" box in order to make these sets denumerable. If the coupling between pseudoscalar and spinor field vanishes, i. e. if $k = 0$, such a representation is valid throughout the whole space. Moreover the commutation rules can now easily be deduced for two arbitrary points $x$ and $\xi$ in the space. The propagation formulas (10.5) and (10.6) with $k = 0$ reduce indeed every operator to its values on a particular space-like surface $\Sigma$ through the point $\xi$, so that (10.14) and (10.15) can be applied. This gives

$$\begin{aligned} \{\Psi_a(x), \Psi_b(\xi)\} &= K_{ab}(x, \xi)\,, \\ i[\varphi(x), \varphi(\xi)] &= D(x, \xi)\,, \end{aligned} \qquad (10.20)$$

and all the other (anti)commutators vanish. Finally the distinction between "positive energy" and "negative energy" states of section 9 can be used to define the vacuum according to Dirac's hole theory. The interpretation of the various field operators in terms of creation and annihilation will thus be the same as in flat space. Therefore all the necessary elements have been assembled from which to compute the effects of coupling between the spinor field and the pseudoscalar field using the same methods as in flat space.

These methods use expansions of the $D$-functions which are similar to (8.14) etc. The integrations over the coordinates can then be performed and one is left with a summation over the parameter of the expansion. This summation has a very intuitive interpretation in terms of intermediate states and virtual processes among them. The difficulty in applying this method to the present case consists in performing the integration over the coordinates. Indeed the solutions of section 3 and 6 do not depend on the space and time coordinates in such a simple manner as the plane waves of flat space. In spite of these mathematical difficulties it may be of some interest to investigate the interaction between quantized wave fields in this more general theory.

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