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Theory of Angular Correlations

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Abstract. The most general angular correlation function may be written in the form $W = Tr\,(\epsilon\varrho)$, where ϱ is the density matrix describing the ensemble of final nuclei and reaction products and the "efficiency matrix" ϵ is completely determined by the counter arrangement and the nature of the particles to be counted. Expansion of ϵ and ϱ into sums of irreducible tensors greatly simplifies the evaluation of the trace. Explicit general rules are derived for the construction of any angular correlation function from a few simple building stones.

I. Introduction.

Since the work of MYERS[1]) and HAMILTON[2]) on angular distributions and angular correlations many papers[3] have been written which deal with the general theory of angular correlation of nuclear radiations. Their main concern is with the derivation of explicit formulae for various correlation functions. The probabilities for transitions between definite pure states specified by a complete set of quantum numbers are calculated, usually by perturbation methods, and the angular dependence is derived from the transformation properties of the relevant matrix elements. In actual experiments one observes not individual systems in a pure state but a statistical ensemble and not all the observables specifying a pure state of the system are actually observed. In the final formula one therefore sums or averages over unobserved quantum numbers.

Considerable simplification in the mathematical formalism can be achieved by using the density matrix ϱ[4]) to describe the ensemble of systems each consisting of all the products of a nuclear reaction or cascade decay[5]). With ϱ, the statistical average of any observable and the probability for any observable effect can be calculated. We are particularly interested in a "yes-no experiment" in which a counter arrangement detects all systems in a state specified by the set of quantum number Q with an efficiency ϵ_Q; $0 \leqslant \epsilon_Q \leqslant 1$. The probability for the answer "yes" is

$$W = \sum_Q \epsilon_Q \,(Q \mid \varrho \mid Q)\,. \tag{1}$$

We may consider ϵ_a to be the eigenvalues of an "efficiency matrix" ϵ and write (1) in the form

$$W = Tr(\epsilon \varrho) \tag{2}$$

where the trace in (2) may be evaluated in any representation. Most convenient is a representation specified by the quantum numbers jm of the total angular momentum and other quantum numbers collectively denoted by $A, A' \ldots$ The evaluation of the trace is facilitated if we introduce the irreducible tensors ϱ_k^n and ϵ_k^n by the definition

$$(jA\,|\varrho_k^n|\,j'A') = \sum_{m\,m'} (-1)^{j'-m'} (j\,j'm - m'|\,k\,n)\,(j\,mA\,|\,\varrho\,|\,j'\,m'A') \tag{3}$$

and

$$(jA\,|\epsilon_k^n|\,j'A') = \sum_{m\,m'} (-1)^{j'-m'} (j\,j'm - m'|\,k\,n)\,(j\,mA|\,\epsilon\,|\,j'\,m'A') \tag{4}$$

where the first factor on the right hand side is a Clebsch-Gordan coefficient for the addition of angular momenta[6]). According to their definition ϵ_k^n and ϱ_k^n are matrices with respect to j and A but not with respect to m; they shall be called efficiency tensors and statistical tensors respectively. Their knowledge is equivalent to the knowledge of the density matrix and the efficiency matrix, for with eq. (A2) of Appendix A there follows from (3) and (4)

$$(j\,mA\,|\,\varrho\,|\,j'\,m'A') = \sum_{k\,n} (-1)^{j'-m'}\,(j\,j'm - m'|\,k\,n)\,(jA\,|\,\varrho_k^n|\,j'A') \tag{5}$$

and

$$(j\,mA\,|\epsilon\,|\,j'\,m'A') = \sum_{k\,n} (-1)^{j'-m'}\,(j\,j'm - m'|\,k\,n)\,(jA|\epsilon_k^n|\,j'A'). \tag{6}$$

From (5) and (6) follows with (A1) and (A3)

$$W = Tr(\epsilon\varrho) = \sum_{j\,j'} \sum_{A\,A'} \sum_{k\,n} (jA\,|\,\epsilon_k^n|\,j'A')\,(jA|\,\varrho_k^n|\,j'A')^*. \tag{7}$$

We also use the fact that the density matrix is hermitian.

The ϵ_k^n will depend explicitly on the geometry of the counter arrangement while ϱ_k^n will depend on the geometric characteristics of the initial states. If we consider the decay of isotropically distributed nuclei the density matrix must be isotropic, that is diagonal in m and independent of m. Hence we find by (3) $(jA \mid \varrho_k^n \mid jA') = 0$ for $k \neq 0$. If we consider a nuclear reaction between non polarized particles we have cylindrical symmetry about the beam. The density matrix is diagonal in m if the direction of the beam is chosen as axis of quantization. There follows from (3) $(j\,A \mid \varrho_k^n \mid jA') = 0$ for $n \neq 0$.

(7) is the most general angular correlation function. In order to get formulae useful for the interpretation of specific experiments we must derive explicit expressions for the ϱ_k^n and the $\in_k^n$. This will be done for the ϱ_k^n in Sec. II and for the $\in_k^n$ in Sec. III.

