Zeitschrift: Helvetica Physica Acta
Band: 26 (1953)
Heft: I

Artikel: On the divergence of perturbation theory for quantized fields
Autor: Thirring, Walter
DOI: https://doi.org/10.5169/seals-112398

On the Divergence of Perturbation Theory for Quantized Fields

by **Walter Thirring.**
Swiss Federal Institute of Technology.

(12. X. 1952.)

Summary. The convergence of perturbation theory is investigated for a scalar field with an interaction term $\lambda\,\psi^3$. It is shown that for a certain energy region the series diverges for all values of λ.

1. Introduction.

The most important progress in field theory in the last years was that it was possible for certain theories to absorb the infinities occuring in the perturbation theory in renormalization constants. The possibility of making a theory finite by renormalization stems from the fact that no new types of infinities appear as one proceeds to higher approximations. Thus the infinities can be absorbed by a finite number of compensating terms in the Lagrangian.

A renormalizable theory would provide a complete mathematical scheme, if the power series in the coupling constant of perturbation theory converged after renormalization. On the other hand, if the series proves to be divergent, there seems to be no significance in the fact that higher orders do not produce new infinities.

Until some time ago it was the general believe that at least for small values for the coupling constant the series converges. This would suffice, as this function of the coupling constant could be extended by analytic continuation to any desired value for the coupling constant. Just recently Dyson found a physical argument which makes it plausible that there is no radius of convergence for perturbation theory in quantum electrodynamics. Yet no mathematical analysis of this problem has been given so far. The expressions of higher order terms in perturbation theory become so complicated that nobody has been able to show whether terms of arbitrary high order are actually small. The purpose of the present paper is to study this problem for a simplified model[1]). It seems reasonable to restrict oneself to scalar fields as it does not seem likely that the question of convergence is tied up with the transformation property of the field[2]).

This leads him to consider two scalar fields Φ and ψ which are coupled by an interaction term $\lambda\, \Phi\, \psi^2$, being linear in one field and bilinear in the other. A closer investigation of this theory shows that as far as convergence is concerned this theory behaves like a scalar field with a non-linear term $\lambda\, \psi^3$.

It can be seen easily that this leads to a renormalizable theory containing a renormalization of mass and field strength. The result of the following analysis will be that the series in powers of λ for the propagation function of the nonlinear field diverges within the energy region where no new particles can be created. Thus it is essential that there are no particles with vanishing mass. The special form of the interaction term does not seem to be very important as long as it is hermitian[3]).

The proof of the divergence rests essentially on the fact that certain terms always have the same sign as long as the interaction term is hermitian. The same property is used in Schwinger's proof that the charge renormalization always decreases the charge. This seems to indicate that the divergence of pertubation theory is a deep lying feature of field theories which cannot be removed by mixing several fields.

The question whether solutions exist other than expansions in powers of the coupling constant will not be considered in this paper. Preliminary investigations on this point seem to indicate that not

---

[1]) It was observed independently by Dr. C. A. Hurst and by the author that it is possible to determine the convergence behaviour of perturbation theory for such a model. The work of Hurst is contained in his (unpublished) thesis Cambridge, January 1952, a paper of his is to appear in the October 1952 issue of the Proc. Camb. Soc. The author announced his results at the Copenhagen conference, June 1952. It turned out that our results parallelled each other, though both works were incomplete in some respect. A mistake in the (unpublished) work of the author was pointed out to him by Dr. G. Källén. The work of Hurst does not deal with renormalization and contains a mistake in sign (in formular 23). But the shortcomings of our works did not overlap and in view of the importance of the subject the author thought it worth to give a complete proof. The present analysis uses in one point an idea of Hurst and is a simplification over both original works.

[2]) One may be inclined to simplify the problem by reducing the number of the four dimensions and investigate a one-dimensional problem. But for this case one can easily construct a convergent theory. The divergent momentum integrals seem to be connected with the 4-dimensional structure of space-time.

[3]) One could carry out the following investigations along the same lines with an interaction term $\lambda\, \psi^4$ though this would be somewhat more involved.

only a power series in powers of λ but a power series in powers of $\lambda - \lambda_0$ diverges as well. This would mean that if there are solutions at all they must be non analytic for all real values of λ. There exists, of course, the possibility that for certain values for λ solutions do exist even for the unrenomalized equations. But it seems very unlikely that for $e^2/4\pi = 1/137$ there exists a solution for quantum electrodynamics. Particles of spin $\frac{1}{2}$ and spin 0 have the same charge and it is hard to believe that these two different theories have solutions for the very same coupling constant.

2. Renormalization of the theory.

At first we shall briefly outline our notation. We use real world coordinates with the metric $g_{00} = 1$, $g_{11} = g_{22} = g_{33} = -1$. The vector product of 4 vectors shall be written in the simple way $pk = p_0 k_0 - p_1 k_1 - p_2 k_2 - p_3 k_3$. The operator P effects the chronological ordering of the following factors. We further use Wicks symbols for the ordered product: ...: where the expression within the double columns has to be ordered in such a way that the positive frequency parts of all operators stand to the right hand side of the negative frequency parts. For the Dalembertian we write $\Box^2 = \partial_i \partial^i$. For the other symbols we use the standardised notations.

