Nuclear Spectroscopy with Harmonic Oscillator Wave-Functions

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Zusammenfassung: Die Beschränkung auf harmonische Zentralpotentiale ermöglicht die Entwicklung einer neuen (vom üblichen Slaterschen Verfahren verschiedenen) Methode zur Berechnung der Matrixelemente der Wechselwirkung zwischen den äusseren Nukleonen im Schalenmodell, welche insbesondere auch den Fall nichtzentraler Kräfte in einfacher Weise zu behandeln gestattet. Als Anwendung wird gerechnet: Die Ordnung der Niveaus in der j^n-Konfiguration, für einige physikalisch interessante Fälle; die von der Spin-Bahn-Kopplung zwischen den Nukleonen herrührende Dublettaufspaltung, und die Lage der untersten Terme von Li^7 bei Berücksichtigung von Spin-Bahn-Kopplung und Tensorkräften.

Introduction.

The discovery of the "magic numbers" and the general success of the shell model in explaining many experimental facts about nuclei have led theoretical physicists to work with the methods of atomic spectroscopy in the calculation of nuclear levels. In this attempt three main difficulties are encountered:

(a) The form of the interaction potential and its exchange character are not sufficiently known. In addition, the form of the wave-functions can be only guessed. These facts make necessary the examination of various forms of the potential.

(b) There exists a lack of sufficient experimental information about nuclear spectra. There are only a few excited states for which the spin and parity assignments have been carried out. Whereas in atomic spectroscopy it was possible from the vast amount of experimental material to determine easily the several parameters used to describe the atomic levels, here these parameters must be mathematically evaluated.

(c) Non-central interactions, such as tensor forces, must here be taken into account, whereas in atomic spectra such interactions give a negligible small contribution.

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Thus, the Slater method, used in atomic spectroscopy, is not practical in nuclear spectroscopy, because:

(a) one must work with different potentials, each of which must be expanded in a series of Legendre polynomials, for which the coefficients are usually complicated functions;

(b) this fact makes impractical the calculation of the matrix elements, which now must be completely carried out; and

(c) for non-central interactions the method looses even its formal simplicity and becomes very much involved.

In the first part of this paper an alternative method is suggested and developed. Although this method is confined to the use of harmonic oscillator wave-functions, this limitation does not seem to be too serious in view of the fact that the form of the wave-functions cannot be determined if the interaction is not known. We use these wave-functions only as a model in order to learn the results for various types of interactions. That our choice of wave-functions is reasonable is seen from the rough agreement of the level order in the oscillator potential with that found experimentally. In any case, one would not expect the exact form of the wave-functions to have a decisive effect on the results. Due to the separability of the potential of two nucleons moving in the central field of a (3-dimensional) harmonic oscillator in the relative and center-of-gravity coordinates, it is possible to write the wave-functions of two interacting nucleons as a finite sum of products of functions which depend on these co-ordinates. This transformation makes it possible, when calculating matrix elements to integrate immediately over the center-of-gravity coordinate, of which the interaction energy is usually independent. What remains is only to calculate matrix elements of the interaction energy (which may be a complicated function of the relative co-ordinate and of the relative momentum), in a scheme of wave-functions written in terms of this coordinate.

After a survey of the shell model and the Slater method (§ 1, 2), the proposed method is described and discussed (§ 3). In § 4 the transformation of the wave-functions from the coordinates of the two nucleons to the relative and center-of-gravity coordinates is considered. The Slater method can be regarded as an approximation procedure in the neighbourhood of the long range limit, whereas our method can serve as a good approximation method in the short range limit, which might be looked upon as the proper approximation in the case of nuclear spectroscopy (this point is discussed in § 6).

In the second part of this paper the determination of the ground state of the j^n configuration is treated. An empirical rule, stated by MAYER[1]), is that the total angular momentum J of the ground state is equal to zero if n is even, and equal to j if n is odd. This rule was then derived theoretically by MAYER[2]) and RACAH[3]) for Majorana forces in the case of δ-type interaction, or, better said, in the short range approximation. A few cases, however, were found experimentally where this rule does not hold. KURATH[4]) and others have suggested that these deviations could be explained by considering the effect of the finite range of the forces. It has already been pointed out by the author[5]) that potentials more singular than that used by KURATH give different results. With the method described in this paper this problem can be treated quite generally. After the examination of several interaction potentials it can be concluded that for physically admissible values of the parameters (range of the forces and extension of the wave-functions), the order of levels is that given by the short range approximation. The break down of the above rule may be due to perturbations from other configurations or to the action of tensor forces.

In § 5 we present the problem and show how our method may be applied to obtain directly the values of the energy levels which are already calculated in the Slater method. The dependence of the results on the form and range of the potential is treated in § 6 for the $(d_{5/2})^3$ configuration. In § 7 the configuration $(f_{7/2})^3$ is treated and discussed. The configuration $(g_{9/2})^3$ is treated in § 8. In several nuclei in which these configurations appear (according to the shell model), the spin of the ground state is not that predicted by the jj-coupling scheme in the short range limit. These cases are especially interesting as they may give direct information about the nuclear interactions (in contrast to the other odd-even nuclei whose level schemes have been classified, in which only the states of the single nucleon are observed, thus giving information about the central field only and providing no direct information about the nuclear forces).

The third part of this paper deals with mutual spin-orbit interactions. An interesting problem is whether the spin-orbit interaction, introduced by CASE and PAIS[6]) in order to explain the results of scattering experiments, can give for heavier nuclei sufficientsplitting between the states $j = l + 1/2$ and $j = l - 1/2$ (of a single nucleon) to satisfy the requirements of the shell model. We state the problem and calculate matrix elements of the mutual spin-orbit interaction in § 9. A few cases of a single nucleon outside closed shells are treated

in § 10, and the resulting doublet splitting is calculated. With reasonable values of the parameters an order of magnitude of a few MeV is obtained. The results are, however, very sensitive to the nuclear radius in the interesting region.

In the fourth part of the paper the Li[7] nucleus is discussed. In this nucleus only one excited state has been found in the energy region between the ground state and several MeV. As early experiments showed that this excited level has a compound nature, it was suggested by INGLIS[7]) that it may be composed of the four excited levels of the jj-coupling configuration $(p_{3/2})_N^3 \, p_{3/2\,P}$ which may perhaps coincide (but he was not able, using only central forces, to obtain such a coincidence). Our method was used to take account also of tensor forces and to check whether their contribution may bring coincidence of those excited levels. The $(p_{3/2})_N \, p_{3/2\,P}$ configuration is treated in § 12 for the case of central interactions with the various exchange operators. The term values were calculated by the sum method combined with the use of the quantum number of the total isotopic spin. In § 13 the matrix elements of the tensor forces interaction for this configuration are calculated. The results (§ 14) show that, for reasonable values of the interaction parameters it is possible to obtain such a coincidence.

As recent experiments have shown that a spin 1/2 for the excited level is compatible with the known facts[8]), the natural explanation arose that the two lowest states are the components of a ^{22}P state (in LS-coupling). LS-coupling for Li[7] is not in contradiction with the shell model, as Li[7] is a light nucleus; it is believed that only for heavier nuclei does jj-coupling take place. It is known that neither the Thomas interaction nor the magnetic spin-orbit interaction can give sufficient splitting between the components of the ^{22}P ground state. The splitting which results from the Case and Pais interaction is calculated for the ^{22}P state of the configuration $p_N^2 \, p_P$ (§ 15). The result is of the right order of magnitude but the splitting obtained is again very sensitive to the nuclear dimensions.

I. The Method of the Harmonic Oscillator Wave-Functions.

§ 1. *The shell model.*

The term "nuclear spectroscopy" is well defined in experimental physics; it means the measurement of nuclear spectra, their classification, and level assignment. On the other hand in theoretical physics this term does not usually mean "the theory of nuclear spectra", but is more limited. It is mostly used in the sense of "the use of the methods of atomic spectroscopy in explaining nuclear spectra".

In the past many attempts have been made to apply the methods of atomic spectroscopy to the calculation of nuclear energy levels. Meanwhile more experimental material has been gathered, especially in the region of high excitation energies, and the theories which successfully explained these facts were statistical in their nature. Only in the last few years has it become clear that there are regularities in the nuclei which are not of a statistical nature (magic numbers). Various "shell models" (which are in fact the former Hartree approximations) have been proposed, the most successful of which is the one based on the strong spin-orbit interaction[1][9]. This model does not only explain the magic numbers and the spins of the ground states of nuclei, but can also be applied to the classification of level schemes in the low energy region[10]. It appears that experiment now forces the theoretician to accept a theory which he earlier abandoned and for the validity of which there are not yet sufficient theoretical grounds.

The basic assumption of any shell model is that the average field which acts on a single nucleon can be approximated by a central field in which the nucleon has a definite state (characterized by quantum numbers n and l). The nucleons occupy the lowest states and may form closed shells. A magic nucleus is a nucleus which contains closed shells only. The characteristics of the ground state are determined by the interaction of the nucleons outside the closed shells. This interaction energy is usually taken as a perturbation on the zero order energy, which is the sum of the energies of the single nucleon in the central field. The zero order energy is the same for all the states of a given configuration (i.e., the nucleons outside closed shells occupy definite states of the central field). In addition to the central field acting on the nucleons there can also be an interaction of the spin of a nucleon with its orbital angular momentum (the usual spin-orbit interaction). If this is large compared to the mutual interaction of the nucleons, the spins (i.e. total angular momenta) of the nucleons j are good quantum numbers. Thus every nucleon has a definite n, l, and j. This is the jj-coupling scheme which is postulated in the shell model of M. G. Mayer.

To the zeroth order the wave-functions of the nucleons are products of the wave functions of the single nucleon in the central field. The wave-functions of the nucleons in the states $n_1 l_1$, $n_2 l_2$, ..., $n_m l_m$, with other quantum numbers a_1, a_2, ..., a_m (where a, for example, could be the magnetic quantum number m) is:

$$\psi_A(x_1, x_2, \ldots, x_m) = \psi_{a_1 n_1 l_1}(x_1)\, \psi_{a_2 n_2 l_2}(x_2) \cdots \psi_{a_m n_m l_m}(x_m). \tag{1}$$

The zero order energy is degenerate, since it is the same for all the states of the configuration. If the mutual interaction of the nucleons is taken into account this degeneracy is partly removed, as known from perturbation theory. The first order energies are then the eigenvalues of that part of the interaction energy which belongs to the configuration.

The appropriate zero-order wave-functions are definite linear combinations of the wave functions ψ_A determined by the quantum numbers of the interaction energy (these are, for example L and S in the case of Russel Saunders coupling, J in the case of jj-coupling, etc.). Therefore the matrix elements of the interaction energy is a sum of matrix elements $(A \mid V \mid B)$. The interaction energy has the form $\sum\limits_{i<j} V_{ij}$ where $V_{ij} = V_{ij}(x_i, x_j)$ describes the interaction of the i-th and j-th nucleons. In evaluating the matrix elements of one of these terms we can integrate over the coordinates x_l, $l \neq i$, $l \neq j$, and obtain:

$$
\left.
\begin{aligned}
&\int\int \cdots \int \psi_{a_1}^*(x_1)\,\psi_{a_2}^*(x_2)\ldots\psi_{a_m}^*(x_m)\,V_{ij}(x_i,x_j)\,\psi_{b_1}(x_1)\,\psi_{b_2}(x_2)\ldots \\
&\ldots\psi_{b_m}(x_m)\,dx_1\,dx_2\ldots dx_m = \delta_{a_1 b_1}\,\delta_{a_2 b_2}\ldots \\
&\ldots\delta_{a_{i-1}b_{i-1}}\,\delta_{a_{i+1}b_{i+1}}\ldots\delta_{a_{j-1}b_{j-1}}\,\delta_{a_{j+1}b_{j+1}}\ldots\delta_{a_m b_m}\times \\
&\times \int\int \psi_{a_i}^*(x_i)\,\psi_{a_j}^*(x_j)\,V_{ij}(x_i,x_j)\,\psi_{b_i}(x_i)\,\psi_{b_j}(x_j)\,dx_i\,dx_j .
\end{aligned}
\right\} \quad (2)
$$

Therefore the matrix elements of V in the zero-order scheme are the sums of matrix elements of the type (2). The coefficients depend only on the operators commuting with the interaction energy (quantum numbers) and may be calculated without the explicit knowledge of the interaction, as they are independent of the exact form of the wave functions. There are well developed methods for obtaining them, as given by CONDON and SHORTLEY[11], G. RACAH[12] and others. The explicit dependence on the form of the interaction is contained in the matrix elements which we shall consider in some detail.