II. The Density Matrix.

The density matrix ϱ in (1) describes the end products of a nuclear reaction or a cascade decay. The initial nuclei are described by the density matrix $(jmA_0 \mid \bar{\varrho} \mid j'm'A_0')$ or by the statistical tensors $(j\,A_0 \mid \bar{\varrho}_k^n \mid j'A_0')$. The probability for a transition from a state Q to a state P during the time t is for large t equal to $t/2\pi|R_{PQ}|^2$ where $S = R + 1$ is the so called HEISENBERG S-Matrix[7]). If interactions with external magnetic fields of macroscopic or atomic origin are negligible[8]) $(jmA|S|j'm'A_0)$ is invariant under rotations of the coordinate system. It is therefore diagonal in j and m and independent of m and may be written in the form $(A|S_j|A_0)$. The density matrices ϱ and $\bar{\varrho}$ are related by the equation[9])

$$(jmA\,|\varrho\,|j'm'A') = \text{const.} \sum_{A_0 A_0'} (jmA_0\,|\bar{\varrho}|\,j'm'A_0') R_j(A,A_0) R_{j'}^*(A',A_0'). \quad (8)$$

Consequently we have with (3)

$$(jA\,|\varrho_k^n|\,j'A') = \text{const.} \sum_{A_0 A_0'} (jA_0\,|\bar{\varrho}_k^n|\,j'A_0') R_j(A,A_0) R_{j'}^*(A',A_0'). \quad (9)$$

In the case of the decay of isotropically distributed metastable nuclei $(jA_0|\bar{\varrho}_k^n|j'A_0')$ is different from zero only for one value of j, $j' = j$, and one value of A_0, $A_0' = A_0$. A_0 can therefore be dropped from the notation and we have

$$(j\,m\,|\,\bar{\varrho}\,|\,j\,m') = \frac{1}{2j+1}\,\delta_{m\,m'} \quad (10)$$

and hence

$$(j\,|\,\bar{\varrho}_k^n\,|\,j) = \sqrt{2j+1}\,\delta_{k0}\,\delta_{n0}\,. \quad (11)$$

If the initial nuclei are partially aligned $(j\,|\,\bar{\varrho}_k^n\,|\,j)$ will be different from zero for $0 \leqslant k \leqslant 2j$, $-k \leqslant n \leqslant k$. These tensor elements describe the state of polarization.

We proceed to determine $\bar{\varrho}_k^n$ for nuclear reactions. j is now the total angular momentum of the target nucleus plus incoming particle which may be a photon or a heavy particle (proton, α-particle, neutron, etc.). If a compound nucleus is formed j is the spin of the compound nucleus. The spin quantum numbers of the target nucleus are $j_t\ m_t$. L is the total angular momentum of the incoming particle

with respect to the target nucleus; the quantum numbers are L, M. The center of mass of the entire system is supposed to be at rest. For particles of definite energy one more quantum number besides L and M is needed to specify a state. This will be the parity $\mathfrak{p}$*) for photons and the orbital angular momentum l for heavy particles. We use ξ as a short notation for both L, $\mathfrak{p}$ and L, l. Instead of the angular momentum quantum numbers the direction of the momentum $\Omega = (\Theta, \varphi)$ and the component of the spin in that direction σ may be used to specify the state of one particle. The transformation matrix between the two representations is given by

$$(\Omega\,\sigma\,|\,\xi M) = \sum_{M'} (0\,\sigma\,|\,\xi\,M')\,D^{L}_{MM'}(\varphi, \vartheta, 0)^* \tag{12}$$

where $D^{L}_{MM'}(\varphi, \vartheta, \chi)$ are an irreducible representation of the rotation through the EULER angles φ, ϑ, χ[10]). For heavy particles with spin s $(0\,\sigma\,|\,L\,l\,M)$ is given by

$$(0\,\sigma\,|\,L\,l\,M) = \sqrt{\frac{2\,l+1}{4\,\pi}}\,(l\,s\,0\,\sigma\,|\,LM) \tag{13}$$

which vanishes unless $M = \sigma$. For magnetic multipole photons, $\mathfrak{p} = (-1)^{L+1}$, one finds

$$(0\,\sigma\,|\,LM(-1)^{L+1}) = -(L1\;0\,\sigma\,|\,LM)\sqrt{\frac{2L+1}{4\,\pi}} = \sigma\sqrt{\frac{2L+1}{8\,\pi}}\,\delta_{\sigma M} \tag{14}$$

(14) vanishes for $\sigma = 0$ as required by the absence of longitudinal photons. For electric multipole photons, $\mathfrak{p} = (-1)^{L}$,

$$\left(0\,\sigma\,|\,LM(-1)^{L}\right) = \sum_{l=L\pm1} (l1\;0\,\sigma\,|\,LM)\sqrt{\frac{2\,l+1}{4\,\pi}}\,c_l \tag{15}$$

where the coefficients c_l are to be determined by the requirement that the right hand side of (15) vanishes for $\sigma = 0$ and $\sum_{l=L\pm1} c_l^2 = 1$. This gives

$$\left(0\,\sigma\,|\,LM(-1)^{L}\right) = \delta_{\sigma M}\sqrt{\frac{2L+1}{8\,\pi}}\,. \tag{16}$$

A beam of particles moving in the direction Ω is described by a density matrix

$$(\omega\xi M\,|\,\varrho(\Omega)|\,\omega'\xi' M') = \sum_{\sigma\sigma'} \delta(\omega-\omega')\,\varrho_{\sigma\sigma'}(\omega)(\Omega\sigma\,|\,\xi M)^*(\Omega\,\sigma'|\,\xi' M') \tag{17}$$

where

$$Tr\;\varrho = \int \delta\,\omega \sum_{\sigma} \varrho_{\sigma\sigma}(\omega) = 1 \tag{18}$$

*) $\mathfrak{p} = (-1)^{L+1}$ for magnetic multipoles, $\mathfrak{p} = (-1)^{L}$ for electric multipoles.