We shall now consider a scalar theory characterised by the Lagrangian

$$L(x) = \frac{1}{2} : \partial_i \psi \partial^i \psi - m^2 \psi^2 : \tag{2,1}$$

from which emerge the field equations and the commutation relations

$$(\Box^2 + m^2)\, \psi = 0 \qquad [\psi(x), \psi(x')] = -i \Delta (x - x') . \tag{2,2}$$

The only other expression we need for the following investigation is the vacuum expectation value for chronologically ordered operators

$$P(0 \,|\, \psi(x)\, \psi(x') \,|\, 0) = -i \int dp \, \frac{e^{-ip(x-x')}}{m^2 - p^2} . \tag{2,3}$$

Now we introduce an interaction term

$$L^{\text{int}}(x) = \lambda : \psi^3(x) : \qquad (2,4)^{1)}$$

which we shall treat as a perturbation in an interaction representa-

[1]) Note that: ψ^3: is hermitian. If one takes ψ^3 as interaction term one gets spurious renormalizations owing to the nonvanishing vacuum expectation value of the interaction term. One would obtain graphs of the type |—o which give an additional mass renormalization.

tion. λ is the coupling constant which has a dimension of a reciprocal length. On calculating the S-matrix with the interaction term (1,4)

$$S' = 1 + i\lambda \int dx_1 \colon \psi^3(1) \colon + \frac{1}{2!} (i\lambda)^2 P \int dx_1\, dx_2 \colon \psi^3(1) \colon\colon \psi^3(2) \colon + \cdots$$
$$\cdots + \frac{1}{n!} (i\lambda)^n P \int dx_1 \ldots dx_n \colon \psi^3(1) \colon \ldots \colon \psi^3(n) \colon \qquad (2,5)$$

one gets graphs where every vertex carries three lines quite similar to quantum electrodynamics. The differences between the graphs in this theory and quantum electrodynamics are:

a) some graphs are identical in this theory while graphs of the same topological structure are different in quantum electrodynamics because we have just one type of field.

b) For all internal lines we have to insert the propagation factor $-i/m^2 - p^2$.

c) There are no factors steming from the vertices nor are there any trace calculations.

One can now easily investigate in which graphs diverging integrals occur over energy momentum vectors of virtual particles. Following the analysis which Dyson has carried out in quantum electrodynamics we can find the primitive divergent graphs simply by comparing the powers of the energy momentum vectors in the numerator and denominator of the expression for the graph. Because from each vertex there emerge three lines, the number of external lines l_e and the number of internal lines l_i are related by

$$2\, l_i + l_e = 3\, n\,. \qquad (2,6)$$

Each internal line contributes a denominator with the squared energy momentum vector to the graph. The number of energy momentum vectors over which one has to integrate is equal to $l_i - n + 1$. The difference K between the powers of denominator and numerator is therefore $K = 2\, l_i - 4\, (l_i - n + 1)$. If we use (2,6) this becomes $K = n + l_e - 4$. In the following table we put K in dependence of n and l_e.

n \ l_e	0	1	2
2	−2	−1	0
3	−1	0	
4	0		

K is less than 1 only for those values of n and l_e indicated by the heavy line. The graphs with $l_e = 0$ sum up to a common phase factor giving the probability that vacuum remains vacuum. They shall not be considered further. The vacuum expectation value of ψ in the HEISENBERG representation does not vanish in this theory and, therefore, these are non-vanishing graphs with just one external line. The graph $n = 2$ $l_e = 1$ does not exist but only the graph $n = 3$ $l_e = 1$ . The last diverging graph is the graph $l_e = 2$ $n = 2$  which corresponds to the selfenergy graph in quantum electrodynamics.

It seems to be interesting to consider a regularized theory where we replace the propagation factor by $1/(m^2 - p^2)^2$. In this case $K = 4\,(n - 1)$ and, therefore, there are no divergent graphs. Yet it turns out that even for a propagation factor $1/(m^2-p^2)^t$, t being an arbitrary integer, the series of perturbation theory diverges. Therefore, in a theory with a form factor which changes the propagation function only in a mild way of the above form, there is but little hope for convergence of perturbation theory.

Our theory differs from quantum electrodynamics in as far as n does not disappear in the expression for K. In our theory types of graphs which are divergent in lower orders become convergent in higher orders[1]).

Though these graphs converge they contain finite renormalizations which have to be separated out as well.

We determine the renormalization constants by the condition that higher order corrections vanish in the non-relativistic limit. This is fulfilled if we cast the propagation function in the general form

$$\Delta'_{F^1} = \frac{-i}{m^2 - p^2}\left[1 + C(m^2 - p^2)\right] \quad \text{with} \quad C(o) = 0\,. \qquad (2{,}7)$$

We next consider sub-graphs which are only connected by one line with the main graph. As we have seen the lowest order graph of this type is divergent. It turns out that inserting these sub-graphs in any line of a graph amounts merely to a mass renormalization. On account of energy momentum conservation the line joining

[1]) This seems to be connected with the fact that the coupling constant has the dimension of a reciprocal length. In renormalizable theories where the coupling constant is dimensionless, n cancels out in the expression for K. Theories where the coupling constant has the dimension of a length are not renormalizable and in this case n remains with the negative sign in K. Thus our theory may be called super-renormalizable.

the sub-graph and the main graph has zero energy momentum vector. One can see this by the following simple example:

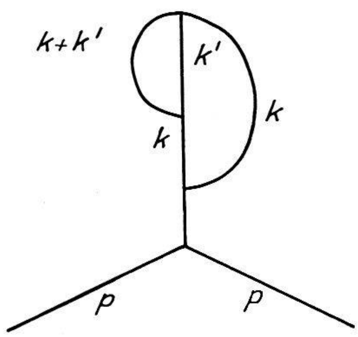

Fig. 1.

The effect of inserting this sub-graph is doubling the propagation factor $-i/m^2-p^2$ and multiplying the whole expression by

$$D_3=\frac{-i}{m^2}\int dk\,dk'\left(\frac{-i}{m^2-k^2}\right)^2\frac{-i}{m^2-k'^2}\;\frac{-i}{m^2-(k+k')^2}\,.$$

The effect of this insertion is exactly compensated by adding a term $D_3 : \psi^2 :$ to the LAGRANGIAN.

There is no finite correction left over after renormalizing this graph. The same holds, of course, if the sub-graph is of a more complicated structure. A closer consideration shows that all graphs which contain sub-graphs, joined to the main graph by only one line, are exactly cancelled if one adds a term $D : \psi^2 :$ to the LAGRANGIAN. D is the sum of all graphs containing one external line only. Henceforth we shall disregard graphs containing this type of sub-graphs.