§ 2. *Survey of the Slater Method.*

The important feature of the matrix element

$$
\int\int \psi_{a_1}^*(\vec{r_1})\,\psi_{a_2}^*(\vec{r_2})\,V_{12}(\vec{r_1}-\vec{r_2})\,\psi_{b_1}(\vec{r_1})\,\psi_{b_2}(\vec{r_2})\,d^3r_1\,d^3r_2 \qquad (3)
$$

is that the ψ are functions of $\vec{r_1}$ and $\vec{r_2}$, and V_{12} is a function of $\vec{r_1} - \vec{r_2}$. The usual way of treating such expressions in atomic spectroscopy was developed by J. SLATER[13], who expressed $V(\vec{r_1} - \vec{r_2})$ as

a function of $\vec{r_1}$ and $\vec{r_2}$. The procedure is to expand $V(\vec{r_1} - \vec{r_2})$ in a series of Legendre polynomials, the argument of which is the angle ω_{12} between $\vec{r_1}$ and $\vec{r_2}$ with coefficients which are functions of $|\vec{r_1}|$ and $|\vec{r_2}|$:

$$V(\vec{r_1} - \vec{r_2}) = \sum_{k=0}^{\infty} f_k(r_1, r_2) \, P_k(\cos \omega_{12}) . \tag{4}$$

$P_k(\cos \omega_{12})$ can be expanded in a finite sum of products of spherical harmonics which are functions of ϑ_1, φ_1 and ϑ_2, φ_2 respectively (these are the polar angles of $\vec{r_1}$ and $\vec{r_2}$). At this stage the angular integrations can be performed, and a sum of radial integrals

$$R^k(a_1, a_2, b_1, b_2) = \int \int R_{a_1}(r_1) R_{a_2}(r_2) f_k(r_1, r_2) R_{b_1}(r_1) R_{b_2}(r_2) \, dr_1 dr_2 \tag{5}$$

remains, the coefficients of which are products of two integrals of the type $\int \Theta_k \Theta_l \Theta_{l'} d \cos \vartheta$ (only finite number of these coefficients do not vanish for definite a_1, a_2, b_1, b_2). We shall mention in this work the F^k only, but all the considerations are true mutatis mutandis for the G^k too

$$[F^k(a_1, a_2) = R^k(a_1, a_2, a_1, a_2) \quad G^k(a_1, a_2) = R^k(a_1, a_2, a_2, a_1)] .$$

This procedure is very useful in atomic spectroscopy because there the interaction is given by $V_{12} = e^2/|\vec{r_1} - \vec{r_2}|$ and there exists the simple expansion:

$$\frac{1}{|\vec{r_1} - \vec{r_2}|} = \sum_{k=0}^{\infty} \frac{r_<^k}{r_>^{k+1}} P_k(\cos \omega_{12}) \quad r_< = \text{Min} \, (r_1, r_2) \quad r_> = \text{Max} \, (r_1, r_2).$$

The wave functions are not easy to compute, and their radial part is the only indeterminate factor of the F^k. In fact, the atomic spectroscopists did not bother themselves with the calculations of the F^k, as in any case many energy levels were measured, so that it was easy to evaluate the F^k from some of them, and check whether these values of the Slater parameters really gave the other levels as well. In nuclear spectroscopy, on the other hand, this is impossible as there are usually only very few levels measured and classified. Therefore if one is to use the Slater-method, the F^k must be mathematically evaluated. Even for the central forces, however, we do not know the exact form of the potential. We should therefore calculate the energy levels for different forms of the potential. However even in simple cases such as the Yukawa potential, the calculation of the F_k is so complicated that Slater-method is of little practical value.

For complicated interactions, such as tensor forces and mutual spin-orbit interaction, the expansion is very complicated and of practically no value (an example of such an expansion is given in a paper of MARVIN[14]).

§ 3. *The alternative method for the oscillator wave functions.*

To overcome these difficulties we make use of the fact that the interaction energy depends only on the relative coordinate $\vec{r} = \vec{r_2} - \vec{r_1}$, and our procedure is to express also the wave functions as functions of $\vec{r} = \vec{r_2} - \vec{r_1}$ and the other coordinate $\vec{R} = (\vec{r_1} + \vec{r_2})/2$ (the coordinate of the center of gravity of the 1$^{\text{st}}$ and 2$^{\text{nd}}$ nucleons). This transformation enables us, when calculating matrix elements, to integrate immediately with respect to R, and we are left with a single integration of $\int \psi_1^*(\vec{r}) \, V(\vec{r}) \, \psi_2(\vec{r}) \, d^3r$ which can usually be carried out without difficulties.

This coordinate transformation is always possible, but the functions of $\vec{r}$ and $\vec{R}$ generally turn out to be very complicated. The success of this procedure depends on the proper choice of the wave-functions. The best choice would be that one which allows us to expand the wave-function $\psi_{a_1}(\vec{r}_1) \, \psi_{a_2}(\vec{r}_2)$ in a finite sum of products of functions which depend on $\vec{r}$ and $\vec{R}$ respectively. This would certainly be the case if the potential of the central field, which is the sum $V_c(r_1^2) + V_c(r_2^2)$, is also separable in the coordinates r and R. The condition for such a decomposition is:

$$V_c(r_1^2) + V_c(r_2^2) = U(R^2) + W(r^2) . \tag{6}$$

Putting $r = 0$ we obtain:

$$2 \, V_c(R^2) = U(R^2) + \text{const.}$$

and putting $R = 0$ we obtain:

$$2 \, V_c(r^2) = W(4 \, r^2) + \text{const.}$$

If we now put in (6) $r_1^2 = r_2^2$ so that $R^2 = r_1^2 \cos^2 \alpha$, $r^2 = 4 \, r_1 \sin^2 \alpha$ where 2α is the angle between $\vec{r}_1$ and $\vec{r}_2$ we obtain:

$$2 \, V_c(r_1^2) = 2 \, V_c(r_1^2 \cos^2 \alpha) + 2 \, V_c\left(\frac{4 \, r_1^2 \sin^2 \alpha}{4}\right) + \text{const.}$$

Differentiation with respect to α yields:

$$0 = - 2 \, r_1^2 \cos \alpha \sin \alpha \, V_c'(r_1^2 \cos^2 \alpha) + 2 \, r_1^2 \sin \alpha \cos \alpha \, V_c'(r_1^2 \sin^2 \alpha)$$

or:

$$V_c'(r_1^2 \cos^2 \alpha) = V_c'(r_1^2 \sin^2 \alpha)$$

which means $V_c'(r^2) = \text{const.} = A$, i.e. $V_c(r^2) = A\,r^2 + \text{const.}$ Therefore the only potential (possessing a derivative) which satisfies this condition is the harmonic oscillator potential:

$$\omega\,r_1^2 + \omega\,r_2^2 = \frac{1}{2}\left(4\,\omega^2\,R^2 + \omega^2\,r^2\right).$$

Taking the wave functions of the harmonic oscillator we can decompose every wave-function $\psi_{a_1}(\vec{r_1})\,\psi_{a_2}(\vec{r_2})$ into a finite sum of terms of the form $\psi_1(\vec{R})\,\psi_2(\vec{r})$ (where ψ_1, ψ_2 are essentially functions of the same kind as ψ_{a_1}, ψ_{a_2}).

In order to utilize this method we must, therefore, use a very special form of the wave functions. This is not such a serious limitation, however, as the exact form of the potential and hence of the wave-functions, is not accurately known. We shall therefore be satisfied with the harmonic oscillator wave-functions, using them as model wave-functions, in order to learn the behaviour of, say, the energy levels in terms of the nuclear radius, range, magnitude and form of the potential.

Compared with other model wave-functions, it seems that, physically, those of the harmonic oscillator are quite good, as the order of levels in the shell model is roughly that of an oscillator potential well.

The potential of a single nucleon bound harmonically, besides being in a central potential, is separable also in Cartesian coordinates. Therefore every function of the three dimensional oscillator can be given as a product of three wave-functions of the one dimensional oscillator depending on the Cartesian coordinates x, y, and z. This representation of the wave-functions has been extensively used[15][16]. The coordinates used in this work are, however, the polar coordinates, and the wave functions are therefore a product of a function of r and a spherical harmonic.

The radial parts of the wave functions

$$\psi\,(r, \vartheta, \varphi) = \frac{R(r)}{r}\;Y_l^{(m)}\,(\vartheta, \varphi)$$

are easily calculated, and can be shown to be of the form:

$$R_{nl}(r) = N_{nl}\,e^{-\frac{\nu}{2}\,r^2}\,r^{l+1}\,v_{nl}\,(r)$$

where N_{nl} is a normalization factor, and v_{nl} is an associated Laguerre polynomial[17] (properly normalized):

$$v_{nl}\,(r) = L_{n+l+\frac{1}{2}}^{l+\frac{1}{2}}(\nu\,r^2)$$

v is given by $v = \omega m/h$, and m and ω appear in the Hamiltonian of the harmonic oscillator:

$$H = \frac{1}{2m} \left(p^2 + m^2 \omega^2 r^2 \right).$$

The explicit expression for these $L_{n+l+1/2}^{l+1/2}$ is

$$L_{n+l+\frac{1}{2}}^{l+\frac{1}{2}}(\varrho) = \sum_{k=0}^{n} (-1)^k \, 2^k \binom{n}{k} \frac{(2l+1)!!}{(2l+2k+1)!!} \, \varrho^k.$$

With this definition the normalization factor becomes:

$$N_{nl}^2 = \frac{2^{l-n+2} (2l+2n+1)!! \, v^{l+3/2}}{\sqrt{\pi} \, n! \, [(2l+1)!!]^2}.$$

The first few functions are:

for $n = 0$

$$R_l(r) = N_l \, e^{-\frac{v}{2} r^2} \, r^{l+1} \qquad \text{and} \qquad N_l^2 = \frac{\sqrt{v} \, v^{l+1} \, 2^{l+2}}{\sqrt{\pi} \, 1 \cdot 3 \cdots (1+2l)},$$

for $n = 1$

$$v_{1l} = 1 - \frac{2v}{2l+3} r^2 \qquad \text{and} \qquad N_{1l}^2 = \frac{\sqrt{v} \, v^{l+1} \, 2^{l+1} \, (2l+3)}{\sqrt{\pi} \, 1 \cdot 3 \cdots (1+2l)},$$

for $n = 2$

$$v_{2l} = 1 - \frac{4v}{2l+3} r^2 + \frac{4v^2}{(2l+3)(2l+5)} r^4$$

$$\text{and} \quad N_{2l}^2 = \frac{\sqrt{v} \, v^{l+1} \, 2^{l-1} \, (2l+3)(2l+5)}{\sqrt{\pi} \, 1 \cdot 3 \cdots (1+2l)}.$$

These are the only wave-functions which will be used in the following. All of them satisfy:

$$\int_0^\infty R_{nl}^2(r) \, dr = 1;$$

the $Y_l^{(m)}(\vartheta, \varphi)$ should therefore also be normalized to unity.

We shall have to calculate integrals of the form

$$I_{nl} = \int_0^\infty R_{nl}^2(r) \, V(r) \, dr$$

The integrals I_{nl} with $n \neq 0$ can be expressed as sums of integrals I_{0l} which we shall write simply as I_l. The respective formulae are:

$$I_{1l} = \frac{2l+3}{2} I_l - (2l+3) I_{l+1} + \frac{2l+5}{2} I_{l+2}$$

$$I_{2l} = \frac{(2l+3)(2l+5)}{8} I_l - \frac{(2l+3)(2l+5)}{2} I_{l+1} + \frac{(2l+5)(6l+13)}{4} \times$$

$$\times I_{l+2} - \frac{(2l+5)(2l+7)}{2} I_{l+3} + \frac{(2l+7)(2l+9)}{8} I_{l+4}.$$

We shall not need here formulae for higher n, but they can be easily calculated. These I_l, as we shall see later, replace the F_k of the Slater method, but unlike the latter can be calculated by a simple integration.

§ 4. *The transformation to the relative and center of gravity coordinates.*

The zero-order Hamiltonian of two nucleons moving in the field of the oscillator potential is:

$$H = \frac{1}{2\,m}\,(p_1^2 + m^2\,\omega^2\,r_1^2) + \frac{1}{2\,m}\,(p_2^2 + m^2\,\omega^2\,r_2^2)\,.$$

We make now the canonical coordinate transformation:

$$\vec{r} = \vec{r}_2 - \vec{r}_1 \qquad\qquad \vec{p} = \frac{\vec{p}_2 - \vec{p}_1}{2}$$

$$\vec{R} = \frac{\vec{r}_2 + \vec{r}_1}{2} \qquad\qquad \vec{P} = \vec{p}_2 + \vec{p}_1$$

which introduces the relative coordinate and the center of gravity coordinate of the two nucleons, with the respective momenta. The Hamiltonian expressed in terms of the new variables is:

$$H = \frac{1}{2\,M}\,(P^2 + M^2\,\omega^2\,R^2) + \frac{1}{2\,\mu}\,(p^2 + \mu^2\,\omega^2\,r^2)$$

where:

$$M = 2\,m, \qquad \mu = \frac{1}{2}\,m\,.$$

This is the Hamiltonian of two harmonic oscillators with masses M and μ. The solution of the corresponding wave-equation can be written as a product of wave-functions of the two oscillator potentials depending on R and r respectively. The angular momenta and the number of nodes which characterize these wave functions will be called L, Λ and N, n respectively.