$\varrho_{\sigma\sigma'}$ describes the polarization of the beam[11]) and the energy distribution in the beam. To simplify the notation it will be convenient to drop the δ-function with respect to the energies and have it understood that the matrix elements are functions of the energy. Since all matrices discussed in the following will be diagonal in the energies this procedure will be used throughout. If the beam is practically monoenergetic it is appropriate to treat ω as a fixed parameter and normalize the density matrix by

$$\sum_{\sigma} \varrho_{\sigma\sigma} = 1 . \tag{19}$$

According to (3), we have with (A3), (C5) and (C4)

$$(\xi \,|\, \varrho_k^n(\Omega) \,|\, \xi') = \sum_{n'} \sum_{\sigma\sigma'} \varrho_{\sigma\sigma'}(\omega)\, (0\,\sigma \,|\, \xi\,\sigma)\, (0\,\sigma' \,|\, \xi'\sigma') \;\times$$
$$\times\, (-1)^{L'-\sigma'} (LL'\,\sigma - \sigma' \,|\, k\,n')\, D^k_{nn'}(\varphi, \vartheta, 0) . \tag{20}$$

If the particles are not polarized $\varrho_{\sigma\sigma'}(\omega) = \varrho(\omega)\, \delta_{\sigma\sigma'}$ and

$$(\xi \,|\, \varrho_k^n \,|\, \xi') = \varrho(\omega)\, C_k(\xi, \xi')\, D^k_{n0}(\varphi, \vartheta, 0) \tag{21}$$

where

$$C_k = \frac{(-1)^{L+k-s}}{4\pi} \sqrt{(2l+1)\,(2l'+1)\,(2L+1)\,(2L'+1)} \times$$
$$\times\, (l\,l'\,0\,0 \,|\, k\,0)\, W(l\,l'\,L\,L', k\,s) \tag{22}$$

for heavy particles and

$$C_k = \frac{1}{4\pi} \sqrt{(2L+1)\,(2L'+1)}\, \frac{1}{2} \times$$
$$\times\, (1 + \mathfrak{p}\,\mathfrak{p}'(-1)^k)\, (-1)^{L'-1}\, (LL'\,1\,-1 \,|\, k\,0) \tag{23}$$

for photons. The definition of the Racah coefficient W which appears in equ. (22) is given in Appendix B.

The target is described by a set of statistical tensors $(j_t \,|\, \varrho_{k_t}^{n_t} \,|\, j_t)$ which vanish for $k_t \neq 0$ if the nuclei are not aligned. The statistical tensors of the entire system can now be constructed as follows

$$(j\,\xi\,j_t \,|\, \overline{\varrho}_k^n \,|\, j'\,\xi'\,j_t) = \sum_{k_t\,n_t} \sum_{k_0\,n_0} (j_t \,|\, \varrho_{k_t}^{n_t} \,|\, j_t)\, (\xi \,|\, \varrho_{k_0}^{n_0} \,|\, \xi') \;\times$$
$$\times\, (j\,j' \,|\, \Gamma(j_t\,j_t'\,LL'\,k) \,|\, k_t\,k_0)\, (k_t\,k_0\,n_t\,n_0 \,|\, k\,n) \tag{24}$$

where Γ is the transformation coefficient defined in (B6) of Appendix B.

For heavy particles it is possible to describe the spin and the orbital motion of the incoming particles by separate density matrices and combine the angular momenta according to the scheme: $\boldsymbol{j}_t + \boldsymbol{s} = \boldsymbol{I}$, $\boldsymbol{I} + \boldsymbol{l} = \boldsymbol{j}$. I is the so called channel spin.

$$(j\,I\,l\,j_t\,s\,|\,\bar{\varrho}^n_k\,|\,j'\,I'\,l'\,j_t\,s) = \sum_{k_t n_t}\sum_{k_s n_s}\sum_{k_0 n_0}(j_t\,|\,\varrho^{n_t}_{k_t}\,|\,j_t)\,(s\,|\,\varrho^{n_s}_{k_s}\,|\,s)\times$$

$$\times\left(l\,|\,\varrho^{n_0}_{k_0}(\Omega)\,|\,l'\right)\sum_{K\,N}(II'\,|\,\Gamma(j_t\,j_t\,s\,s\,K)\,|\,k_t\,k_s)\times$$

$$\times\,(k_t\,k_s\,n_t\,n_s\,|\,KN)\,(j\,j'\,|\,\Gamma(I\,I'\,l\,l'\,k)\,|\,K\,k_0)\,(K\,k_0\,N\,n_0\,|\,k\,n) \qquad (25)$$

where

$$(l\,|\,\varrho^n_k(\Omega)\,|\,l') = \frac{(-1)^{l'}}{4\pi}\sqrt{(2\,l+1)\,(2\,l'+1)}\,(l\,l'\,0\,0\,|\,k\,o)\,D^k_{n0}(\varphi,\vartheta,0)\,. \qquad (26)$$