The renormalization of self-energy graphs can be carried out in the same way as in quantum electrodynamics. We call a self-energy part proper, if it cannot be divided into two parts by cutting one line. Then the propagation function Δ_F' for the non-linear field which contains the effects of all self-energy graphs can be expressed by

$$\Delta_F'=\Delta_F+\Delta_F\,\Sigma\,\Delta_F' \tag{2,8}$$

where Σ is the contribution from all proper self-energy graphs. In momentum space Σ depends only on the invariant p^2 and we expand Σ in powers p^2-m^2. Then the above expression becomes

$$\Delta_F'(p^2)=\Delta_F(p^2)+\Delta_F(p^2)\left[\Sigma(m^2)-(m^2-p^2)\,\Sigma'(m^2)+\right.$$
$$\left.+(m^2-p^2)^2\,R(p^2)\right]\Delta_F'(p^2)\,.$$

The first term can be compensated by a mass renormalization[1]) term in the LAGRANGIAN $\Sigma(m^2): \psi^2:$. The second term has to be compensated by an addition

$$\Sigma'(m^2): \partial_i \psi^i \psi - m^2 \psi^2: \tag{2.9}$$

to the LAGRANGIAN. It represents a renormalization of the field strength[2]). After separating out these two terms the remaining expression Δ_F' is of the desired form (2,7).

Renormalization works here in a similar though in a somewhat simpler way as in quantum electrodynamics. In the next section we shall investigate the expression for Σ more closely. Perturbation theory gives an expansion for $\Sigma(p^2)$ and our aim will be to show that the series diverges for a certain domain of p^2.

3. General expressions for Δ_F'.

We now consider the contribution of a self-energy graph of n^{th} order to Δ_F' —— $\boxed{\Sigma_n}$ —— .

We forget at the moment about renormalization. Then the contribution will be

$$\frac{-i}{m^2-p^2}\,\Sigma_n(p^2)\,\frac{-i}{m^2-p^2}\,.$$

Σ_n consists of $3\,n/2 - 1$ internal lines each contributing a factor $-i/m^2 - k^2$, if k is the energy momentum vector of the corresponding line. There are $n/2$ energy momentum vectors over which one has to integrate, the others are fixed by conservation laws. For carrying out the integration over $n/2$ k's we combine the denominators in the expression for Σ_n with the identity

$$\frac{1}{a_0^{r_0} a_1^{r_1} \dots a_n^{r_n}} = \frac{({}_i\Sigma r_i - 1)!}{\Pi_i (r_i - 1)!} \int_0^1 dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{n-1}} dx_n \times$$

$$\times \frac{(1-x_1)^{r_0-1}(x_1-x_2)^{r_1-1}\dots(x_n)^{r_n-1}}{[a_0 + x_1(a_1-a_0) + \dots + x_n(a_n - a_{n-1})]^{\Sigma_i r_i}}\,. \tag{3,1}$$

We shall use the abbreviated notation $\int dx$ for the integral over

[1]) The compensating terms given here are not quite correct. This will not be analysed, however, as it has no influence on the results of this paper.

[2]) There is a formal complication as this term in the Lagrangian yields normal-dependent terms in the Hamiltonian. This point will not be discussed here as it has no bearing on our main problem.

the $x_1 \ldots x_n$ including the normalization factor $n!$ and the $(1-x_1)^{r_0} \ldots (x_n)^{r_{n-1}}$. This casts the expression for Σ into the form

$$(-i)^{3n/2-1} \int dx \int \frac{dk_1}{(2\pi)^4} \cdots \frac{dk\, n/2}{(2\pi)^4} \times$$
$$\times \left[m^2 - \left(k_i Q_{ij}(x) k_j + 2 r_i(x) p k_i + s(x) p^2\right)\right]^{-3n/2+1}. \qquad (3,2)$$

The quadratic form Q_{ij}, r_i and s depend on the x only. Because m^2 occurs in each denominator, it is not multiplied by an x-dependent factor. For carrying out the integration over k, we have to get rid of the terms with (pk_i). This is accomplished by the substitution

$$k_i \to k_i - Q_{ie}^{-1} r_e p .$$

Here we consider Q as a matrix of the rank $n/2$ and Q^{-1} is the reciprocal matrix. This transforms the bracket of the integral into

$$(\ ) \to k_i Q_{ij} k_j + p^2 (s - r_i Q_{ij}^{-1} r_j) .$$

Because Q is a symmetric quadratic form with real coeficients it can be made diagonal by a transformation with a determinant 1. After this transformation we are left with an integral

$$(-i)^{3n/2-1} \int dx \int \frac{dk_1}{(2\pi)^4} \cdots \frac{dk\, n/2}{(2\pi)^4} [m^2 - \beta_i(x) k_i^2 - p^2 \alpha(x)]^{-3n/2+1} \qquad (3,3)$$

where β_i are the Eigenvalues of Q and $\alpha(x) = s(x) - r_i(x)\, Q_{ij}(x) r_j(x)$. The integral over k will be carried out with the aid of the well-known formula

$$\int \frac{dk}{[L \pm a k^2]^{n+2}} = \frac{i\pi^2}{a^2 L^n n(n+1)} .$$

This leads us to the result

$$\Sigma_n(p^2) = (-i)^{n-1} (4\pi)^{-n} \frac{(n/2-2)!}{(3n/2-2)!} \int \frac{dx}{|Q(x)^2|} \frac{1}{[m^2 - p^2 \alpha(x)]^{n/2-1}} ,$$

where Q is the product of the Eigenvalues of Q_{ij}. The integral, therefore, does not depend on the Eigenvalues separately, but on the determinant only. The Eigenvalues of Q can be shown to be positive. (Appendix a.) An important property of α is $o < \alpha(x) < 1$.

This comes from the fact that there are no real processes for $p^2 < m^2$. The creation of new particles by an external field with the energy momentum p is given by the real part of $\Sigma_n(p^2)$. The integral for Σ is real as long as there are no zeros of the denominator. If poles occur in the integrand one has to take the path of integration in complience with the path of integration for the Δ_F-function. It amounts to adding a small negative imaginary part to the mass and taking the principle values for all integrations. As no particles can be created for $p^2 < m^2$ there cannot be a real part of Σ and, therefore, no poles in the integrand. A direct proof will be given in appendix a.