If we want to express the wave-function $\psi^{m_1}_{n_1 l_1}(\vec{r}_1)\,\psi^{m_2}_{n_2 l_2}(\vec{r}_2)$ of two nucleons with definite quantum numbers $n_1\,l_1\,m_1$ and $n_2\,l_2\,m_2$, as a sum of products $\psi^M_{NL}(\vec{R})\,\psi^m_{n\Lambda}(\vec{r})$ we must find out what values of N, L, M and n, Λ, m should be taken into account. It is obvious that the integrals (quantum numbers) of the system must have the same values on both sides. We thus have the following restrictions:

(1) The z-component of the orbital angular momentum.

$$m_1 + m_2 = M + m\,.$$

(2) The energy.

$$\hbar \, \omega \, (2 \, n_1 + l_1 + 2 \, n_2 + l_2 + 3) = \hbar \, \omega \, (2 \, N + L + 2 \, n + \Lambda + 3)$$

which implies

$$2 \, n_1 + 2 \, n_2 + l_1 + l_2 = 2 \, N + 2 \, n + L + \Lambda \, .$$

(3) The symmetry. Instead of the wave-function $\psi_{n_1 l_1}^{m_1} (\vec{r_1}) \, \psi_{n_2 l_2}^{m_2} (\vec{r_2})$ (in the case that n_1, l_1, m_1, differ from n_2, l_2, m_2) it is useful to take the symmetric and anti-symmetric combinations:

$$\frac{1}{\sqrt{2}} \left(\psi_{n_1 l_1}^{m_1}(\vec{r_1}) \, \psi_{n_2 l_2}^{m_2}(\vec{r_2}) + \psi_{n_1 l_1}^{m_1}(\vec{r_2}) \, \psi_{n_2 l_2}^{m_2}(\vec{r_1}) \right) \tag{7}$$

$$\frac{1}{\sqrt{2}} \left(\psi_{n_1 l_1}^{m_1}(\vec{r_1}) \, \psi_{n_2 l_2}^{m_2}(\vec{r_2}) - \psi_{n_1 l_1}^{m_1}(\vec{r_2}) \, \psi_{n_2 l_2}^{m_2}(\vec{r_1}) \right) \tag{8}$$

which are multiplied by $+1$ and (-1) respectively under the transformation $\vec{r_1} \to \vec{r_2}, \vec{r_2} \to \vec{r_1}$ (which induces the transformation $\vec{r} \to -\vec{r}$, $\vec{R} \to \vec{r}$), and therefore $\psi_L(\vec{R}) \, \psi_\Lambda(\vec{r})$ is multiplied by $(-1)^\Lambda$ (the parity of $\psi_\Lambda(r)$]. Therefore:

Symmetrical (antisymmetrical) wave-functions contain only even (odd) values of Λ in their expansion.

(4) Parity. The transformation $\vec{r_1} \to -\vec{r_1}$ corresponds to $\vec{r} \to 2\vec{R}$, $\vec{R} \to \vec{r}/2$. The radial part of the function $\psi_L(\vec{R}) \, \psi_\Lambda(\vec{r})$, which is the only part that changes, is $f(\sqrt{2\nu} \, R) \, g(\sqrt{\nu} \, r/\sqrt{2})$ (the factors 2 and 1/2 multiplying the ν of the oscillator potential come from $M = 2 \, m$, $\mu = m/2$). Therefore it is changed under the transformation according to:

$$f(\sqrt{2 \, \nu} \, R) \, g \left(\sqrt{\frac{\nu}{2}} \, r \right) \to f \left(\sqrt{2 \, \nu} \, \frac{r}{2} \right) g \left(\sqrt{\frac{\nu}{2}} \, 2 \, R \right) = f \left(\sqrt{\frac{\nu}{2}} \, r \right) g (\sqrt{2 \, \nu} \, R)$$

and consequently:

$$\psi_L(\vec{R}) \, \psi_\Lambda(\vec{r}) \to \psi_L(\vec{r}) \, \psi_\Lambda(\vec{R}) \, .$$

In the same way it is seen that the transformation $\vec{r_2} \to -\vec{r_2}$ or $\vec{r} \to -2 \, \vec{R}, \vec{R} \to -\vec{r}/2$ results in:

$$\psi_L(\vec{r_1}) \, \psi_\Lambda(\vec{r}) \to (-1)^{L+\Lambda} \, \psi_L(\vec{r}) \, \psi_\Lambda(\vec{R}) \, .$$

As a result, the expansion of $\psi_{l_1}(\vec{r_1}) \, \psi_{l_2}(\vec{r_2})$ should contain $\psi_L(\vec{R}) \, \psi_\Lambda(\vec{r})$ and $\psi_L(\vec{r}) \, \psi_\Lambda(\vec{R})$ only in the combination:

$$\psi_L(\vec{R}) \, \psi_\Lambda(\vec{r}) + (-1)^{l_1} \psi_L(\vec{r}) \, \psi_\Lambda(\vec{R}) \, . \tag{9}$$

It is multiplied in the transformation $\vec{r_1} \rightarrow -\vec{r_1}$ by $(-1)^{l_1}$, and in the transformation $\vec{r_2} \rightarrow -\vec{r_2}$ by

$$(-1)^{L+\Lambda}(-1)^{l_1} = (-1)^{l_1+l_2}(-1)^{l_1} = (-1)^{l_2}$$

(as $2N + 2n + L + \Lambda = 2n_1 + 2n_2 + l_1 + l_2$).

When we build the symmetric and antisymmetric functions, we see that if l_1 and l_2 are both even or both odd (and hence L and Λ have the same property) both $\psi_L(\vec{R})\,\psi_\Lambda(\vec{r})$ and $\psi_L(\vec{r})\,\psi_\Lambda(\vec{R})$ may enter into those functions, and therefore only in the combination (9). If one of l_1, l_2 is even and the other is odd, the symmetric function contains only $\psi_L(\vec{R})\,\psi_\Lambda(\vec{r})$ (with Λ even and L odd) and the antisymmetric function contain only $\psi_\Lambda(\vec{R})\,\psi_L(\vec{r})$. Therefore the coefficient of $\psi_L(\vec{R})\,\psi_\Lambda(\vec{r})$ in the symmetric function should be $(-1)^{l_1}$ times the coefficient of $\psi_\Lambda(\vec{R})\,\psi_L(\vec{r})$ in the antisymmetric function.

Subject to the above conditions, the transformation is most conveniently calculated in the following manner. The wave-functions (7) and (8) are written down in the case $\varphi_1 = \varphi_2 = \varphi = \Phi$. It is then easy to express them as functions of $\vec{R}$ and $\vec{r}$ and compare these expressions with a linear combination of the admissible wave-functions $\psi_{NL}^M(\vec{R})\,\psi_{n\Lambda}^m(\vec{r})$ (according to the above conditions). By equating respective coefficients, the factors of this expansion may be easily found. When doing so no account should be taken of the exponential factor, as

$$e^{-\frac{\nu}{2}r_1^2}\,e^{-\frac{\nu}{2}r_2^2} = e^{-\frac{\nu}{2}(r_1^2+r_2^2)} = e^{-\frac{\nu}{2}\left(\frac{4R^2+r^2}{2}\right)} = e^{-\frac{2\nu}{2}R^2}\,e^{-\frac{\nu/2}{2}r^2}.$$

II. On the Order of Levels in j^n Configurations.

§ 5. *The evaluation of Slater parameters for the harmonic oscillator wave-functions.*

The method described in I will now be used to obtain some results on level spacings and especially on the order of levels in a few interesting cases together with a discussion of their dependence on the range and form of the potential. An important problem in the shell model based on strong spin-orbit coupling, which leads to the jj-coupling scheme, is the determination of the spin (total angular momentum) of the ground state of the j^n configuration. It was pointed out by FEENBERG[18] that for $n = 2$ the spin of the ground state is zero (for Majorana forces). His argument was generalized by RACAH[19] who showed that for any n the ground state has the

minimum spin allowed by the Pauli principle, namely (for an odd n) 3/2 in the case of three particles (or holes) in a shell, and 1/2 in other cases. This result is apparently in contradiction with the empirical rule that the spin of the ground state is equal to the spin of the odd nucleon. However, RACAH's argument was based (as that of FEENBERG) on the long range approximation, in which one assumes that the interaction potential can be approximated by a square well in the region where the amplitudes of the wave-functions are important. On the other hand, results of MAYER[2]) showed that with a δ-potential the spin of the ground state is $J = j$ if n is odd and $J = 0$ if n is even. (This is in fact a short-range approximation, where one assumes that the potential is different from zero only in a region so small that the change of the wave-functions in that region is small compared to the wave-functions themselves.) This result was subsequently proved by RACAH[3]) in a general way.

If we pass continuously (i.e. by a continuous change of some parameters) from the short range limit to that of the long range, we see that the order of levels changes continuously, and for different values of the parameters one obtains different spins of the ground state. The dependence of the spin of the ground state on the range of the potential has been treated in some special cases by KURATH[4]) and the author[5]). In the following, our method is used to obtain general results which are applicable to every potential, and additional interesting forms of the potential are treated.

The method of the harmonic oscillator wave functions can be combined with the vast amount of results calculated in atomic spectroscopy: in these results all the energy levels are expressed in terms of the F_k; on the other hand it is possible to express them in terms of the integrals I_l. It is therefore always possible to write down the F_k as functions of the I_l. This allows us to calculate the values of the F_k, when using any potential, by first computing the values of the I_l by direct integration. The use of this procedure makes all the formulae derived in the F_k formalism useful for calculations of the energy levels.

If one is interested in central interactions only, it is not necessary to calculate the wave-functions of the configuration in terms of $\vec{R}$ and $\vec{r}$, as only the Slater parameters F_k are required and with the above transformation they can be directly calculated in terms of the I_l.

The F^k (which differ only by a constant factor from the F_k) are defined by:

$$F^k(a_1, a_2) = \int_0^\infty \int_0^\infty R_{a_1}^2(r_1) R_{a_2}^2(r_2) f_k(r_1, r_2) \, dr_1 \, dr_2$$

where the $f_k(r_1, r_2)$ are the coefficients of the expansion (4). The f_k are therefore given by

$$f_k(r_1, r_2) = \frac{2\,k+1}{2} \int_{-1}^{+1} V(|\vec{r_2} - \vec{r_1}|)\; P_k(\cos \omega_{12})\; d \cos \omega_{12}$$

(the factor $2/2\,k + 1$ is the square of the normalization factor of the Legendre polynomials), and thus:

$$F^k = \frac{2\,k+1}{2} \int_{-1}^{+1} \int_0^\infty \int_0^\infty V(|\vec{r_2} - \vec{r_1}|) \frac{R_{a_1}^2(r_1)}{r_1^2} \frac{R_{a_2}^2(r_2)}{r_2^2} \times$$
$$\times P_k(\cos \omega_{12})\, r_1^2\, dr_1\, r_2^2\, dr_2\, d \cos \omega_{12}\,.$$

This expression can be integrated also over other angles $d\Omega$ of which the integrand is independent, thus completing $r_1^2\, r_2^2\, dr_1\, dr_2\, d\cos\omega_{12}$ to the differential of the volume element in the space of the vectors $\vec{r_1}$ and $\vec{r_2}$. We obtain, after multiplication by a proper normalization factor N_Ω,

$$F^k(a_1, a_2) = N_\Omega \frac{2\,k+1}{2} \int\int V(|\vec{r_2} - \vec{r_1}|) \frac{R_{a_1}^2(r_1)}{r_1^2} \frac{R\,(r_2)}{r_2^2} \times$$
$$\times P_k(\cos \omega_{12})\, d^3 r_1\, d^3 r_2\,. \qquad (10)$$

This integral can now be transformed to an integral over $\vec{R}$ and $\vec{r}$. As $V(|\vec{r_2} - \vec{r_1}|)$ is a function of $|\vec{r}|$ only, it remains to express

$$\frac{R_{a_1}^2(r_1)}{r_1^2} \frac{R_{a_2}^2(r_2)}{r_2^2}\; P_k(\cos \omega_{12})$$

in terms of $\vec{R}$ and $\vec{r}$. This is always possible with the help of the relation

$$r_1\, r_2 \cos \omega_{12} = (\vec{r_1}, \vec{r_2}) = \frac{1}{4}\, (4\,R^2 - r^2)\,.$$

The result is a function of R, r and the angle α between the vectors $\vec{R}$ and $\vec{r} - \varphi_k(R, r, \alpha)$. So the above expression becomes:

$$F^k = \frac{2\,k+1}{2}\, N_\Omega \int\int V(r)\; \varphi_k(R, r, \alpha)\, d^3 R\, d^3 r = \frac{2\,k+1}{2}\, N_\Omega \times$$
$$\times \int_{\Omega'} \int_{-1}^{+1} \int_0^\infty \int_0^\infty V(r)\; \varphi_k(R, r, \alpha)\, R^2\, dR\, r^2\, dr\, d\cos \alpha\, d\Omega' = \frac{2\,k+1}{2} \times$$
$$\times \int_{-1}^{+1} \int_0^\infty \int_0^\infty V(r)\; \varphi_k(R, r, \alpha)\, R^2\, dR\, r^2\, dr\, d\cos \alpha\,. \qquad (11)$$

Where $d\Omega'$ stands for the differentials which form with $R^2\,dR\,r^2\,dr$ $\times\,d\cos\alpha$ the volume element $d^3R\,d^3r$; the integrand is independent of them and the integration results merely in the dropping of the normalization constant N_Ω. The integration over α can be easily done and it is then possible to carry out the R-integration, which leaves F^k expressed in terms of the r-integrals $-I_l$. In this way the Slater parameters are calculated in terms of simple integrals without bothering about the expansion of the potential in a series of Legendre polynomials and without the complicated integrations in which such an expansion results.