This representation is particularly useful only if both target and beam are unpolarized, that is if

$$(j_t\,|\,\varrho^{n_t}_{k_t}\,|\,j_t) = \delta_{k_t 0}\,\delta_{n_t 0}\sqrt{2\,j_t+1} \qquad (27)$$

and

$$(s\,|\,\varrho^{n_s}_{k_s}\,|\,s) = \delta_{k_s 0}\,\delta_{n_s 0}\,\varrho\,(\omega)\,. \qquad (28)$$

With (B6), (B4) and (C3) one obtains then from (25) and (26)

$$(j\,I\,l\,j_t s\,|\,\varrho^n_k(\Omega)\,|\,j'\,I'\,l'\,j_t\,s) =$$

$$= \delta_{II'}\sqrt{\frac{(2\,j+1)\,(2\,j'+1)\,(2\,l+1)\,(2\,l'+1)}{(2\,s+1)\,(2\,j_t+1)\,(2\,I+1)}}\,\frac{(-1)^{l'}}{4\pi}\,(l\,l'\,0\,0\,|\,k\,0)\times$$

$$\times\,W(I\,j\,l'\,k,l\,j')\sqrt{\frac{4\pi}{2\,k+1}}\,Y^n_k(\Omega)^*\,. \qquad (29)$$

It is, of course, always possible to transform a $jLls$ representation to a $jIls$ representation by (B5).

III. The Efficiency Matrix.

In order to evaluate explicit expressions for the efficiency tensors we must specify the quantum numbers denoted by A, A' in detail. Let ξ stand for the quantum numbers L, $\mathfrak{p}$ or L, l of any emitted particle. Different particles in the same process are distinguished by subscripts $1, 2, 3\ldots$. We denote now the total angular momentum of the system by $\boldsymbol{j}_a$ and define $\boldsymbol{j}_b$ as the angular momentum of the nucleus after emission of the particle 1, $\boldsymbol{j}_b = \boldsymbol{j}_a - \boldsymbol{L}_1$. Similarly $\boldsymbol{j}_c = \boldsymbol{j}_b - \boldsymbol{L}_2$. j_a, j_b $j_c\ldots$ are the nuclear spins of successive levels in a

cascade decay. We specify the quantum numbers A in successive steps: j_a, $A = j_a$, ξ_1, j_b, B where B stands for the remaining quantum numbers, if any. j_b, $B = j_b$, ξ_2, j_c, C. This procedure may be repeated until all emitted particles are described explicitly.

Since the efficiency matrix for the entire system is equal to the product of the efficiency matrices of the individual fragments we have

$$(j_a A \,|\, \epsilon^{n_a}_{k_a} | j'_a A') = \sum_{k_1 n_1} \sum_{k_b n_b} (j_b B \,|\, \epsilon^{n_b}_{k_b} | j'_b B') \, (\xi_1 \,|\, \epsilon^{n_1}_{k_1}(\Omega_1) \,|\, \xi'_1) \times$$

$$\times \, (j_a j'_a \,|\, \Gamma(j_b j'_b L_1 L'_1 k_a) \,|\, k_b k_1) \, (k_b k_1 n_b n_1 \,|\, k_a n_a) \tag{30}$$

where

$$(\xi \,|\, \epsilon^n_k \,|\, \xi') = \sum_{\sigma\sigma'} \sum_{n'} \epsilon_{\sigma\sigma'}(\bar{\omega}) \, (0\,\sigma \,|\, \xi\,\sigma) \, (0\,\sigma' \,|\, \xi'\,\sigma')$$

$$\times \, (-1)^{L'-\sigma'} \, (LL'\,\sigma - \sigma' \,|\, k\,n') \, D^k_{nn'}(\varphi, \vartheta, 0) \,. \tag{31}$$

The derivation of (31) is quite analogous to that of (20). $\bar{\omega}$ is the particle energy in the laboratory system. If the counter efficiency does not depend on the polarization of the emitted particle we have $\epsilon_{\sigma\sigma'}(\bar{\omega}) = \epsilon(\bar{\omega}) \, \delta_{\sigma\sigma'}$ and

$$(\xi \,|\, \epsilon^n_k \,|\, \xi') = \epsilon(\bar{\omega}) \, C_k(\xi, \xi') \, D^k_{n0}(\varphi, \vartheta, 0) \,. \tag{32}$$

According to our derivation Ω_1 in equ. (30) is the direction of propagation of particle 1 in the center of mass reference frame. For high energy nuclear reactions this reference frame is not identical with the laboratory frame. Ω_1 is then a function of the corresponding solid angle Φ_1 in the laboratory system depending on the energies of the incoming and outgoing particles ω_0 and ω_1. If no other particles are observed we have

$$(j_b B \,|\, \epsilon^{n_b}_{k_b} \,|\, j'_0 B') = \delta_{k_b 0} \, \delta_{n_b 0} \sqrt{2 j_b + 1} \, \delta_{j_b j_{b'}} \, \delta_{BB'} \,. \tag{33}$$