A graph of n^{th} order of the form —$\boxed{\Sigma_n}$— provides us with a function of p^2 which is of the following form:

$$\frac{-i}{m^2-p^2}\left[\frac{(4\pi)^{-n}}{m^2-p^2}\,\frac{(n/2-2)!}{(3n/2-2)!}\int\frac{dx}{|Q|^2}\,\frac{1}{[m^2-p^2\alpha]^{n/2-1}}\right]. \qquad (3,4)$$

Here we have taken into account the factor i^n of the general expression (2,5). We note that the factor in the square bracket is positive definite for $p^2 < m^2$. One can see from general considerations[1]) that $-i\,\Sigma\,(p^2)/m^2 - p^2$ is positive definite, so that the sum of the contributions of all graphs is bound to be positive (within this energy region). Our result tells us that the contribution of each graph separately is positive.

We now consider an irreducible self-energy graph and carry out renormalization according to the prescription given in the preceding section. The expression given by an irreducible self-energy graph will be of the general form (3,4). We first separate out the mass renormalization.

$$\frac{1}{m^2-p^2}\int\frac{dx}{|Q|^2}\left\{\frac{1}{(m^2-p^2\alpha)^{n/2-1}}-\frac{1}{[m^2(1-\alpha)]^{n/2-1}}\right\}=$$

$$=-\left(\frac{n}{2}-1\right)\int_0^1 dy_1\int\frac{dx}{|Q|^2}\,\frac{1}{[m^2(1-\alpha)+(m^2-p^2)\,y_1\alpha]^{n/2}}. \qquad (3,5)$$

In the case of $n=2$ one has to renormalize before carrying out the k-integration. The result is again of the form (3,5). A second subtraction allowing for the renormalization of the field strength gives us

$$-\left(\frac{n}{2}-1\right)\int_0^1 dy_1\int\frac{dx}{|Q|^2}\left\{\frac{1}{[m^2(1-\alpha)+(m^2-p^2)\,\alpha\,y_1]^{n/2}}-\frac{1}{[m^2(1-\alpha)]^{n/2}}\right\}=$$

$$=\left(\frac{n}{2}-1\right)\frac{n}{2}\int_0^1 dy_1\int_0^1 dy_2\int\frac{dx}{|Q|^2}\,\alpha^2\,\frac{(m^2-p^2)\,y_1}{[m^2(1-\alpha)+(m^2-p^2)\,\alpha\,y_1\,y_2]^{n/2+1}}. \qquad (3,6)$$

---

1) KÄLLEN, Helv. Phys. Acta, **25**, 417 (1952).

After renormalization the contribution of an irreducible self-energy graph $-\boxed{\Sigma_n}-$ is of the form

$$-\frac{i}{m^2-p^2}\left[\frac{m^2-p^2}{(4\pi)^n}\,\frac{n/2!}{(3n/2-2)!}\int\frac{dx\,dy\,\alpha^2\,y_1}{|Q|^2[m^2(1-\alpha)+(m^2-p^2)\,\alpha\,y_1\,y_2]^{2n+1}}\right]. \quad (3,7)$$

After the double renormalization the expression in the square bracket is again positive for $p^2 < m^2$.

If one deals with reducible graphs, which are graphs containing subgraphs of the self-energy type, renormalization replaces the subgraph by a propagation function of the form (3,7). Thus one has to insert into the skeleton graph instead of the propagation function $-i/m^2-p^2$ a more general function of the form

$$\int\frac{-i}{[m^2\,a-p^2\,b]^{n/2+1}} \quad \text{with } 1>a>\frac{3}{4},\ \frac{1}{4}>b>0.$$

Inserting this propagation function into the lines of an irreducible self-energy graph leads us back to an expression of the type (3, 4).

There is a difficulty arising, if we encounter self-energy graphs of the form —o—o—, where two or more self-energy graphs are one on top of the other. In this case one cannot use the line on the right side and on the left side of a self-energy graph in the process of renormalization and there remains m^2-p^2 in the numerator. Then one has to apply more general integration formulas, a case which will be studied in the appendix *b*. This difficulty would not arise, if one considered a regularized theory with a propagation factor $-i/(m^2-p^2)^2$.

The result of this section is that the expression for Δ'_{F_1} is of the form $-i[1+\Sigma G(p^2)]/m^2-p^2$ where the sum runs over all self-energy graphs which are renormalized according to the general pattern. The contribution of each graph is positive for $p^2 < m^2$. This gives us the advantage that we get a lower bound for G by picking out certain terms of the sum which are easy to calculate. This will be done in the next section where we give an estimation of certain irreducible self-energy graphs. Our aim is to find self-energy graphs of n^{th} order which are of such a simple structure that they can be well estimated, yet their number should be large enough to effect the divergence of the series.

4. Estimation of terms of n^{th} order.

For estimating contributions of n^{th} order to G we have to look for graphs for which one can find a lower bound for α, which is greater than nothing. This leads more or less uniquely to considering graphs of the following type:

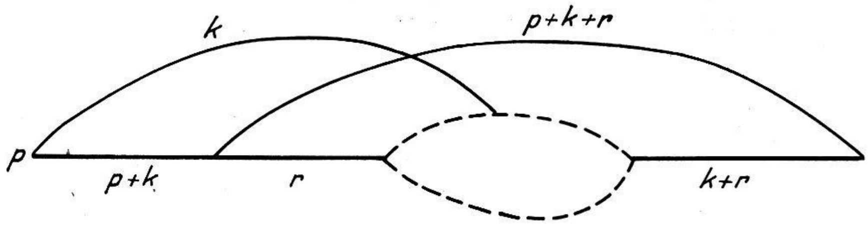

Fig. 2.