In order to see how this procedure works, we give explicitly the expression of $r_1^k\,r_2^k\,P_k(\cos\omega_{12})$ in terms of R, r and α. This expression results from the calculation of a configuration in which $l_1 = l_2$ (and therefore k is even). Putting

$$P_k(\cos\omega_{12}) = \sum_{n=0}^\infty c_{2n}\cos^{2n}\omega_{12}$$

$k = 2\,m$, we obtain:

$$r_1^k\,r_2^k\,P_k(\cos\omega_{12}) = \sum_{n=0}^m c_{2n}\,r_1^{k-2n}\,r_2^{k-2n}\,r_1^{2n}\,r_2^{2n}\cos^{2n}\omega_{12} =$$

$$= \sum_{n=0}^m c_{2n}(r_1^2\,r_2^2)^{m-n}\left(\frac{4\,R^2-r^2}{4}\right)^{2n} = \sum_{n=0}^m c_{2n}\left(\frac{4\,R^2-r^2}{4}\right)^{2n}\times$$

$$\times\left[\left(\frac{4\,R^2-r^2}{4}\right)^2 + R^2\,r^2 - (\vec{R},\,\vec{r})^2\right]^{m-n}.$$

Use was made here of the relation

$$r_1^2\,r_2^2 = \left(\frac{4\,R^2-r^2}{4}\right)^2 + R^2\,r^2 - (\vec{R},\,\vec{r})^2 = \left(\frac{4\,R^2-r^2}{4}\right) + R^2\,r^2\sin^2\alpha.$$

As an example we evaluate in detail F^2 of the configuration p^2:

$$F^2 = N_1^4\,\frac{5}{2}\int_{-1}^{+1}\int_0^\infty\int_0^\infty V(r)\,e^{-\nu(r_1^2+r_2^2)}\,r_1^2\,r_2^2\left(\frac{3}{2}\cos^2\omega_{12}-\frac{1}{2}\right)\times$$

$$\times\,r_1^2\,dr_1\,r_2^2\,dr_2\,d\cos\omega_{12} =$$

$$= \left(\frac{\sqrt{\nu}\,\nu^2\,8}{\sqrt{\pi}\cdot3}\right)^2\frac{5}{2}\int_{-1}^{+1}\int_0^\infty\int_0^\infty V(r)\,e^{-\nu\frac{4\,R^2+r^2}{2}}\left[\frac{3}{2}\left(\frac{4\,R^2-r^2}{4}\right)^2 -\right.$$

$$\left.- \frac{1}{2}\left(\left(\frac{4\,R^2-r^2}{4}\right)^2 + R^2\,r^2\sin^2\alpha\right)\right]R^2\,dR\,r^2\,dr\,d\cos\alpha =$$

$$= \left(\frac{\sqrt{\nu}\,\nu^2\,8}{\sqrt{\pi}\cdot 3}\right)^2 5 \int_0^\infty V(r)\, e^{-\frac{\nu}{2}r^2}\left[\frac{1}{4}\left(\frac{4\sqrt{\pi}\,15}{\sqrt{2\nu}\,(2\nu)^3\,16}+\frac{\sqrt{\pi}}{4\sqrt{2\nu}\cdot 2\nu\cdot 4}r^4\right)-\right.$$

$$\left.-\frac{5\sqrt{\pi}\cdot 3}{6\sqrt{2\nu}\,(2\nu)^2\,8}r^2\right]r^2\,dr = \left(\frac{\sqrt{\nu}\,\nu^2\,8}{\sqrt{\pi}\cdot 3}\right)^2\times$$

$$\times 5\left[\frac{1}{4}\left(\frac{4\sqrt{\pi}\cdot 15}{\sqrt{2\nu}\,(2\nu)^3\,16}\cdot\frac{\sqrt{\pi}}{\sqrt{\nu/2}\cdot\nu/2\cdot 4}I_0+\frac{\sqrt{\pi}}{4\sqrt{2\nu}\cdot 2\nu\cdot 4}\cdot\frac{\sqrt{\pi}\cdot 15}{\sqrt{\nu/2}\,(\nu/2)^3\,16}I_2\right)-\right.$$

$$\left.-\frac{5\sqrt{\pi}\,3}{6\sqrt{2\nu}\,(2\nu)^2\,8}\cdot\frac{\sqrt{\pi}\,3}{\sqrt{\nu/2}\,(\nu/2)^2\,8}I_1\right] = \frac{25}{12}\left[(I_0+I_2)-2I_1\right].$$

In this way the F^k for the configurations d^n, f^n, and g^n were obtained; these are later used in the discussion of the jj-coupling configurations $(d_{5/2})^3$, $(f_{7/2})^3$, and $(g_{5/2})^3$.

§ 6. The $(d_{5/2})^3$ configuration with various potentials.

The simplest configuration where the short and long range approximations give different results is $(d_{5/2})^3$. In the δ-limit the state with $J = 5/2$ is the ground state, and in the long range limit the state with $J = 3/2$ is the lowest. If one passes from one limit to the other a cross-over of these two levels occurs. This case is also of an experimental interest as a $J = 3/2$ state of the configuration is postulated[1] for the ground state of Na[21]. This case was treated by KURATH[4] who used oscillator wave-functions and a Gaussian potential. It was also calculated by the author[5] using the Slater method with Coulomb field wave-functions and a Yukawa potential. The results were different, since the Gaussian potential can be well approximated by a square well when its range is increased, whereas the Yukawa potential has a singularity at the origin and does not yield the long range limit when its range tends to infinity. We shall discuss in this paragraph the cross-over and its dependence on the range and form of the potential.

The energy states of the configuration $(d_{5/2})^n$ were calculated by INGLIS[20] in the case of ordinary (WIGNER) forces between the particles. The calculation of the levels in the case of Majorana interaction can be carried out in the same manner (an example of such a calculation is given later in the case of Li[7]).

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The results are:

$$E_{9/2} = \frac{36}{25} F^0 - \frac{72}{5 \cdot 49} F^2 - \frac{1329}{25 \cdot 441} F^4$$

$$E_{5/2} = \frac{4}{5} F^0 - \frac{56}{5 \cdot 49} F^2 - \frac{651}{5 \cdot 441} F^4$$

$$E_{3/2} = \frac{3}{5} F^0 + \frac{12}{5 \cdot 49} F^2 - \frac{627}{5 \cdot 441} F^4.$$

The sign convention is that the F_k are positive and the Majorana potential is attractive (negative) in symmetrical states.

From the definition of the F_k it follows that in the short range limit $F^k = (2k + 1) F_0$, while in the long range approximation $F^k \ll F_0$. From these relations the order of the levels $J = 5/2$ and $J = 3/2$ in the two limits can be immediately seen. We calculated the F^k in terms of the I_l using the procedure of § 5 with the following results:

$$3 \cdot 5 F^0 = \frac{63}{16} (I_0 + I_4) + \frac{7}{4} (I_1 + I_3) + \frac{29}{8} I_2$$

$$7 \cdot 3 \frac{1}{49} F^2 = \frac{9}{16} (I_0 + I_4) - \frac{1}{2} (I_1 + I_3) - \frac{1}{8} I_2$$

$$7 \cdot 5 \frac{1}{49} F^4 = \frac{3}{16} (I_0 + I_4) - \frac{3}{4} (I_1 + I_3) + \frac{9}{8} I_2.$$

Remark: In the above results only the combination $(I_0 + I_4)$, $(I_1 + I_3)$ appear, thus forming with I_2 three independent parameters which replace F_0, F_2, and F_4 of the Slater method. The reason for this fact can be seen as follows: In the wavefunction $\psi_{l_1}(r_1) \, \psi_{l_2}(r_2)$ the product $r_1^{l_1} \cdot r_2^{l_2}$ appears as a factor, and therefore when we express it as a function of $\vec{r}$ and $\vec{R}$ the sum of the powers of R and r in every term is equal to $l_1 + l_2$. The transformation $\vec{r_1} \to -\vec{r_1}$ which induces $\vec{r} \to 2 \vec{R}$, $\vec{R} \to \vec{r}/2$ transforms, as we have already seen, $f(\sqrt{2\nu} R) \, g(\sqrt{\nu/2} r)$ into $f(\sqrt{\nu/2} r) \, g(\sqrt{2\nu} R)$. As a result, to every term in which $R^m \, r^{l_1 + l_2 - m}$ appears there corresponds another term which has the same angular part and contains $r^m \, R^{l_1 + l_2 - m}$ as a factor. When the angular and R-integrations are performed these two terms give in the r-integration the integrals $I_{l_1 + l_2 - m}$ and I_m with the same factors. Therefore only the combination $(I_m + I_{l_1 + l_2 - m})$ appears in the result. It should be remembered that there exists only one set of the I_l the combinations of which form the F_k and G_k of the various configurations.

Putting these values in the above formulae we obtain the energy levels expressed by the I_l:

$$E_{9/2} = \frac{1}{25} \left(-\frac{117}{16} (I_0 + I_4) + \frac{165}{4} (I_1 + I_3) - \frac{255}{8} I_2 \right)$$

$$E_{5/2} = \frac{1}{5} \left(-\frac{63}{16} (I_0 + I_4) + \frac{63}{4} (I_1 + I_3) - \frac{157}{8} I_2 \right)$$

$$E_{3/2} = \frac{1}{5} \left(-\frac{36}{16} (I_0 + I_4) + \frac{54}{4} (I_1 + I_3) - \frac{156}{8} I_2 \right).$$

From the definition of the I_l it is seen that in the short range approximation $I_l \ll I_0$ $(l > 0)$ (because for $l > 0$, R_l vanishes at the origin), and in the long range limit all the I_l are equal.

This property of the I_l makes our method useful for approximations in the short range limit. In the first approximation only the coefficient of I_0 should be calculated (use is made of this fact in § 15). On the other hand the Slater method can serve as an approximation in the long range limit. Near this limit $F^k \ll F^0$ $(k > 0)$, so that the first approximation is given by F^0. From this point of view these two methods are complementary, and hence it is natural to work with the I_l in nuclear spectroscopy where there are good reasons to believe that the short range approximation is justified.

We have mentioned before that the cross-over should depend not only on the range of the potential but also strongly on its form. A flat potential like a square well or the Gaussian potential give the long range limit as its range tends to infinity, but a potential which is singular at the origin does not yield the long range limit even if its range tends to infinity (the important parts of the potential are in the neighbourhood of the origin). The I_l $(l > 0)$, though they are no longer zero, do not approach the value of I_0 (alternatively the F^k, $k > 0$, are smaller than F^0 but do not tend to zero). As far as the order of the levels is considered, such potentials represent an intermediate case between the two limits.

We shall compare potentials in which the range parameter appears in the same manner but their forms are different, namely:

(a) the Gaussian potential $V e^{-r^2/r_0^2}$ which for $r_0 \to \infty$ gives the long range limit (this potential was used by KURATH[4]);

(b) the potential $V e^{-r^2/r_0^2} / r/r_0$ which gives in the limit $r_0 \to \infty$ the Coulomb potential, and,

(c) the potential $V e^{-r^2/r_0^2} / r^2/r_0^2$ which is still more singular at the origin.

Although the difference in behaviour was already noted[5], we discuss it again as we shall use here the same wave-functions for all of them. The I_l corresponding to these potentials are calculated to be:

(a) $$I_l = N_l^2 \int_0^\infty e^{-\nu r^2} V(r) \, r^{2l+2} \, dr = N_l^2 \, V \int_0^\infty e^{-\left(\nu + \frac{1}{r_0^2}\right) r^2} r^{2l+2} \, dr =$$

$$= V \left(\frac{\lambda^2}{1+\lambda^2}\right)^{l+3/2}$$

(b) $I_l = N_l^2 \int_0^\infty e^{-\nu r^2} V(r)\, r^{2l+2}\, dr = N_l^2\, V r_0 \int_0^\infty e^{-\left(\nu + \frac{1}{r_0^2}\right) r^2} r^{2l+1}\, dr =$

$\qquad = V\lambda\, \dfrac{n_l^2\, l!}{2\sqrt{\pi}} \left(\dfrac{\lambda^2}{1+\lambda^2}\right)^{l+1} \qquad\qquad n_l^2 = \dfrac{\sqrt{\pi}}{\sqrt{\nu}\, \nu^{l+1}}\, N_l^2$

(c) $I_l = N_l^2 \int_0^\infty e^{-\nu r^2} V(r)\, r^{2l+2}\, dr = N_l^2\, V r_0^2 \int_0^\infty e^{-\left(\nu + \frac{1}{r_0^2}\right) r^2} r^{2l}\, dr =$

$\qquad = V\lambda^2\, \dfrac{2}{2l+1} \left(\dfrac{\lambda^2}{1+\lambda^2}\right)^{l+\frac{1}{2}}.$

λ is defined by $\lambda = r_0 \sqrt{\nu}$; it is the ratio of the potential range to $1/\sqrt{\nu}$ which determines the extension of the wave-function. If λ tends to infinity it may be seen that I_l of the Gaussian potential are all equal, while in the limit $\lambda \to \infty$ for the case (b) (which gives the Coulomb potential), I_l is proportional to $n_l^2\, l!$; and in case (c) I_l is proportional to $2/2\,l+1$.

The cross over of the levels $J = 5/2$ and $J = 3/2$ occurs where the I fulfil the equation:

$$27\,(I_0 + I_4) - 36\,(I_1 + I_3) + 2\, I_2 = 0\,. \qquad\qquad (13)$$

Relative to the value of I, the values of the I in the three cases are:

(a) $I_l \sim \left(\dfrac{\lambda^2}{1+\lambda^2}\right)^l$

(b) $I_l \sim \dfrac{n_l^2\, l!}{4} \left(\dfrac{\lambda^2}{1+\lambda^2}\right)^l$

and

(c) $I_l \sim \dfrac{1}{2\,l+1} \left(\dfrac{\lambda^2}{1+\lambda^2}\right)^l.$

Using these relations we find that the cross over occurs in case (a) at the value $\lambda = 1{\cdot}326$, whereas in case (b) it occurs only at $\lambda = 4{\cdot}34$. The effect of the form of the potential appears more pronounced in case (c); in this case no cross-over occurs as λ varies from zero to infinity. [For the left hand side of (13) for $\lambda = 0$ is positive, and it remains so even when the I_l attain their maximum value $\sim 1/(2\,l+1)$.]