On the other hand if more particles are emitted and observed we proceed in analogy to (30)

$$(j_b B \,|\, \epsilon^{n_b}_{k_b} \,|\, j'_b B') = \sum_{k_2 n_2} \sum_{k_c n_c} (j_c C \,|\, \epsilon^{n_c}_{k_c} \,|\, j'_c C') \, (\xi_2 \,|\, \epsilon^{n_2}_{k_2}(\Omega_2) \,|\, \xi'_2) \times$$

$$(j_b j'_b \,|\, \Gamma(j_c j'_c L_2 L'_2 k_b) \,|\, k_c k_2) \, (k_c k_2 n_c n_2 \,|\, k_b n_b) \,. \tag{34}$$

If particle 2 is the last particle observed $\epsilon^{n_c}_{k_c}$ is given by (33) with proper change of subscripts, otherwise we may reduce it further

according to the scheme employed in (30) and (34). Ω_2 is the direction of the relative velocity of the particle 2 and the nucleus c. This is in general not identical with the direction Φ_2 observed in the laboratory system. Ω_2 is a function of Φ_2 depending on the energies ω_0, ω_1 and ω_2. Ω_1 and Ω_2 are the observed angles only if the recoil of the nucleus is negligible at all stages of the process.

IV. Summary of Results.

The development of the preceding sections allows us to construct explicit angular correlation functions from (7). We use two sets of subscripts to distinguish between the various particles and nuclei. Subscript $a, b, c, \ldots$ label respectively the entire system and the residual nuclei after emission of particles labeled by subscripts 1, 2, 3... The target nucleus in a nuclear reaction is labeled by the subscript t and the incident particle by 0. With (7) and (9) we write the general correlation function in the form

$$W = \text{const.} \int \delta\omega \sum_{j_a j_a'} \sum_{AA'} \sum_{A_0 A_0'} \sum_{k_a n_a} (j_a A \,|\, \epsilon_{k_a}^{n_a} \,|\, j_a' A') \times$$
$$\times (j_a A_0 \,|\, \bar{\varrho}_{k_a}^{n_a} \,|\, j_a' A_0')^* R_{j_a}^*(A, A_0)\, R_{j_a'}(A', A_0'). \tag{35}$$

$\int d\omega$ indicates integration over all energy variables. In the case of cascade decay of metastable nuclei $j_a = j_a'$ and $\bar{\varrho}_{k_a}^{n_a}$ is given by (11) if the nuclei are not aligned; otherwise $\bar{\varrho}_{k_a}^{n_a}$ describes the alignment. In the case of a nuclear reaction $\bar{\varrho}_{k_a}^{n_a}$ is given by (24) with (20) or (21). Numerical values for the I's can be found from (B6) and BIEDENHARN's Tables for the RACAH coefficients[12]).

Explicit expressions for $\epsilon_{k_a}^{n_a}$ are obtained in a series of steps described by eq. (30) ff. The energy variables ω_i appear in the following places: (1) the energy distribution in the beam $\varrho_{\sigma\sigma'}(\omega_0)$. (2) the energy dependence of the counter efficiencies $\epsilon_{\sigma\sigma'}(\bar{\omega}_i)$. (3) the energy dependence of the angles Ω_i introduced through the transformation to laboratory angles Φ_i. (4) the energy dependence of the matrix elements R. For sharp levels the energy dependence of the matrix elements can be effectively replaced by δ-functions; for broad levels some other plausible assumptions must be made. Selection rules and plausible assumptions about the matrix elements greatly simplify the formulae to be compared with experiment in concrete cases. The discussion of these aspects is, however, outside the scope of the present paper[13]).

Two examples should suffice to illustrate the form in which correlation functions appear. Let us first consider the directional correlation between two successive radiations from a metastable nucleus. Assuming that the intermediate level has a definite spin j_b we obtain with the rules described above

$$W(\Omega_1, \Omega_2) = \text{const.} \sum_{k_b n_b} \sum_{k_1 n_1} \sum_{k_2 n_2} \sum_{\xi_1, \xi_1'} \sum_{\xi_2, \xi_2'} C_{k_1}(\xi_1, \xi_1')\, C_{k_2}(\xi_2, \xi_2') \times$$

$$\times D^{k_1}_{n_1 0}(\varphi_1 \vartheta_1 0)\, D^{k_2}_{n_2 0}(\varphi_2 \vartheta_2 0)(j_a j_a | \Gamma(j_b j_b L_1 L_1' 0) | k_b k_1)\, (k_b k_1 n_b n_1 | 00) \times$$

$$\times (j_b j_b | \Gamma(j_c j_c L_2 L_2' k_b) | 0 k_2)\, (0 k_2 0 n_2 | k_b n_b) \times$$

$$\times R^*_{j_a}(\xi_1, \xi_2)\, R_{j_a}(\xi_1', \xi_2'). \tag{36}$$

With A(3), C(3) and C(4) there follows[14])