The part within the broken lines is a vertex part K of the order $n - 3$. Of these graphs we take those in which the line with the energy-momentum-vector k enters in the middle of the graph and where the $n/2 - 2$ points on the left hand side are joined with the $n/2 - 2$ points on the right hand side:

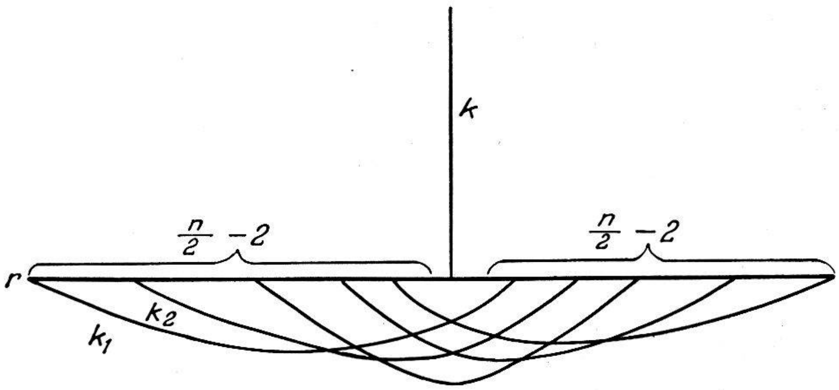

Fig. 3.

The number of those vertex parts is equal to $(n/2 - 2)!$ corresponding to the possibilities of joining the $n/2 - 2$ points on the left hand side with the points on the right hand side[1]). Here and in the following we consider only topologically different graphs as the number of connected graphs which differ only by the labelling of the points equals $n!$. This cancels exactly the $n!$ in the denominator of our general expression. The total number of contributions of the n^{th} order is less than

$$(3n-1)(3n-3)\ldots 1 = \frac{(3n)!}{(3n/2)!\, 2^{3n/2}} . \qquad (4,1)$$

[1]) Here we disregard the possibility of choosing the three ψ's of each vertex in different ways. This gives us a factor of the order $3^{3n/2}$ which is, however, of no importance in the question of convergence.

This is the number of possible pairings of $3n$ ψ-factors. The number of graphs of n^{th} order is less than that, as pairings of ψ'^{s} belonging to the same point do not contribute. This number is to be compared with the number of graphs we have selected. This is equal to $n!\,(n/2-2)!$, being about the $3^{-3n/2}\,2^{-n/2}$ *th* part of the total number. For calculating K we use the formula (3,1). The denominator will be of the general form

$$m^2 - k_i Q_{ij} k_j - 2\,k_i (a_1^i r + a_2^i k) - b_1 r^2 - 2\,b_2 r k - b_3 k^2. \qquad (4,2)$$

Q and a and b depending on x only. After carrying out the integration over $dk_1 \ldots dk_{n/2-2}$ we arrive at the following expression:

$$K = (4\,\pi)^{-n+4}\,\frac{(n/2-3)!}{(3\,n/2-7)!}\int dx \times$$

$$\times\,[m^2 - d_1 r^2 - d_2 r k - d_3 k^2]^{-n/2+2}\,|\,Q(x)\,|^{-2} \qquad (4,3)$$

with

$$\begin{aligned} d_1 &= b_1 - a_1^i\,Q_{ij}^{-1}\,a_1^j \\ d_2 &= b_2 - a_1^i\,Q_{ij}^{-1}\,a_2^j \\ d_3 &= b_3 - a_2^i\,Q_{ij}^{-1}\,a_2^j\,. \end{aligned} \qquad (4,4)$$

The condition that there are no real processes for $k = 0$, $r^2 < m^2$ and for $r = 0$, $k^2 < m^2$ and $r = -k$, $k^2 < 0$ supplies us the inequalities

$$0 < d_1 < 1 \qquad 0 < d_3 < 1 \qquad d_2 < \frac{d_1 + d_3}{2} \qquad (4,5)$$

If one combines this expression with the external line carrying the energy momentum vector k and carrying out the integration over k one is led by a similar argument to the inequality

$$d_2^2 < d_1\,d_3. \qquad (4,6)$$

When combining this expression with the denominators for the remaining lines of the complete graph fig. 5 we first take those with the energy momentum vector r and $r + k$:

$$\frac{1}{m^2 - r^2}\,\frac{1}{m^2-(r+k)^2}\,K = (4\,\pi)^{-n+4}\,\frac{(n/2-3)!}{(3\,n/2-7)!}\int dx\,dy \times$$

$$\times\,\frac{|\,Q_{(x)}\,|^{-2}}{[m^2 - (c_1 r^2 + 2\,c_2\,r\,k + c_3\,k^2)]^{n/2}} \qquad (4,7)$$

with

$$c_1 = 1 - y_2\,(1 - d_1) \quad c_2 = y_1 - y_2\,(1 - d_2) \quad c_3 = y_1 - y_2\,(1 - d_3) \qquad (4,8)$$

In view of the inequalities (4,6) we get

$$\begin{array}{ll} 0 < c_1 < 1 & c_2 < \frac{c_1+c_3}{2} \\ 0 < c_3 < 1 & 0 < c_2^2 < c_1\, c_3 \,. \end{array} \tag{4,9}$$

In (4,7) we have used the abbreviated notation $\int$ dy for

$$\left(\frac{n}{2}-2\right)\left(\frac{n}{2}-1\right)\int\limits_0^1 dy_1 \int\limits_0^{y_1} dy_2\, y_2^{n/2-3}\,.$$

We join the remaining denominators to the expression (4,7) with the help of three auxiliary variables z.

$$[m^2-(p+k)^2]^{-1}\,[m^2-(p+r+k)^2]^{-1}\,[m^2-k^2]^{-1}\,[m^2-c_1 r^2-2\,c_2 r k-c_3 k^2]^{-n/2}$$
$$\diagdown_{z_1}\diagup \qquad\qquad \diagdown_{z_2}\diagup \qquad \diagdown_{z_3}\diagup \tag{4,10}$$

Then we can carry out the integration over r and k and arrive at an expression

$$(4\pi)^{-n}\frac{\left(\frac{n}{2}-3\right)!}{\left(\frac{3n}{2}-7\right)!\left(\frac{n}{2}+2\right)\left(\frac{n}{2}+1\right)\frac{n}{2}\left(\frac{n}{2}-1\right)}\int dx\,dy\,dz\,\frac{|Q_{(x)}|^{-2}\,|Q'|-2}{[m^2-\alpha p^2]\,\frac{n}{2}-1} \tag{4,11}$$