This strong dependence on the form shows that for such problems it is not enough to consider the range only. Potentials of different forms which may be adjusted to fit some condition will not usually give here the same results.

In the preceding section we discussed the effect of the form of the potential on the order of the levels and the place of the cross-

over, using the same wave-functions; we shall now give the results for the Yukawa potential using harmonic oscillator wave-functions and compare them to those obtained with the Coulomb field wave-functions[5]).

For the Yukawa potential, $V(r) = V e^{-r/r_0} / r/r_0$, the I_l are calculated as follows:

$$I_l = N_l^2 \int_0^\infty V(r)\, e^{-\nu r^2}\, r^{2l+2}\, dr = N_l^2\, V r_0 \int_0^\infty e^{-\nu r^2 - r/r_0}\, r^{2l+1}\, dr =$$

$$= \frac{N_l^2}{\sqrt{\nu}\,\nu^{l+1}}\, V\lambda \int_0^\infty e^{-\xi^2 - \xi/\lambda}\, \xi^{2l+1}\, d\xi$$

where $\xi = \sqrt{\nu}\, r$ and $\lambda = \sqrt{\nu}\, r_0$. We introduce the constant $\mu = 1/2\,\lambda$ and $x = \xi + \mu$, and obtain:

$$I_l = \frac{N_l^2}{\sqrt{\nu}\,\nu^{l+1}}\, V\lambda\, e^{\mu^2} \int_\mu^\infty e^{-x^2} (x - \mu)^{2l+1}\, dx\,.$$

This gives for the integrals which we need:

$$I_0 = V\lambda \left[\frac{2}{\sqrt{\pi}} - 2\,\mu\, (1 - \Phi(\mu))\, e^{\mu^2} \right]$$

$$I_1 = \frac{2}{3}\, V\lambda \left[\frac{2}{\sqrt{\pi}}\, (1 + \mu^2) - 2\,\mu \left(\frac{3}{2} + \mu^2 \right) (1 - \Phi(\mu))\, e^{\mu^2} \right]$$

$$I_2 = \frac{8}{15}\, V\lambda \left[\frac{2}{\sqrt{\pi}} \left(1 + \frac{9}{4}\,\mu^2 + \frac{1}{2}\,\mu^4 \right) - 2\,\mu \left(\frac{15}{8} + \frac{5}{2}\,\mu^2 + \frac{1}{2}\,\mu^4 \right) \times$$

$$\times\, (1 - \Phi(\mu))\, e^{\mu^2} \right]$$

$$I_3 = \frac{16}{35}\, V\lambda \left[\frac{2}{\sqrt{\pi}} \left(1 + \frac{29}{8}\,\mu^2 + \frac{5}{3}\,\mu^4 + \frac{1}{6}\,\mu^6 \right) -$$

$$-\, 2\,\mu \left(\frac{35}{16} + \frac{35}{8}\,\mu^2 + \frac{7}{4}\,\mu^4 + \frac{1}{6}\,\mu^6 \right) (1 - \Phi(\mu))\, e^{\mu^2} \right]$$

$$I_4 = \frac{128}{315}\, V\lambda \left[\frac{2}{\sqrt{\pi}} \left(1 + \frac{325}{64}\,\mu^2 + \frac{115}{32}\,\mu^4 + \frac{35}{48}\,\mu^6 + \frac{1}{24}\,\mu^8 \right) -$$

$$-\, 2\,\mu \left(\frac{315}{128} + \frac{105}{16}\,\mu^2 + \frac{63}{16}\,\mu^4 + \frac{3}{4}\,\mu^6 + \frac{1}{24}\,\mu^8 \right) (1 - \Phi(\mu))\, e^{\mu^2} \right]$$

Φ is the error function

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt\,.$$

These values of I_l for various values of μ were substituted into formulae (12), (13) to give the level spacings corresponding to each μ.

The cross-over of the levels $J = 5/2$ and $J = 3/2$ occurs at a value $\lambda = 9$, i.e. for $r_0/r_l = 9 \sqrt{2} \sim 13$ where $r_l = \sqrt{2\nu}$ is the parameter appearing in the wave function of a single d-nucleon. This should be compared to the value $r_0/r_l \sim 10$, obtained with the use of the Coulomb field wave-functions, where r appears in these functions in the exponent: $R_l(r) = N_l\, r^{l+1}\, e^{-r/r_l}$.

It is interesting to see that the results are essentially the same, thus supporting the opinion that the exact form of the wave-function is not very important.

The results of this paragraph show that the appearence of the spin $3/2$ in the ground state of Na^{21} is unlikely to be due only to the effect of Majorana forces, as the value of λ which should then be assumed is too big, especially if we assume that the potential is a "deep hole" potential, such as that of YUKAWA, rather that a flat one.

§ 7. *The configuration* $(f_{7/2})^3$.

This configuration [and equivalently $(f_{7/2})^5$] has six states with spins $J = 15/2,\ 11/2,\ 9/2,\ 7/2,\ 5/2,\ 3/2$. For Majorana forces in the short range approximation the state with $J = 7/2$ is the ground state, and in the long range limit the state with $J = 3/2$ is the lowest; above it lies $J = 5/2$ and the state with $J = 7/2$ is next to it. The cross-over of these three states was found by KURATH[4]) who used a Gaussian potential. We shall in the following discuss this configuration using also other forms of the potential.

The matrix elements of the Majorana interaction in the $(j_1,\ j_2,\ m_{j_1},\ m_{j_2})$ scheme were found in the way described in § 12. The integrals $J(m_1,\ m_2;\ m_3,\ m_4)$ were calculated in terms of the Slater parameters F_k by the usual method. As there are no two states with equal J in this case the sum method was used to obtain the energies of the various states with the following results:

$$
\left.
\begin{aligned}
E_{15/2} &= -90\,F_0 + 2700\,F_2 + 7020\,F_4 + 7974\,F_6 \\
E_{11/2} &= -62\,F_0 + 1580\,F_2 + 2736\,F_4 + 42302\,F_6 \\
E_{9/2} &= -51\,F_0 + 425\,F_2 + 6201\,F_4 + 32765\,F_6 \\
E_{7/2} &= -42\,F_0 + 1610\,F_2 + 6930\,F_4 + 82082\,F_6 \\
E_{5/2} &= -35\,F_0 + 665\,F_2 - 2583\,F_4 + 80717\,F_6 \\
E_{3/2} &= -30\,F_0 - 660\,F_2 + 4752\,F_4 + 50622\,F_6
\end{aligned}
\right\} \quad (14)
$$

The F_k in the above expressions are defined by:

$$F_0 = \frac{F^0}{7^2}, \qquad F_2 = \frac{F^2}{3^2\,5^2\,7^2}, \qquad F_4 = \frac{F^4}{3^2\,7^2\,11^2}, \qquad F_6 = \frac{5^2\,F^6}{3^2\,7^2\,11^2\,13^2}.$$

Using the method described above, these F^k are expressed in the terms of I_l as is given in the following formulae:

$$F^6 = \frac{5577}{2240}\left[I_0 + I_6\right) - 6\,(I_1 + I_5) + 15\,(I_2 + I_4) - 20\,I_3\right]$$

$$F^4 = \frac{297}{2240}\left[13\,(I_0 + I_6) - 34\,(I_1 + I_5) + 19\,(I_2 + I_4) + 4\,I_3\right]$$

$$F^2 = \frac{3}{448}\left[143\,(I_0 + I_6) - 66\,(I_1 + I_5) - 15\,(I_2 + I_4) - 124\,I_3)\right]$$

$$F^0 = \frac{1}{2240}\left[429\,(I_0 + I_6) + 198\,(I_1 + I_5) + 387\,(I_2 + I_4) + 212\,I_3\right].$$

Taking the Gaussian potential we obtain the following results which are in agreement with those of Kurath: for $\lambda < 1\cdot27$ the lowest state is $J = 7/2$; at $\lambda = 1\cdot27$ the level with $J = 5/2$ becomes the ground level and is the lowest one in the region $1\cdot27 < \lambda < 1\cdot35$; for $\lambda > 1\cdot35$ the $J = 3/2$ state is the ground state. Thus there is a region where the ground state has the spin 5/2. The configuration $(f_{7/2})^n$ is predicted by the shell model for all nuclei having N or Z (neutron or proton number) between 20 and 28. The only odd-even nuclei in this region, of which the spins have been measured are the following odd-proton nuclei: $_{21}\mathrm{Sc}^{45}$ and $_{27}\mathrm{Co}^{59}$ have the spin 7/2 as required by the shell model, $_{23}\mathrm{V}^{51}$ has also the spin 7/2 whereas $_{25}\mathrm{Mn}^{55}$ has the spin 5/2. It was suggested[1] that the ground state of the last nucleus is a $J = 5/2$ state of the $(f_{7/2})^5$ configuration. Kurath[4] tries to explain the occurrence of the 5/2 ground state by suggesting that r_0 of the nuclear forces and r_l of the nucleon orbit satisfy $1\cdot27 < \lambda < 1\cdot35$. Even for the Gaussian potential such a value of r_0 is too big if one determines r_l roughly from the nuclear radius, but a more serious objection is raised if we consider potentials which are singular at the origin. Taking the potential $V(r) = Ve^{-r^2/r_0^2}/r/r_0$ we find that for $\lambda < 3\cdot3$ the ground state is $J = 7/2$, at $\lambda = 3\cdot3$ a cross-over of this with the $J = 3/2$ level occurs and for $\lambda > 3\cdot3$ the ground state is $J = 3/2$. The level $J = 5/2$ crosses the level $J = 7/2$ only at $\lambda = 9\cdot5$ where $J = 3/2$ is already lower. We see that here the use of a $\sim 1/r$ potential not only shifted the points of cross-over but changes completely the situation: there is no more any region of λ where $J = 5/2$ is the ground state. As far as scattering experiments are concerned, there is no indication that the Gaussian potential is better than the Yukawa potential. If one considers it

just as a model the above calculations show that the results depend strongly on the model used and it is a little bit early to draw from them any physical conclusions. We must conclude that it is not likely that the effect of Majorana forces can alone explain the appearance of the 5/2 ground state of Mn^{55}.

§ 8. *The configuration* $(g_{9/2})^3$.

This is the simplest configuration j^n in which two states with the same J appear. The states of the $(g_{9/2})^3$ configuration are: two states with $J = 9/2$ and eight more states with $J = 21/2, 17/2, 15/2, 13/2, 11/2, 7/2, 5/2,$ and $3/2$. The sum method cannot give the term values of the two $J = 9/2$ states but only their sum. The separation of these energies can be done by diagonalization of the energy matrix, but with the elaborate methods of RACAH[12]) this becomes much easier. One can define two states with $J = 9/2$, one with the Seniority $v = 1$ and the other with $v = 3$. As the interaction energy does not commute with the Seniority operator, there is a non-vanishing matrix element connecting these states and the energies of the ground $J = 9/2$ and of the excited $J = 9/2$ states can be found by diagonalization of the two-rowed matrix.

The energy levels with $J \neq 9/2$ and $J = 9/2, v = 1, J = 9/2, v=3$ as well as the matrix element V connecting the last two states were calculated by RACAH[21] in terms of the Slater's F_k. We write here down only the results for $J \leqslant 9/2$ as the other levels lie above these and are of less physical interest.

State	Wigner forces factors of					Majorana forces factors of				
	F_0	F_2	F_4	F_6	F_8	F_0	F_2	F_4	F_6	F_8
$J=9/2$ $v=1$	3	33	1287	429	2431	$-8/9$	407/14	24453/14	1144	24739/18
$J=9/2$ $v=3$	3	-43	-93	-163	-3417	$-8/9$	75/14	3153/14	1014	7879/18
$V/\sqrt{429}$	0	-1	105	-31	51	0	1/2	$-105/2$	31/2	$-51/2$
$J=7/2$	3	56	-1488	-595	-2295	$-7/9$	100/7	3336/7	542	12379/9
$J=5/2$	3	-14	962	-728	-2652	$-56/81$	613/63	$-4693/7$	5629/9	69992/81
$J=3/2$	3	-24	-1488	417	-4947	$-17/27$	$-284/21$	4272/7	2518/3	15089/27

The F_k are defined by:

$$F_0 = F^0,$$

$$F_2 = \frac{4\,F^2}{3^2\,11^2}, \qquad F_4 = \frac{F^4}{11^2\,13^2}, \qquad F_6 = \frac{16\,F^6}{3^2\,11^2\,13^2}, \qquad F_8 = \frac{7^2\,F^8}{11^2\,13^2\,17^2}.$$

The two states with $J = 9/2$ are given by the solutions of the secular equation as follows:

$$E^g_{9/2} = \frac{1}{2}(E^1_{9/2} + E^3_{9/2}) - \frac{1}{2}[(E^1_{9/2} - E^3_{9/2})^2 + 4V^2]^{1/2}$$

$$E^e_{9/2} = \frac{1}{2}(E^1_{9/2} + E^3_{9/2}) + \frac{1}{2}[(E^1_{9/2} - E^3_{9/2})^2 + 4V^2]^{1/2}.$$

If one examines the values given above it appears that $|V|$ is small in comparison to $|E^1_{9/2} - E^3_{9/2}|$ (it vanishes both in the short range and the long range limits where $v = 1$ and $v = 3$ are exactly the ground and excited states with $J = 9/2$). Therefore it is sufficient to take only the first term of the expansion in powers of $V/|E^1_{9/2} - E^3_{9/2}|$ (which is the second approximation of perturbation theory). This gives:

$$E^g_{9/2} = E^1_{9/2} - \frac{V^2}{|E^1_{9/2} - E^3_{9/2}|} = E^1_{9/2} + \frac{V^2}{E^1_{9/2} - E^3_{9/2}} ;$$

$$E^e_{9/2} = E^3_{9/2} + \frac{V^2}{|E^1_{9/2} - E^3_{9/2}|} = E^3_{9/2} - \frac{V^2}{E^1_{9/2} - E^3_{9/2}} .$$

The $(g_{9/2})^{3,5,7}$ configurations are of a special interest, since it was observed by GOLDHABER and SUNYAR[22]) that in nuclei which have such a configuration according to the shell model there is a state with spin 7/2 and even parity. This state, 7/2+, lies in a few cases under the level $J = 9/2$ which is usually observed in these nuclei (and described as $g_{9/2}$ according to the shell model) and above it in the other cases. An assignment $g_{7/2}$ for this level is in contrast with the shell model as this state (of the single nucleon) should lie 1–2 MeV above the $g_{9/2}$ state (because of the strong spin-orbit interaction). GOLDHABER and SUNYAR[22]) assume therefore that this state is the $J = 7/2$ state of the $(g_{9/2})^n$ configuration.