$$W(\Omega_1, \Omega_2) = \text{const.} \sum_k \sum_{\xi_1 \xi_1'} \sum_{\xi_2 \xi_2'} C_k(\xi_1 \xi_1')\, C_k(\xi_2, \xi_2')\, P_k(\cos \Theta) \times$$

$$\frac{(-1)^k}{\sqrt{2k+1}}\, (j_a j_a | \Gamma(j_b j_b L_1 L_1' 0) | k k) \times$$

$$\times (j_b j_b | \Gamma(j_c j_c L_2 L_2' k) | 0 k) R^*_{j_a}(\xi_1 \xi_2) R_{j_a}(\xi_1', \xi_2') \tag{37}$$

where Θ is the angle between the directions Ω_1 and Ω_2.

As a second example we take a nuclear reaction with two emerging particles. The beam is nonpolarized and the counters are insensitive to polarization[15]).

$$W(\Omega_0 \Omega_1 \Omega_2) = \text{const.} \int \delta\omega_0 \int \delta\omega_1 \int \delta\omega_2\, \varrho(\omega_0)\, \epsilon^{(1)}(\bar{\omega}_1)\, \epsilon^{(2)}(\bar{\omega}_2) \times$$

$$\times \delta(\omega_0 - \omega_1 - \omega_2 + \Delta E) \sum_{\xi_1 \xi_1'} \sum_{\xi_2 \xi_2'} \sum_{\xi_3 \xi_3'} \sum_{j_a j_a'} \sum_{j_b j_b'} \sum_{k_0} \sum_{k_1} \sum_{k_2} C_{k_0}(\xi_0 \xi_0') \times$$

$$\times C_{k_1}(\xi_1 \xi_1')\, C_{k_2}(\xi_2 \xi_2')\, D^{k_0 *}_{n_0 0}(\varphi_0 \vartheta_0 0)\, D^{k_1}_{n_1 0}(\varphi_1 \vartheta_1 0)\, D^{k_2}_{n_2 0}(\varphi_2 \vartheta_2 0) \times$$

$$\times (j_a j_a' | \Gamma(j_t j_t L_0 L_0' k_0) | 0 k_0)\, (j_a j_a' | \Gamma(j_b j_b' L_1 L_1' k_0) | k_2 k_1) \times$$

$$\times (k_2 k_1 n_2 n_1 | k_0 n_0)\, (j_b j_b' | \Gamma(j_c j_c L_2 L_2' k_2) | 0 k_2)\, R^*_{j_a}(\xi_0 \xi_1 \xi_2 j_b) \times$$

$$\times R_{j_a'}(\xi_0' \xi_1' \xi_2' j_b'). \tag{38}$$

V. Internal Conversion.

According to the theory of internal conversion[16]) the matrix element for the ejection of a conversion electron can be written in the form

$$\sum_{LM\mathfrak{p}} (\Omega\sigma\,|\,\alpha\,|\,LM\mathfrak{p}\,i)\,(LM\mathfrak{p}\,b\,|\,H\,|\,a) \tag{39}$$

where i denotes the initial state of the electron. a and b are nuclear states and $(LM\,\mathfrak{p}\,b|H|\,a)$ is the matrix element for the emission of a photon in the transition $a \to b$. α depends only on an electronic wave function and universal functions but not on nuclear wave functions. Since

$$\sum_i (\Omega\sigma\,|\,\alpha\,|\,LM\mathfrak{p}\,i)^*\,(\Omega\sigma'\,|\,\alpha\,|\,L'M'\,\mathfrak{p}'\,i) =$$

$$= \sum_{NN'} \sum_i (0\sigma\,|\,\alpha\,|LN\,\mathfrak{p}\,i)^*(0\sigma'\,|\alpha|L'N'\,\mathfrak{p}'\,i) D^L_{MN}(\varphi\,\vartheta\,0)\,D^{L'}_{MN'}(\varphi\,\vartheta\,0)^* \tag{40}$$

we may apply all previous results to correlations with conversion electrons provided we replace (31) by

$$(\xi\,|\,\in^n_k\,|\,\xi') = \sum_{\sigma\sigma'} \sum_{NN'} \sum_{n'} \sum_i \in_{\sigma\sigma'} (0\sigma\,|\,\alpha\,|\,LN\,\mathfrak{p}\,i)\,(0\sigma\,|\,\alpha\,|\,L'N'\,\mathfrak{p}'\,i) \times$$

$$\times (-1)^{L'-N'}\,(LL'\,N-N'\,|\,k\,n')\,D^k_{n\,n'}(\varphi\,\vartheta\,0)\,. \tag{41}$$

If the polarization of the conversion electron is not observed $\in_{\sigma\sigma'} = \delta_{\sigma\sigma'}$. For K-conversion of pure multipole radiation the coefficients in (41) have been calculated by Rose, Biedenharn and Arfken[17]). The result can be written in the form

$$(\xi\,|\,\in^n_k\,|\,\xi) = b_k(\xi)\,C_k(\xi_1\,\xi)\,D^k_{n0}(\varphi\,\vartheta\,0) \tag{42}$$

where C_k is given by (23) and b_k is defined by

$$b_k = \frac{4\pi \sum_\sigma \sum_N \sum_i |(0\sigma\,|\,\alpha\,|\,LN\,\mathfrak{p}\,i)|^2\,(LLN-N\,|\,k\,0)\,(-1)^{L-N}}{(2L+1)\,(-1)^{L+1}\,(LL\,1-1\,|\,k\,0)}\,. \tag{43}$$

The coefficient b_k denoted b_ν by Rose, Biedenharn and Arfken has been evaluated numerically. $\sum_i$ in (43) is taken over K-electrons only.