The quadratic form of the denominators before integrating over r and k is given by

	p	k	r
p	$1-z_2$	$1-z_2$	z_1-z_2
k		$1-z_3+z_3c_3$	$z_1-z_2+z_3c_2$
r			$z_1-z_2+z_3c_1$

From this we calculate the expressions

$$\begin{aligned} Q' &= (1-z_3+z_3\,c_3)\,(z_1-z_2+z_3\,c_1) - (z_1-z_2+z_3\,c_2)^2 \\ \alpha &= \{(1-z_2)\,[(1-z_3+z_3\,c_3)\,(z_1-z_2+z_3\,c_1)-(z_1-z_2+z_3\,c_2)^2] - \\ &\quad -(1-z_2)^2\,(z_1-z_2+z_3\,c_1)+2\,(1-z_2)\,(z_1-z_2)\,(z_1-z_2-z_3\,c_3) - \\ &\quad -(z_1-z_2)^2\,(1-z_3+z_3\,c_3)\}\{(1-z_3+z_3\,c_3)\,(z_1-z_2+z_3\,c_1) - \\ &\quad -(z_1-z_2+z_3\,c_2)^2\}^{-1}. \end{aligned} \tag{4,12}$$

The last step is carrying out the double renormalization and multiplying with the last denominator which contains the energy momentum vector p only. Thus we are finally led to

$$H = \left(\frac{\lambda}{4\pi}\right)^n \frac{\left(\frac{n}{2}-3\right)!\,\frac{n}{2}!}{\left(\frac{3n}{2}-7\right)!\left(\frac{n}{2}+2\right)!} \int \frac{dx\,dy\,dz\,du}{|Q(x)|^2\,|Q'|^2} \times$$

$$\times \frac{u_1\,\alpha^2\,(m^2-p^2)}{[m^2(1-\alpha)+\alpha\,u_1\,u_2\,(m^2-p^2)]^{n/2+1}}\,. \qquad (4{,}13)$$

In this expression the y- and z-dependence of α and Q' is known whilst the x-dependence is unknown. The minimum-value of α, however, with respect to the $3n/2-6$ variables x is greater than zero in view of (4,9). We obtain a suitable lower bound for H by replacing all x-dependent quantities except Q in the numerator and denominator by their minimum or maximum value respectively. Then we can break the integral for H into two parts:

$$H > \left(\frac{\lambda}{4\pi}\right)^n \frac{(n/2-3)!\,(n/2)!}{(3n/2-7)!\,(n/2+2)!} \int \frac{dz\,dy\,du\,u_1\,\alpha^2\,(m^2-p^2)_{\min}}{|Q'|^2\,\{m^2(1-\alpha)+\alpha\,u_1\,u_2\,(m^2-p^2)\}^{n/2+1}_{\max}} \times$$

$$\times \int dx\,|Q(x)|^{-2}\,. \qquad (4{,}14)$$

Furtheron we concentrate on estimating the integral over dx. This can be done by giving a close lower bound for (4,3). For this goal we observe that in the region $p^2 < m^2$ there are no displaced poles (in the terminology of Dyson, Phys. Rev. 75, 1736 (1949) in the k_0-integrations. Thus we put in (4,2) $r = k = 0$ and rotate the path of integration for each k_0-variable to the imaginary axis. Then each propagation factor becomes positive definite and we can use the fact that the geometric mean value is less than the arithmetic mean value for combining the denominators[1]) for K:

$$\frac{1}{m^2-k_1^2} \cdots \frac{1}{m^2-k_{3t}^2} > \frac{(3\,t)^{3t}}{(3\,tm^2-k_1^2-\cdots-k_{3t}^2)^{3t}}\,. \qquad (4{,}15)$$

Here and in the following we put $n/2-2=t$. We obtain a lower bound for (4,3) by using (4,15) instead of (3,1). It proves useful not to use the conservation laws from the vertices for expressing $2\,t$

---

[1]) This method is essentially due to Hurst: (Compare introduction).

k-variables as linear combinations of the basic t k-variables. We postpone this and write the k-Integral with matrix notation

$$(3\,t)^{3\,t} \int d^{(4\,.\,3\,t)}\, K \, \frac{\delta^{(4\,.\,2\,t}\,(AK)}{3\,t\,m^2 - K^2)^{3\,t}} \,. \tag{4,16}$$

The $4 \cdot 3\,t$-dimensional vector K is composed of the $3\,t$ fourvectors k_i. The $4 \cdot 2\,t$-fold δ-function expresses the conservation-laws stemming from the $2\,t$ vertices. They are written in matrix notation $A_{ij} k_j = 0$, where A is a $4 \cdot 2\,t$ by $4 \cdot 3\,t$ rectangular matrix. Each row of A contains at most three times ± 1 and 0 everywhere else. This is because there are three lines at each vertex and the conservation law puts the sum of the three energy-momentum-vectors equal to 0. The $4 \cdot 2\,t$ rows of A may be considered as a system of $4 \cdot 2\,t$ linear independent vectors a_i which span a subspace S. The δ-function demands that K must be orthogonal to all a, or that the integration has to be carried out over the complementary subspace T of $4 \cdot t$ dimensions. K can be split into a component K_S in S and a component K_T in T. For the domain of integration $K^2 = K_T^2$ holds. In order to eliminate the δ-functions we choose an orthogonal set of $4 \cdot t$ vectors n_i in T and make the linear transformation

$$MK = K' \qquad \text{with } M = \overset{4\cdot 3\,t}{\left.\begin{array}{|c|} \hline N \\ \hline A \\ \hline \end{array}\right.} \begin{array}{l} 4\cdot 2\,t \\ 4\cdot t \end{array} \tag{4,17}$$