The general method described in § 5 was used to obtain the F^k expressed in terms of the I_l; the results are:

$$F^8 = \frac{2431}{16128}[(I_0 + I_8) - 8(I_1 + I_7) + 28(I_2 + I_6) - 56(I_3 + I_5) + 70 I_4]$$

$$F^6 = \frac{1759}{16128}[17(I_0 + I_8) - 76(I_1 + I_7) + 116(I_2 + I_6) - 52(I_3 + I_5) - 10 I_4]$$

$$F^4 = \frac{143}{26880}[255(I_0 + I_8) - 480(I_1 + I_7) + 68(I_2 + I_6) - 32(I_3 + I_5) + 378 I_4]$$

$$F^2 = \frac{11}{16128}[1105\,(I_0 + I_8) - 260\,(I_1 + I_7) + 52\,(I_2 + I_6) -$$
$$- 764\,(I_3 + I_5) - 266\,I_4]$$

$$F^0 = \frac{1}{80640}[12155\,(I_0 + I_8) + 5720\,(I_1 + I_7) + 10868\,(I_2 + I_6) +$$
$$+ 6248\,(I_3 + I_5) + 10658\,I_4].$$

With the help of these formulae we calculated the order of the levels, for Wigner and Majorana forces with the potential $V(r) = V e^{-r^2/r_0^2}/r/r_0$. As is seen in the table above, the coefficient of F_2 in the case of Wigner forces, is larger for the state $J = 7/2$ than for the other states. Therefore in the long range approximation this is the lowest state up to the long range limit where all the levels coincide. As the potential used does not yield this limit, the resulting order of the levels is: $9/2^g$, $7/2$, $5/2$, $3/2$, $9/2^e$. The spacing of the levels $9/2^g$ and $7/2$ is, for $\lambda \leqslant 1\cdot5$, smaller by a factor ~ 4 than the distance between the $9/2^g$ and the $5/2$ levels. For Majorana forces with the same potential the lowest state is $9/2^g$ up to $\lambda = 3\cdot2$ where a cross-over of this level occurs with the $J = 3/2$ level, which is the ground level for $\lambda > 3\cdot2$. The $7/2$ level does not cross the $9/2^g$ for any λ. When a potential of the form $(1 + P_x)\,V(r)/2$ is used (equal Wigner and Majorana forces), the situation is like that in the case of Wigner forces, except that the levels $9/2^g$ and $7/2$ are not so close. Also for the Gaussian potential $V(r) = V\,e^{-r^2/r_0^2}$ there is in the case of Majorana forces no region of λ in which the ground state has $J = 7/2$.

The question of the relative spacings of the other levels $J = 5/2$ and $J = 3/2$ is critical in this case because they should be found experimentally if they are not much higher. In order to clarify this point, we shall refer to an interesting case discussed by GOLDHABER and SUNYAR[22]. $_{36}Kr^{83}$ has an even number of protons and 47 neutrons which are equivalent to three holes in the $g_{9/2}$ sub-shell. The spin of the ground state is measured to be $9/2$. The isomeric transition is from a $p_{1/2}$ state not to the ground state but to a low lying $7/2+$ state, 9 keV above the ground $9/2$ state. The distance between the levels $p_{1/2}$ and $7/2+$ is $32\cdot2$ keV. If a state with $J = 5/2$ or $J = 3/2$ were lying in this spacing, not too close to the upper state, a transition to it would have been preferred upon the observed transition.

It is interesting that the $7/2$ state is close to the ground state also for small λ, so that perturbations from other configurations or tensor forces and other non-central interactions may bring it still lower. This question, however, should be considered in more detail.

III. Mutual Spin-Orbit Interactions.

§ 9. *Matrix elements of the mutual spin-orbit interaction.*

It is well known that neither the Thomas interaction nor the magnetic spin-orbit interaction are big enough to account for the wide separation between the states $j = l + 1/2$ and $j = l - 1/2$ of an odd nucleon as required by Mayer's shell model. CASE and PAIS[6]) introduced in a phenomenological way a strong spin-orbit interaction, in order to preserve charge symmetry of the nuclear forces in analyzing high energy nucleon-nucleon scattering. They also made a very rough estimation and found that this interaction could give a doublet splitting of the right order of magnitude necessary for the shell model.

Although there is not yet a conclusive evidence for the existence of such a strong spin-orbit interaction, it is interesting to calculate the effects of such an interaction in cases of a single nucleon outside closed shells.

The spin-orbit interaction of CASE and PAIS was assumed to be of the form:

$$V_{12} = V(|\vec{r_2} - \vec{r_1}|) \, (\vec{s}^{(1)} + \vec{s}^{(2)}, \vec{L}_{12}) \qquad (17)$$

where $\vec{s}^{(1)}$ and $\vec{s}^{(2)}$ are the spins of the two interacting nucleons, and $\vec{L}_{12}$ is their relative angular momentum

$$\hbar \, \vec{L}_{12} = (\vec{p_2} - \vec{p_1}) \times (\vec{r_2} - \vec{r_1}) .$$

It is immediately seen why it is extremely difficult to use here the Slater method. Such interactions were generally treated[8]) with the simplification of summation over all nucleons but one, so that $\sum\limits_{i \neq 1} V(|r_i - r_1)| \, (\vec{s}^{(i)} + \vec{s}^{(1)}, \vec{L}_{i,1})$ became $\tilde{V}(r_1) \, (\vec{s}^{(1)}, \vec{L_1})$ (in the case of an odd number of nucleons).

Using the method described above the solution of the problem becomes very simple, as $\vec{L}_{12}$ is exactly the $\vec{\Lambda}$ introduced before. To obtain the matrix elements, the operator $(\vec{s}^{(1)} + \vec{s}^{(2)}, \vec{L}_{12})$ is applied to a wave-function $\psi_{L_1}(\vec{R}) \, \psi_{\Lambda_1}(\vec{r})$; this yields a sum of functions of the same type. When we multiply this sum by another wave-function $\psi_{L_2}(\vec{R}) \, \psi_{\Lambda_2}(\vec{r})$ and integrate (the integration on the R-coordinate can be immediately done) we obtain the result as a sum of the integrals I_l [of course, I_0 will not appear as $\vec{\Lambda} \, \psi_{n0}^0(\vec{r})$ vanishes].

In order to calculate the matrix elements of (17) it is more convenient not to work with the A_x and A_y components of a vector,

but instead with the combinations $A_{\pm} = A_x \pm i\, A_y$. With this notation (17) can be written in the form:

$$V(r)\left[\frac{1}{2}\,(s^{(1)}_+ + s^{(2)}_+)\,\varLambda_- + \frac{1}{2}\,(s^{(1)}_- + s^{(2)}_-)\,\varLambda_+ + (s^{(1)}_z + s^{(2)}_z)\,\varLambda_z\right]. \quad (18)$$

The wave-functions with definite n_1, l_1 and n_2, l_2 in the (m_l, m_s)-scheme are:

$$(m_1^{\pm}, m_2^{\pm}) = (m_1, m_2)\,\delta\left(\pm\frac{1}{2}\Big|\sigma_1\right)\delta\left(\pm\frac{1}{2}\Big|\sigma_2\right) =$$
$$= \psi^{m_1}(r_1)\,\psi^{m_2}(r_2)\,\delta\left(\pm\frac{1}{2}\Big|\sigma_1\right)\delta\left(\pm\frac{1}{2}\Big|\sigma_2\right)$$

where σ_1 and σ_2 are the spin coordinates of the nucleons 1 and 2. The operators s_+, s_-, and s_z, when applied on these functions, give, as is well known:

$$s_+\,\delta(m_s|\sigma) = \left(\frac{1}{2} - m_s\right)\delta(-m_s|\sigma)$$

$$s_-\,\delta(m_s|\sigma) = \left(\frac{1}{2} + m_s\right)\delta(-m_s|\sigma) \qquad s_z\,\delta(m_s|\sigma) = m_s\,\delta(m_s|\sigma)\,.$$

The corresponding equations for a general L are:

$$L_{\pm}\,\psi_l^m = (l \mp m)^{\frac{1}{2}}\,(l \pm m + 1)^{\frac{1}{2}}\,\psi_l^{m \pm 1} \qquad L_z\,\psi_l^m = m\,\psi_l^m.$$

As (17) is linear in $\vec{s}^{(1)}$ and $\vec{s}^{(2)}$, the only non-vanishing matrix elements are those which connect states differing at most in one of the spin eigenvalues. As an example we calculate

$$\begin{aligned}
J(m_1^+, m_2^-) &= \sum_{\sigma_1, \sigma_2}\int\int (m_1, m_2)^* \,\delta\left(\frac{1}{2}\Big|\sigma_1\right)\delta\left(-\frac{1}{2}\Big|\sigma_2\right) V_{12}(m_1, m_2) \times \\
&\quad \times \delta\left(\frac{1}{2}\Big|\sigma_1\right)\delta\left(-\frac{1}{2}\Big|\sigma_2\right) d^3 r_1\, d^3 r_2 = \\
&= \sum_{\sigma_1, \sigma_2}\int\int (m_1, m_2)^* \,\delta\left(\frac{1}{2}\Big|\sigma_1\right)\delta\left(-\frac{1}{2}\Big|\sigma_2\right) V(r)\left(\frac{1}{2} - \frac{1}{2}\right) \\
&\quad \times \varLambda_z(m_1, m_2)\,\delta\left(\frac{1}{2}\Big|\sigma_1\right)\delta\left(\frac{1}{2}\Big|\sigma_2\right) d^3 r_1\, d^3 r_2 = 0 \\[4pt]
J(m_1^+, m_2^+) &= \int\int V(r)\,(m_1, m_2)^*\,\varLambda_z(m_1, m_2)\,d^3 r_1\, d^3 r_2 \\[4pt]
K(m_1^+, m_2^+) &= \int\int V(r)\,(m_1, m_2)^*\,\varLambda_z(m_2, m_1)\,d^3 r_1\, d^3 r_2 \\[4pt]
J(m_1^-, m_2^-) &= -\int\int V(r)\,(m_1, m_2)^*\,\varLambda_z(m_1, m_2)\,d^3 r_1\, d^3 r_2 \\[4pt]
K(m_1^-, m_2^-) &= -\int\int V(r)\,(m_1, m_2)^*\,\varLambda_z(m_2, m_1)\,d^3 r_1\, d^3 r_2\,.
\end{aligned} \right\} \quad (19)$$

These matrix elements will be used later on. If there is a difference in one of the spin values, Λ_+ and Λ_- may appear; for example:

$$\sum_{\sigma_1,\,\sigma_2} \int\int (m_1^\pm, m_2^-)^* \, V_{12}(m_1^\pm, m_2^+) \, d^3 r_1 \, d^3 r_2 =$$

$$= \int\int V(r) \, (m_1, m_2)^* \frac{1}{2} \Lambda_+ (m_1, m_2) \, d^3 r_1 \, d^3 r_2 .$$

§ 10. *Doublet splitting for a nucleon outside closed shells.*

We shall have to calculate matrix elements of 18) in the (m_l, m_s)-scheme and make applications for the doublet splitting for a nucleon outside closed shells (the effect of this interaction on the energy levels of Li[7] will be treated later). If the doublet splitting in such a case is large compared to the interaction energy between two nucleons in this unfilled shell, the jj-coupling scheme would result.