Appendix A.

Clebsch Gordan Coefficients[18][19].

The Clebsch Gordan coeffcients for the addition of angular momenta $\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{c}$ are denoted by $(ab\,\alpha\beta\,|\,c\gamma)$. α, β, γ are the 3-components of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$. $(ab\,\alpha\beta\,|\,c\gamma)$ vanishes unless $\alpha+\beta=\gamma$ and c is one of the numbers $a+b,\ a+b-1,\ \ldots,\ |a-b|$. The Clebsch Gordan coefficients are real and satisfy the orthogonality relations

$$\sum_{\alpha\beta}(ab\alpha\beta\,|\,c\gamma)\,(ab\alpha\beta\,|\,c'\gamma') = \delta_{cc'}\,\delta_{\gamma\gamma'} \tag{A1}$$

and

$$\sum_{c}(ab\alpha\beta\,|\,c\gamma)\,(ab\alpha'\beta'\,|\,c\gamma) = \delta_{\alpha\alpha'}\,\delta_{\beta\beta'}\,\delta_{\gamma\alpha+\beta}\,. \tag{A2}$$

They also satisfy the following relations

$$(ab\alpha\beta\,|\,c\gamma) = (-1)^{a+b-c}(ba\beta\alpha\,|\,c\gamma) = (-1)^{a+b-c}(ab{-}\alpha{-}\beta\,|\,c{-}\gamma) \tag{A3}$$

and

$$(ab\alpha\beta\,|\,c\gamma) = \sqrt{\frac{2c+1}{2b+1}}\,(-1)^{a-\alpha}\,(ac\alpha{-}\gamma\,|\,b{-}\beta) =$$

$$= \sqrt{\frac{2c+1}{2a+1}}\,(-1)^{b+\beta}\,(cb{-}\gamma\,\beta\,|\,a{-}\alpha)\,. \tag{A4}$$

Appendix B.

Racah Coefficients[20][13] and Transformations between Angular Momentum Coupling Schemas.

The Racah coefficient $W(abcd;\,ef)$ is related to the Clebsch Gordan coefficients by

$$W(abcd;\,ef)\,\delta_{cc'}\,\delta_{\gamma\gamma'} = \sum_{\alpha\beta\delta}\sum_{\varepsilon\varphi}\frac{1}{\sqrt{(2l+1)\,(2f+1)}}\,(ab\alpha\beta\,|\,e\varepsilon)\times$$

$$\times\,(ed\varepsilon\delta\,|\,c\gamma)\,(bd\beta\delta\,|\,f\varphi)\,(af\alpha\varphi\,|\,c'\gamma')\,. \tag{B1}$$

It satisfies the symmetry relations

$$W(abcd;\,ef) = W(badc;\,ef) = W(cdab;\,ef) =$$

$$= W(acbd;\,fe) = (-1)^{b+c-e-f}\,W(aefd;\,bc) \tag{B2}$$

and the sum rule

$$\sum_{g}(2g+1)\,(-1)^{a+b+c+d-e-f-g}\,W(abdc;\,eg)\;W(acdb;\,fg) =$$

$$= W(abcd;\,ef)\,. \tag{B3}$$

If any one variable vanishes the value of the RACAH coefficient can be written down immediately

$$W(abcd;\,0f) = \frac{(-1)^{b+c-f}\,\delta_{ab}\,\delta_{cd}}{\sqrt{(2b+1)\,(2c+1)}}\,. \tag{B4}$$

Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ be three angular momentum vectors; $\boldsymbol{d} = \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$; $\boldsymbol{e} = \boldsymbol{a} + \boldsymbol{b}$, $\boldsymbol{f} = \boldsymbol{a} + \boldsymbol{c}$. The transformation matrix between the coupling schemes $abecd$ and $acfbd$ is given by

$$\begin{aligned}(e\,|\,\Gamma(abcd)\,|\,f) = \sum_{\alpha\beta\gamma}\sum_{\varepsilon\varphi}(ba\,\beta\alpha\,|\,e\varepsilon)\,(ec\,\varepsilon\gamma\,|\,d\delta)\times\\ \times\,(ac\,\alpha\gamma\,|\,f\varphi)\,(bf\,\beta\varphi\,|\,d\delta) = \sqrt{(2e+1)\,(2f+1)}\;W(abcd;\,ef)\,.\end{aligned} \tag{B5}$$