where the rows of the $4 \cdot t$ by $4 \cdot 3\,t$ matrix N consist of the normalized vectors n_i. This transforms the δ-functions into (now explicitely written) $\delta\,(K_1') \ldots . \,\delta\,(K'_{4 \cdot 2\,t})$. We further note that $K_T'^2 = K_T^2$, as M is only non-orthogonal in S. Thus the integral (4,16) is transformed into (disregarding i's)

$$(3\,t)^{3\,t}\, |M|^{-1} \int d^{(4 \cdot 3\,t)}\, K' \, \frac{\delta^{(4 \cdot 2\,t)}\,(K')}{(3\,t\,m^2 - K'^2)^{3\,t}} =$$

$$= |M|^{-1} (3\,t)^{3\,t} \int d^{(4 \cdot t)}\, K' \, \frac{1}{(3\,t\,m^2 - K^2)^{3\,t}} =$$

$$= |M|^{-1} (3\,t)^{3\,t} (4\,\pi)^{-2\,t} \frac{(t-1)!}{(3\,t-1)!}\, (3\,t\,m^2)^{-t} \,. \tag{4,18}$$

For estimating $|M|$ we may use Hadamards inequality which yields $|M|^2 < 3^{4 \cdot 2\,t}$, as there are $4 \cdot 2\,t$ a's with $a^2 < 3$ and $n_1^2 = 1$. This gives us the estimate

$$K > (4\,\pi)^{-2\,t_3 - 4\,t}\, (3\,t)^{2\,t} \frac{(t-1)!}{(3\,t-1)!}\, m^{-2\,t} \,. \tag{4,19}$$

Comparing this expression with (4,3) we obtain

$$\int dx \, |Q(x)|^2 > \left(\frac{t}{3}\right)^{2t}. \tag{4,20}$$

The first part of (4,14) can be estimated by observing that in view of the inequalities (4,9) the minimum value and the maximum value of Q with respect to x is given by

$$\alpha_{\min}(z) = \frac{(1-z_1)(z_1-z_2)(z_2-z_3)}{(1-z_3)(z_1-z_2)-(z_1-z_2)^2}, \; Q'_{\max} = (1-z_3)(z_1-z_3)-(z_1-z_3)^2. \tag{4,21}$$

As $\alpha_{\min}$ is positive definite, the integral

$$\int \frac{dy\, dz\, du\, \{u_1 \alpha^2 (m^2-p^2)\} \min}{\{|Q'|^2 (m^2(1-\alpha)+\alpha u_1 u_2 (m^2-p^2)^{n/2+1}\} \max} = C(p^2)\, m^{-n} \tag{4,22}$$

is positive for $m^2 > p^2$. The integrand is independent of y and, therefore, $\int dy$ can be replaced by 1[1]). The remaining integral does not contain n nor is it dependent on the structure of the graph. This casts the estimate for H into its final form

$$\begin{aligned} H &> \left(\frac{\lambda}{4\pi m}\right)^n C(p^2)\, e^{n-2}\, 3^{-5n/2+11} \left(\frac{2}{n}\right)^2 = \\ &= \left(\frac{\lambda}{4\pi m}\right)^n C(p^2)\, A^{-n}\, n^{-2}\, g \\ A &= \frac{3^{5/2}}{e}, \quad g = 4\cdot 3^{11}\, e^{-2}. \end{aligned} \tag{4,23}$$

Now we know that there are $n/2 - 2$ factorial graphs with the lower bound H. Therefore the series Δ'_F diverges at least[2]) as

$$\left(\frac{\lambda}{4\pi m A}\right)^n C\, \frac{(n/2-2)!}{n^2}\, g\,.$$

The series does not converge for any value of λ. If any solution exists at all it must be nonanalytic for $\lambda = 0$. On the other hand we know that for $\lambda = 0$ there is a solution for the equations namely the free field. If there is any solution for the $\lambda \neq 0$ it seems, therefore, doubtful whether it has any relation to the solution for $\lambda = 0$. It is to be expected that if there are solutions for $\lambda \neq 0$ that they have quite an unphysical behaviour.

---

[1]) One can estimate easily that, for instance, $C(o) > 1/5040$ and $C(p^2) > C(o)$ for $p^2 > 0$.

[2]) Hurst has found an upper bound for the terms of the series which shows that the series diverges actually in this manner.

It remains to be discussed whether the series converges in the domain $p^2 > m^2$. One might suspect that if the series converges for $p^2 > m^2$ one could extend this expression to the region $p^2 < m^2$ by analytic continuation with respect to p. Though the author has not been able to prove anything in the case considered here, one can construct examples for which the series diverges for all values of the external momenta. If one investigates a theory with the non-linearity $\lambda \psi^4$ the convergence of the k-integration of an expression is only determined by the number of external momenta and not by the order of the graph. The present analysis can be carried through in this theory along similar lines, although things are more complicated in this case. But perturbation theory diverges for this theory on account of the same reasons. If one considers a graph with eight external lines, as for example

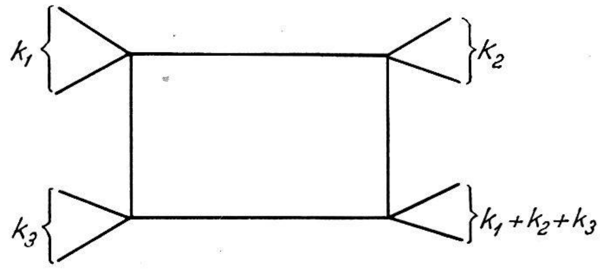

Fig. 4.

he gets an expression of the form $\int dx (m^2 d_{ij}(x)\, k_i k_j)^{-2}$. This is positive for all values of the k's and the same holds for contributions of higher order. The series diverges for the real part of this matrix element (which corresponds to the principal value of the above integral) irrespective of the value of k. But the real and the imaginary part of matrix elements are connected by certain equations and the series will diverge for the imaginary part too.

To sum up one can say that the chances for quantized fields to become a mathematical consistent theory are rather slender.

This work was carried out partly under a Fellowship of UNESCO and partly under a fellowship of the Swiss Federal Institute of Technology. The author wants to express his thanks to Prof. Pauli for his hospitality in his Institute and for making the work of Hurst available to him. The author is very much indepted to Dr. Källen for pointing out a mistake in the original manuscript.