We start from the (m_l, m_s)-scheme and use the sum method to obtain the energies of the doublet components. We use the fact that M_J (as well as J) is an integral, and also the Landé interval rule which holds for these interactions. This rule states that the energy of the state $^{2S+1}L_J$ is equal to

$$\zeta(S, L) \, \frac{J(J+1) - L(L+1) - S(S+1)}{2} .$$

The expansion of the wave-functions (7), (8) in terms of the wave-functions $\psi_L(\vec{R}) \, \psi_\Lambda(\vec{r})$ is easily calculated by the method described above. The results we need are given in the Appendix. We treat the following cases:

A. A single p-nucleon outside the closed s-shell.

The wave-functions are $(m^\pm; 0^+0^-; 0^+0^-)$ where the last two quantum numbers refer to two s-protons, the other two to two s-neutrons and the $m^\pm$ refers to the odd p-nucleon. When computing the diagonal matrix elements which belong to these functions we see that the only non-vanishing contributions come from the ps-terms. These are exchange terms if the two nucleons have the same charge, and ordinary terms otherwise. The matrix elements were calculated with the help of formulae (19). For example:

$$J(\pm 1^+, 0^+) - K(\pm 1^+, 0^+) = \int\int V(r) \, (1, 0)^* \, \Lambda_z(1, 0) \, d^3 r_1 \, d^3 r_2 =$$

$$= \int\int V(r) \, \psi_1^{\pm 1}(r)^* \, \psi_0^0(R)^* \, \Lambda_z \, \psi_1^{\pm 1}(r) \, \psi_0^0(R) \, d^3 r \, d^3 R =$$

$$= \int V(r) \, \psi_1^{\pm 1}(r)^* \, \Lambda_z \, \psi_1^{\pm 1}(r) \, d^3 r =$$

$$= \pm \int V(r) \, \psi_1^{\pm 1}(r)^* \, \psi_1^{\pm 1}(r) \, d^3 r = \pm \, I_1$$

and similar calculations for other matrix elements. The results for the ps-diagonal elements are:

$$J(\pm 1^+, 0^+) - K(\pm 1^+, 0^+) = \pm I_1 \qquad J(\pm 1^+, 0^+) = \pm \frac{1}{2} I_1$$

$$J(0^+, 0^+) = K(0^+, 0^+) = 0$$

$$J(\pm 1^-, 0^-) - K(\pm 1^-, 0^-) = \mp I_1 \qquad J(\pm 1^-, 0^-) = \mp \frac{1}{2} I_1$$

$$J(0^-, 0^-) = K(0^-, 0^-) = 0.$$

With these results we build the following table:

| M_J | Wave-functions | ps diagonal matrix elements | | Sum of the energies |
		ordinary	exchange	
$\frac{3}{2}$	$(1^+; 0^+ 0^-; 0^+ 0^-)$	$(1^+, 0^+)$	$(1^+, 0^+)$	$\frac{3}{2} I_1$
$\frac{1}{2}$	$(0^+; 0^+ 0^-; 0^+ 0^-)$ $(1^-; 0^+ 0^-; 0^+ 0^-)$	$(0^+, 0^+)$ $(1^-, 0^-)$	$(0^+, 0^+)$ $(1^-, 0^-)$	$-\frac{3}{2} I_1$

As M_J is an integral the following equations can be written:

$$E(^2P_{3/2}) = \frac{3}{2} I_1$$

$$E(^2P_{3/2}) + E(^2P_{1/2}) = -\frac{3}{2} I_1$$

from which it follows:

$$E(^2P_{3/2}) = \frac{3}{2} I_1$$

$$E(^2P_{1/2}) = -3 I_1$$

in accordance with the interval rule. The splitting is accordingly:

$$\Delta E = \frac{9}{2} I_1.$$

B. A hole in the p-shell and a closed s-shell.

The relevant pp-matrix elements are calculated to be:

$$J(\pm 1^+, 0^+) - K(\pm 1^+, 0^+) = \pm \frac{1}{2} I_1$$

$$J(0^+, 0^+) - K(0^+, 0^+) = 0 = J(\pm 1^+, \mp 1^+) - K(\pm 1^+, \mp 1^+).$$

For the configuration $s^4 p^5$ the corresponding table is:

M_J	Wave-functions	Diagonal matrix elements			Sum of the energies		Total
		ps ordinary	ps exchange	pp exchange	ps	pp	
$\frac{3}{2}$	$(1^+, 1^-, -1^+, 0^+, 0^-;$ $0^+0^-; 0^+0^-)$	$(1^+, 0^+)$ $(1^-, 0^-)$ $(-1^+, 0^+)$	$(1^+, 0^+)$ $(1^-, 0^-)$ $(-1^+, 0^+)$	$(1^+, 0^+)$ $(1^-, 0^-)$ $(-1^+, 0^+)$	$-\frac{3}{2} I_1$	$-\frac{1}{2} I_1$	$-2 I_1$

From this table we obtain $E(^2P_{3/2}) = -2 I_1$, and with the help of the interval rule $E(^2P_{1/2}) = 4 I_1$. The doublet is inverted and the splitting is:

$$\Delta E = 6 I_1.$$

C. A single p-proton outside a closed s-shell and a closed neutron p-shell.

The additional matrix elements for this case are (ordinary):

$$J(1^+, 1^+) = I_2 \quad J(1^+, 0^+) = \frac{1}{4} I_1 + \frac{1}{4} I_2 \quad J(0^+, 0^+) = J(1^+, -1^+) = 0$$

from which we obtain:

M_J	Wave-functions	pp (ordinary) diagonal elements	sum of energies		Total
			pp	ps	
$\frac{3}{2}$	$(1^+; 1^+, 1^-, 0^+, 0^-, -1^+,$ $-1^-; 0^+, 0^-; 0^+ 0^-)$	$(1^+ 1^+) (1^+ 0^+)$	$\frac{1}{4} I_1 + \frac{5}{4} I_2$	$\frac{3}{2} I_1$	$\frac{7}{4} I_1 + \frac{5}{4} I_2$

This table gives

$$E(^2P_{3/2}) = \frac{7}{4} I_1 + \frac{5}{4} I_2,$$

and

$$E(^2P_{1/2}) = -\frac{7}{2} I_1 - \frac{5}{2} I_2.$$

Hence the splitting is:

$$\Delta E = \frac{21}{4} I_1 + \frac{15}{4} I_2.$$

D. A single d-nucleon outside the closed s- and p-shells.

The relevant matrix elements are:

$$ds$$

$$J(2^+, 0^+) - K(2^+, 0^+) = I_1 \qquad J(2^+, 0^+) = \frac{1}{2} I_1 + \frac{1}{2} I_2$$

$$dp$$

$$J(2^+, 1^+) - K(2^+,1^+) = \frac{1}{4} I_1 + \frac{9}{4} I_3 \qquad J(2^+, 1^+) = \frac{1}{8} I_1 + \frac{1}{4} I_2 + \frac{9}{8} I_3$$

$$J(2^+, 0^+) - K(2^+, 0^+) = \frac{1}{2} I_1 + \frac{1}{2} I_3 \qquad J(2^+, 0^+) = \frac{1}{4} I_1 + \frac{1}{2} I_2 + \frac{1}{4} I_3$$

$$J(2^+, -1^+) - K(2^+, -1^+) = \frac{1}{4} I_1 + \frac{1}{5} I_{11} + \frac{1}{20} I_3 = \frac{3}{4} I_1 - I_2 + \frac{3}{4} I_3$$

$$J(2^+, -1^+) = \frac{3}{8} I_1 - \frac{1}{4} I_2 + \frac{3}{8} I_3$$

with the help of which we obtain:

M_J	Wave-functions	diagonal matrix elements				sum of energies		Total sum
		ds		dp		ds	dp	
		ord.	exch.	ord.	exch.			
$\frac{5}{2}$	$(2^+; 1^+, 1^-, 0^+, 0^-,$ $-1^+, -1^-; 1^+, 1^-,$ $0^+, 0^-, -1^+, -1^-;$ $0^+, 0^-; 0^+0^-)$	$(2^+,0^+)$	$(2^+,0^+)$	$(2^+,1^+)$	$(2^+,1^+)$			
				$(2^+,0^+)$	$(2^+,0^+)$	$\frac{3}{2} I_1$ $+ \frac{1}{2} I_2$	$\frac{9}{4} I_1 - \frac{1}{2} I_2$ $+ \frac{21}{4} I_3$	$\frac{15}{4} I_1$ $+ \frac{21}{4} I_3$
				$(2^+,-1^+)$	$(2^+,-1^+)$			

Therefore

$$E(^2D_{5/2}) = \frac{15}{4} I_1 + \frac{21}{4} I_3, \qquad E(^2D_{3/2}) = -\frac{45}{8} I_1 - \frac{63}{8} I_3,$$

and the splitting is:

$$\Delta E = \frac{75}{8} I_1 + \frac{105}{8} I_3.$$

We thus see that:

for p-nucleon interacting with the 4 s-nucleons, the splitting is

$$\frac{9}{2} I_1;$$

for p-proton interacting with the 4 s-nucleons and the 6 p-neutrons, the splitting is

$$\frac{21}{4} I_1 + \frac{15}{4} I_2;$$

for d-nucleon interacting with the 4 s-nucleons and the 12 p-nucleons, the splitting is

$$\frac{75}{8} I_1 + \frac{105}{8} I_3.$$

CASE and PAIS gave in their paper[6]) an estimated form of the potential $V(r)$ to suit the experimental data of the scattering:

$$V(r) = V_0 \frac{1}{x} \frac{d}{dx} \frac{e^{-x}}{x}$$

where $x = r/r_0$.

The values of r_0 and V_0 were given roughly as $r_0 \sim 1{,}1 \times 10^{-13}$ cm, $V_0 \sim 24$ MeV. Using this potential the I_1 can be easily calculated. The result is:

$$I_1 = \frac{V_0}{3} \left[\frac{2}{\sqrt{\pi}} \frac{2\mu^2 - 1}{4\mu^3} - \left(1 - \Phi(\mu)\right) e^{\mu^2} \right]$$

where

$$\mu = \frac{1}{2\sqrt{\nu}\, r_0}.$$

For this value of r_0 and reasonable value of $\sqrt{\nu}$ the I_l, $l \geqslant 2$, are much smaller than I_1. The integral is very sensitive to the value of μ (and for a fixed value of r_0—to the value of $\sqrt{\nu}$). Some interesting values are:

$$\mu = 1{,}0 \quad 1{,}2 \quad 1{,}3 \quad 1{,}4 \quad 2{,}0$$

$$I_1 = 1{,}16 \quad 0{,}57 \quad 0{,}40 \quad 0{,}31 \quad 0{,}07 \text{ MeV}$$

We can fix the value of ν by determining the nuclear radius in terms of it:

$$R^2 = \overline{r^2} = N_l^2 \int\limits_0^\infty e^{-2\nu r^2} r^{2l+4}\, dr = \frac{2l+3}{4\nu}.$$

Using the relation $R \sim 1{\cdot}5 \times A^{1/3} \times 10^{-13}$ cm we obtain for the cases considered:

A. $\Delta E \sim 4$ MeV   **C.** $\Delta E \sim 2$ MeV   **D.** $\Delta E \sim 3{\cdot}5$ MeV.

The first case cannot give any information on He^5 or Li^5 as there are no bound states for these nuclei. It is interesting to see that with the parameters roughly determined from the scattering experiments the order of magnitude of a few MeV may be obtained as required by the shell model (although this calculation should not be considered more than an estimate).

*

IV. The Low States of Li⁷.

§ 11. *The Li⁷ nucleus.*

The most striking fact about the Li⁷ nucleus is the existence of
only one low lying level, 0·48 MeV above the ground state, and the
absence of any other level up to a few MeV. The most natural ex-
planation of such a situation is that of the LS-coupling scheme,
where the ground state and the first excited level are assumed to be
the components of a ^2P. However, various reactions were believed
to point out that the excited level has a more complicated character
and in certain reactions it behaves like a level with a spin 5/2. It was
therefore suggested by INGLIS[7]) that the jj-coupling scheme might
give the right explanation. In this scheme the ground level of the
configuration $(p_{3/2})^2_N\, p_{3/2\,P}$ has $J = 3/2$ and there are four higher
levels with the spins 1/2, 5/2, 7/2 and another 3/2 (which lies some-
what higher than the other three). If there are only Wigner forces
between the nucleons the levels $J = 1/2$, 5/2, 7/2 coincide, and it
was assumed by INGLIS that the excited level might be such a com-
pound one. However, even a rough coincidence did not result from
his calculations and the situation became still worse when Majorana
forces were included too[23]). There exists, however, another possibil-
ity, that such a coincidence of the first excited levels would result
from the effect of non-central forces, such as tensor forces, and hence
in the next sections the calculation of the configuration $(p_{3/2})^2_N\, p_{3/2\,P}$
is carried out in the extreme jj-coupling, taking into account tensor
forces and mutual spin-orbit interactions in addition to the central
forces.

Recent experimental results show that the assignment $J = 1/2$
to the excited level of Li⁷ is very probable[8]), and therefore the ex-
planation could be once again based on the LS-coupling scheme[24]).
The splitting of the two levels should then arise from the spin-orbit
interactions. As the Thomas term gives too small a contribution,
we calculate the splitting due to the Case and Pais interaction. It
should be mentioned that the lack of other levels below 5 MeV is
not very well understood in either coupling scheme. Such calcu-
lations should therefore, not be interpreted as successful explanation
but only as a preliminary survey of the various possibilities.