Let $\boldsymbol{a}, \boldsymbol{a}'\ \boldsymbol{b}\ \boldsymbol{b}'$ be four angular momentum vectors $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$; $\boldsymbol{a}' + \boldsymbol{b}' = \boldsymbol{c}'$; $\boldsymbol{c} - \boldsymbol{c}' = \boldsymbol{d}$; $\boldsymbol{a} - \boldsymbol{a}' = \boldsymbol{e}$; $\boldsymbol{b} - \boldsymbol{b}' = \boldsymbol{f}$. The transformation matrix between the couling schemes $abc\ a'b'c'\ d$ and $aa'e\ bb'fd$ is given by

$$\begin{aligned}(cc'\,|\,\Gamma(aa'\,bb'\,d)\,|\,ef) = \sum_{a\alpha'}\sum_{\beta\beta'}\sum_{\gamma\gamma'}\sum_{\varepsilon\varphi}(ab\,\alpha\beta\,|\,c\gamma)\,(a'b'\,\alpha'\beta'\,|\,c'\gamma')\times\\ \times\,(-1)^{a'-\alpha'}\,(aa'\,\alpha-\alpha'\,|\,e\varepsilon)\,(-1)^{b'-\beta'}\,(bb'\,\beta-\beta'\,|\,f\varphi)\times\\ \times\,(-1)^{c'-\gamma'}\,(cc'\,\gamma-\gamma'\,|\,d\delta)\,(ef\,\varepsilon\varphi\,|\,d\delta) =\\ = \sqrt{(2e+1)\,(2f+1)\,(2c+1)\,(2c'+1)}\sum_{\varkappa}(2\varkappa+1)\;W(c'cef;\,d\varkappa) =\\ \times\,W(acbf';\,b\varkappa)\;W(b'c'ae;\,a'\varkappa)\,.\end{aligned} \tag{B6}$$

Appendix C.

The Representations of the Rotation Group.

We list here some often used formulae involving the representations $D^L_{MM'}(\varphi, \vartheta, \chi)$ of the rotation group. If $F^M_L(\Omega)$ is any irreducible tensor depending on the direction $\Omega \equiv (\vartheta, \varphi)$ we have the relation

$$F^M_L(\Omega) = \sum_{M'} F^{M'}_L(0)\,D^L_{MM'}(\varphi, \vartheta, 0)^*. \tag{C1}$$

This holds in particular for spherical harmonics and with

$$Y^M_L(0) = \delta_{M0}\sqrt{\frac{2L+1}{4\pi}} \tag{C2}$$

there follows

$$D^L_{M0}(\varphi\,\vartheta\,\chi)^* = \sqrt{\frac{4\pi}{2L+1}}\;Y^M_L(\vartheta, \varphi) \qquad D^L_{00}(\varphi\,\vartheta\,\chi) = P_L(\cos\vartheta) \tag{C3}$$

where $P_L(\cos\vartheta)$ is a Legendre Polynomial. $D^L_{MM'}$ satisfies the relations

$$D^L_{MM'}(\varphi\vartheta\chi)^* = (-1)^{M-M'} D^L_{-M-M'}(\varphi\vartheta\chi) \tag{C4}$$

and

$$D^{L_1}_{M_1M_1'}(\varphi\vartheta\chi)\, D^{L_2}_{M_2M_2'}(\varphi\vartheta\chi) =$$

$$= \sum_L \sum_{MM'} (L_1 L_2 M_1 M_2 \,|\, LM)\,(L_1 L_2 M_1' M_2' \,|\, LM')\, D^L_{MM'}(\varphi\vartheta\chi)\ . \tag{C5}$$

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[4]) For a definition of the density matrix and a definition of its formal properties see for instance R. C. Tolman "Statistical Mechanics", Oxford, Chapter IX.

[5]) Recently U. Fano (National Bureau of Standards Rep. 1214 and Phys. Rev. to be published) has investigated the geometric properties of density matrices describing nuclei in successive levels of a cascade decay and discussed the bearing of these properties on the derivation of angular correlation functions. (In a special case this approach has also been used by H. A. Tolhoek and S. R. de Groot, Phys. Rev. **83**, 189, 845 (L) (1951).) The present work was done mostly without knowledge of Fano's results. Only the initial and final states of the overall reaction are described explicitly by density matrices. The development is carried through to explicit correlation functions of greater generality than those indicated by Fano's discussion. We are indebted to Dr. Fano for sending us his manuscript prior to publication.

[6]) See Appendix A.

[7]) J. A. Wheeler, Phys. Rev. **52**, 1107 (1937). – W. Heisenberg, Z. f. Phys. **120**, 513, 673 (1943).

[8]) The criteria for the validity of this assumption have been discussed by G. Goertzel, Phys. Rev. **70**, 897 (1946). – K. Alder, Helv. Phys. Acta **25**, 235 (1952) and H. Frauenfelder, Annual Review of Nuclear Science **3**, (1953).

[9]) This follows directly in a representation in which both ϱ and $\bar{\varrho}$ are diagonal; (8) itself is then obtained by appropriate change of representation. See also F. Coester, Phys. Rev. to be published, eq. (19) and L. Wolfenstein and J. Ashkin, Phys. Rev. **85**, 947 (1952). The notation $R_j(A, A_0)$ instead of $(A \mid R_j \mid A_0)$ will be convenient in the following.

[10]) Some often used properties of $D^L_{MM'}$ are listed for convenience in Appendix C.

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