Appendix a.

In this appendix we shall show that the Eigenvalues β_i and α (3,3) are positive. Considering p as one of the k's we prove this by showing that the quadratic form of the k's in the denominator is positive definite. The quadratic form is obtained by combining the denominators

$$d_0(1-x_1)+d_1(x_1-x_2)+\cdots+d_n x_n = k_i k_j Q_{ij} \tag{a,1}$$

where $d_i = (\Sigma k_i c_i)^2$ and the c are $0, \pm 1$. Q is a real symmetrical quadratic form and has, therefore, real Eigenvectors. If x_i is the normalized Eigenvector belonging to the Eigenvalue β, β equals $x_i x_j Q_{ij}$. If one takes for k the four k_i vectors $\begin{pmatrix} x_i \\ 0 \\ 0 \\ 0 \end{pmatrix}$, all having time direction we can write $\beta = k_i k_j Q_{ij}$. For these k's all the d's in (a,1) are positive and β is bound to be positive as all the terms on the left hand side of (a,1) are positive.

For proving $0 < \alpha(x) < 1$ it suffices to observe that $0 < s(x) < 1$ and $r_i Q_{ik} r_k > 0$ as we have shown above that $\alpha(x) > 0$. The first of the above statements follows from the fact that s is of the form $\Sigma_i (x_{a_i} - x_{a_i+1})$ with $a_i > a_k$ for $i > k$. Keeping in mind that Q is positive definite we have in the diagonal system of $Q: r_i Q_{ik}^{-1} r_k = r_i'^2 \beta_i^{-1} > 0$ which is our second assertion.

Appendix b.

Here we shall investigate the case where in the k-integration some k remain in the numerator. Differenciating the general formula for k-integration with respect to a one gets

$$\int \frac{dk\, k^2}{[L - a k^2]^r} = \frac{-2 i \pi^2}{a^3 L^{r-3} (r-1)(r-2)(r-3)}. \tag{b, 1}$$

Now we assume for simplicity that only one k occurs in the numerator and for this k we have a propagation factor of the form

$$\frac{m^2 - k_1^2}{[\beta m^2 - \alpha k_1^2]^s}.$$

If one combines this line with the other lines of the graph we arrive at an expression of the form,

$$\int \frac{dk (m^2 - k_1^2)}{[b m^2 - (k Q k + 2 p k r + s p^2)]^t}. \tag{b, 2}$$

We use matrix notation and write $k Q k$ for $k_i Q_{ij} k_j$. By the substitution $k_1 \to k_1 - Q_{11}^{-1} r_1 p$ we get rid of $p k_i$ in the denominator and a second substitution for all the other k's and an unitary transformation U makes the denominator diagonal. The numerator becomes in this process

$$m^2 - k_s^2 U_{s1}^2 - \left(\frac{r_1}{Q_{11}}\right)^2 p^2,$$

if we drop linear in the k as they vanish by integrating. Applying the formula (b, 1) we are left with

$$\frac{1}{(t-1)(t-2)} \frac{1}{|Q|^2} \frac{1}{(b m^2 - \alpha p^2)^h} \left[m^2 - \left(\frac{r_1}{Q_{11}}\right)^2 p^2 + 2 \frac{b m^2 - \alpha p^2}{t-3} \frac{(U_{1s})^2}{\beta s}\right]. \quad \text{(b, 3)}$$

The β are the Eigenvalues of Q. Now we know the inequality $Q_{11}(x) > r_1(x)$. It can be seen immediately if one introduces the Feynman variables in the way similar (3, 1). Then Q_{11} is the total length of all lines containing k_1 while r_1 is the length of the lines containing r and k_1. As the β are positive the square bracket in (b, 3) is positive for $p^2 < m^2$.

The reason for the signs in favour of our general condition lies in the fact that we have a sign in (*b*, 1) opposite to the sign of an integral where no k^2 occurs in the numerator.

If there are two integration variables in the numerator, $(m^2 - k_1^2)$ $(m^2 - k_2^2)$ the term of the fourth order too yields the correct sign. This can be seen by the following analysis. The real unitary transformation transforms the term of fourth order in k into a term $(k_s k_{s'}) (k_r k_{r'}) U_{1s'} U_{1s} U_{2r} U_{2r'}$. As the denominator then depends on the k squared only we make the substitution

$$k_s^\alpha k_{s'}^\alpha, k_r^\beta k_{r'}^\beta \to \delta_{ss'} \delta_{rr'} k_s^2 k_r^2 + \frac{1}{4} \delta^{\alpha\beta} \frac{1}{4} \delta^{\alpha\beta} \delta_{rs} \delta_{r's'} k_r^2 k_s^2 +$$
$$+ \frac{1}{4} \delta^{\alpha\beta} \frac{1}{4} \delta^{\alpha\beta} \delta_{rs'} \delta_{sr'} k_r^2 k_s^2 .$$

Then the numerator of this term becomes

$$k_r^2 k_s^2 \left[U_{1s}^2 U_{2r}^2 + \frac{1}{2} U_{1r} U_{1s} U_{2r} U_{2s}\right].$$

The square bracket is again positive as it may be written in the form

$$\frac{1}{2} U_{1s}^2 U_{2r}^2 + \frac{1}{4} (U_{1r} U_{2s} + U_{1s} U_{2r})^2 .$$

In the same way he can convince himself that in more complicated cases the signs of the terms are so as to make the expression positive for $p^2 < m^2$.

A more general argument runs as follows. We replace the propagation factor between self-energy graphs by

$$\frac{1}{(m^2 - K^2)^r},$$

$$\downarrow$$

and consider the expression as a function of the r. For r, $2 < r < \infty$, there are no k^2 in the numerator and the expression can be calculated in the same way as in (3,2).

The expression thus obtained is positive for all values of r and goes to infinity for those values of r for which the k-integration becomes divergent. As this formal expression is an analytic function in the r which is identical with our expression for all integral r between 2 and ∞, it has to be identical with our expression for $r = 1$.