§ 12. *The configuration $(p_{3/2})^2_N\, p_{3/2\,P}$ with central forces.*

The energy values of the various states of the jj-coupling scheme
can be calculated by the sum method from the diagonal matrix
elements of the $(j_1,\, j_2,\, m_{j_1},\, m_{j_2})$-scheme, in the same way as in

atomic spectroscopy. The wave-function of a single nucleon with given n, l, $j = l + 1/2$ and m_j is given by:

$$u(n\,l\,j\,m_j|1) = (2l+1)^{-\frac{1}{2}}\left[\left(l+m_j+\frac{1}{2}\right)^{\frac{1}{2}} u\left(n\,l\,m_j-\frac{1}{2}\Big|1\right)\delta\left(\frac{1}{2}\Big|\sigma_1\right) + \right.$$
$$\left. +\left(l-m_j+\frac{1}{2}\right)^{\frac{1}{2}} u\left(n\,l\,m_j+\frac{1}{2}\Big|1\right)\delta\left(-\frac{1}{2}\Big|\sigma_1\right)\right] \qquad (20)$$

(the phases are fixed according to CONDON and SHORTLEY[25])), or briefly:

$$u(n\,l\,j\,m_j|1) = f\,u\left(n\,l\,m_j-\frac{1}{2}\Big|1\right)\delta\left(\frac{1}{2}\Big|\sigma_1\right) + g\,u\left(n\,l\,m_j+\frac{1}{2}\Big|1\right)\delta\left(-\frac{1}{2}\Big|\sigma_1\right).$$

For $l = 1$ and $j = 3/2$ the f and g are:

m_j	$\dfrac{3}{2}$	$\dfrac{1}{2}$	$-\dfrac{1}{2}$	$-\dfrac{3}{2}$
f	1	$\sqrt{\dfrac{2}{3}}$	$\sqrt{\dfrac{1}{3}}$	0
g	0	$\sqrt{\dfrac{1}{3}}$	$\sqrt{\dfrac{2}{3}}$	1

We define the integral J and the exchange integral K of the interaction by:

$$\left.\begin{aligned}
J(n\,l\,j\,m_j,\,n'\,l'\,j'\,m_j') &= \sum_{\sigma_1\sigma_2}\int\int u^*(n\,l\,j\,m_j|1)\,u^*(n'\,l'\,j'\,m_j'|2)\times \\
&\quad\times V_{12}\,u(n\,l\,j\,m_j|1)\,u(n'\,l'\,j'\,m_j'|2)\,d^3r_1\,d^3r_2 \\
K(n\,l\,j\,m_j,\,n'\,l'\,j'\,m_j') &= \sum_{\sigma_1\sigma_2}\int\int u^*(n\,l\,j\,m_j|1)\,u^*(n'\,l'\,j'\,m_j'|2)\times \\
&\quad\times V_{12}\,u(n\,l\,j\,m_j|2)\,u(n'\,l'\,j\,m_j'|1)\,d^3r_1\,d^3r_2\,.
\end{aligned}\right\} \qquad (21)$$

Exchange operators can be defined for the various central interactions in terms of the spin-exchange operator P_σ and the space-exchange operator P_x.

Type of force			
WIGNER (ordinary)	MAJORANA	HEISENBERG (char. exch.)	BARTLET
1	P_x	$-P_x P_\sigma$	P_σ

When the summation over the spin coordinates is carried out there remains a sum of integrals on space coordinates only, the coefficients

of which are products of f, g (of m_j), and of f', g' (of m_j). For ordinary (Wigner) interaction we obtain (the common n, l, and j were dropped out):

$$
\left.
\begin{aligned}
J(m_j, m_j') &= f^2 f'^2 J\left(m_j - \tfrac{1}{2},\ m_j' - \tfrac{1}{2};\ m_j - \tfrac{1}{2},\ m_j' - \tfrac{1}{2}\right) + \\
&+ f^2 g'^2 J\left(m_j - \tfrac{1}{2},\ m_j' + \tfrac{1}{2};\ m_j - \tfrac{1}{2},\ m_j' + \tfrac{1}{2}\right) + \\
&+ f'^2 g^2 J\left(m_j + \tfrac{1}{2},\ m_j' - \tfrac{1}{2};\ m_j + \tfrac{1}{2},\ m_j' - \tfrac{1}{2}\right) + \\
&+ g^2 g'^2 J\left(m_j + \tfrac{1}{2},\ m_j' + \tfrac{1}{2};\ m_j + \tfrac{1}{2},\ m_j' + \tfrac{1}{2}\right) \\[2mm]
K(m_j, m_j') &= f^2 f'^2 J\left(m_j - \tfrac{1}{2},\ m_j' - \tfrac{1}{2};\ m_j' - \tfrac{1}{2},\ m_j - \tfrac{1}{2}\right) + \\
&+ 2 f f' g g' J\left(m_j - \tfrac{1}{2},\ m_j' + \tfrac{1}{2};\ m_j' - \tfrac{1}{2},\ m_j + \tfrac{1}{2}\right) + \\
&+ g^2 g'^2 J\left(m_j + \tfrac{1}{2},\ m_j' + \tfrac{1}{2};\ m'_j + \tfrac{1}{2},\ m_j + \tfrac{1}{2}\right)
\end{aligned}
\right\} \quad (22)
$$

where J are the integrals on space coordinates defined by:

$$
J(m_1, m_2;\ m_3, m_4) =
$$
$$
\int \int V(r)\, u^*(m_1|1)\, u^*(m_2|2)\, u(m_3|1)\, u(m_4|2)\, d^3 r_1\, d^3 r_2 \qquad (23)
$$

some of them are the integrals encountered with among the matrix elements in the (m_l, m_l')-scheme, e.g.

$$
J(m_1, m_2;\ m_1, m_2) = J(m_1, m_2)
$$
$$
J(m_1, m_2;\ m_2, m_1) = K(m_1, m_2).
$$

It is easy to see that in order to obtain the matrix elements of the Majorana interaction, which is spin-independent in the above representation, we must take the same coefficients as in (22) and only multiply $V(r)$ in (23) by the operator P_x. This results in a change in the order of the last two quantum numbers in $J(m_1, m_2, m_3, m_4)$ which thus becomes $J(m_1, m_2, m_4, m_3)$. [This therefore transforms $J(m_1, m_2)$ into $K(m_1, m_2)$ and vice versa.]

In order to obtain the matrix elements of the Bartlet interaction, we must exchange the spin coordinates in (21). This can be achieved by first doing an exchange of both the space and the spin coordinates and then another space exchange. As a result J^B has the same coefficients as K^W but the last two quantum numbers are exchanged. This means:

$$
J^B = K^M \quad \text{and} \quad K^B = J^M.
$$

The matrix elements of the Heisenberg interaction should be related to those of the Wigner interaction in the same way, but for a change in sign:

$$J^H = -K^W \quad \text{and} \quad K^H = -J^W.$$

It is therefore enough to calculate the case of ordinary forces. This gives the matrix elements in the $(j_1, j_2, m_{j_1}, m_{j_2})$-scheme:

$$J\left(\frac{3}{2}, \frac{3}{2}\right) = J(1, 1) \qquad\qquad K\left(\frac{3}{2}, \frac{3}{2}\right) = K(1, 1)$$

$$J\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{2}{3} J(1, 0) + \frac{1}{3} J(1, 1) \qquad K\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{2}{3} K(1, 0)$$

$$J\left(\frac{3}{2}, -\frac{1}{2}\right) = \frac{1}{3} J(1, -1) + \frac{2}{3} J(1, 0) \qquad K\left(\frac{3}{2}, -\frac{1}{2}\right) = \frac{1}{3} K(1, -1)$$

$$J\left(\frac{3}{2}, -\frac{3}{2}\right) = J(1, -1) \qquad\qquad K\left(\frac{3}{2}, -\frac{3}{2}\right) = 0$$

$$J\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{4}{9} J(0, 0) + \frac{4}{9} J(1, 0) + \frac{1}{9} J(1, 1)$$

$$K\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{4}{9} K(0, 0) + \frac{4}{9} J(1, 0) + \frac{1}{9} K(1, 1)$$

$$J\left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{4}{9} J(0, 0) + \frac{2}{9} J(1, 0) + \frac{2}{9} J(0, -1) + \frac{1}{9} J(1, -1)$$

$$K\left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{2}{9} K(1, 0) + \frac{2}{9} K(0, -1) + \frac{4}{9} J(0, 0; -1, 1).$$

The matrix elements in the scheme (m_l, m_l'), i.e. $J(m_1, m_2, m_3, m_4)$ are easily calculated in terms of the I_l from the wave-functions of the pp-configuration A 3; they are (for ordinary forces):

$$J(1, 1) = \frac{1}{2}(I_0 + I_2) \qquad\qquad J(1, 0) = \frac{1}{4}(I_0 + I_2) + \frac{1}{2} I_1$$

$$J(0, 0) = \frac{3}{4}(I_0 + I_2) - \frac{1}{2} I_1 \qquad J(1, -1) = \frac{1}{2}(I_0 + I_2)$$

$$K(1, 1) = \frac{1}{2}(I_0 + I_2) \qquad\qquad K(1, 0) = \frac{1}{4}(I_0 + I_2) - \frac{1}{2} I_1$$

$$K(0, 0) = \frac{3}{4}(I_0 + I_2) - \frac{1}{2} I_1 \qquad K(1, -1) = \frac{1}{2}(I_0 + I_2) - I_1$$

$$J(0, 0; -1, 1) = -\frac{1}{4}(I_0 + I_2) + \frac{1}{2} I_1.$$

To pass from a direct to an exchange integral in this scheme the integrals I which arise from functions symmetric in the (space) coordinates of the two nucleons (those of even Λ, like I_0, I_2, I_{10}, etc.) should be kept unchanged, while those arising from antisymmetric functions (odd Λ, like I_1) should be multiplied by -1. In

terms of these matrix elements the diagonal elements of the $(j_1, j_2, m_{j_1}, m_{j_2})$-scheme can be written down.

In order to calculate the energy levels we write down the matrix elements corresponding to the various M_J [when writing $(m_{j_1}, m_{j_2}; m_{j_3})$ — m_{j_1} and m_{j_2} are the two neutrons' quantum numbers and m_{j_3} is the proton quantum number]. We obtain the following table:

M_J	States in the $(m_{j_1}, m_{j_2}; m_{j_3})$-scheme	Diagonal matrix elements	
		exchange	ordinary
$\frac{7}{2}$	$(\frac{3}{2}, \frac{1}{2}; \frac{3}{2})$	$(\frac{3}{2}, \frac{1}{2})$	$(\frac{3}{2}, \frac{3}{2})(\frac{3}{2}, \frac{1}{2})$
$\frac{5}{2}$	$(\frac{3}{2}, -\frac{1}{2}; \frac{3}{2})(\frac{3}{2}, \frac{1}{2}; -\frac{1}{2})$	$(\frac{3}{2}, \frac{1}{2})(\frac{3}{2}, -\frac{1}{2})$	$(\frac{3}{2}, \frac{3}{2})(\frac{3}{2}, \frac{1}{2})(\frac{3}{2}, -\frac{1}{2})(\frac{1}{2}, \frac{1}{2})$
$\frac{3}{2}$	$(\frac{3}{2}, -\frac{3}{2}; \frac{3}{2})(\frac{3}{2}, -\frac{1}{2}; \frac{1}{2})(\frac{3}{2}, \frac{1}{2}; -\frac{1}{2})(\frac{1}{2}, -\frac{1}{2}; \frac{3}{2})$	$(\frac{3}{2}, \frac{1}{2})(\frac{3}{2}, -\frac{1}{2})$ $(\frac{3}{2}, -\frac{3}{2})(\frac{1}{2}, -\frac{1}{2})$	$(\frac{3}{2}, \frac{3}{2})(\frac{3}{2}, \frac{1}{2})(\frac{3}{2}, -\frac{1}{2})(\frac{1}{2}, -\frac{1}{2})$ $(\frac{3}{2}, -\frac{3}{2})(\frac{3}{2}, -\frac{1}{2})(\frac{3}{2}, \frac{1}{2})(\frac{1}{2}, -\frac{1}{2})$
$\frac{1}{2}$	$(\frac{3}{2}, -\frac{3}{2}; \frac{1}{2})(\frac{1}{2}, -\frac{1}{2}; \frac{1}{2})(\frac{3}{2}, -\frac{1}{2}; -\frac{1}{2})(\frac{3}{2}, \frac{1}{2}; -\frac{3}{2})$ $(\frac{1}{2}, -\frac{3}{2}; \frac{3}{2})$	$(\frac{3}{2}, \frac{1}{2})(\frac{3}{2}, -\frac{1}{2})$ $(\frac{3}{2}, -\frac{3}{2})(\frac{1}{2}, -\frac{1}{2})$ $(\frac{1}{2}, -\frac{3}{2})$	$(\frac{3}{2}, -\frac{3}{2})(\frac{3}{2}, \frac{1}{2})(\frac{3}{2}, -\frac{1}{2})(\frac{1}{2}, \frac{1}{2})$ $(\frac{3}{2}, -\frac{3}{2})(\frac{3}{2}, \frac{1}{2})(\frac{3}{2}, -\frac{1}{2})(\frac{1}{2}, \frac{1}{2})$ $(\frac{1}{2}, -\frac{1}{2})(\frac{3}{2}, -\frac{1}{2})$

Thus we see that there are 5 independent states, namely those with $J = 7/2$ and $J = 5/2$, the two with $J = 3/2$ and the one with $J = 1/2$. The sum method can only give the sum of the two states with $J = 3/2$ but here we can separate the two levels with the help of another quantum number—the isotopic spin τ. 2 neutrons and 1 proton have the projection $\tau_z = -1/2$ of the isotopic spin $\vec{\tau}$, and therefore the states may have either $\tau = 1/2$ or $\tau = 3/2$. The states which have $\tau = 3/2$ are also the states of the configuration $(p_{3/2})^3_P$ $(\tau_z = 3/2)$. Now $(p_{3/2})^3_P$ has only one state—$J = 3/2$ $(\tau = 3/2)$ the energy of which can be immediately obtained from an analogous table:

M_J	States in the $(m_{j_1}, m_{j_2}, m_{j_3})$-scheme	Diagonal matrix elements	
		exchange	ordinary
$\frac{3}{2}$	$(\frac{3}{2}, \frac{1}{2}, -\frac{1}{2})$	$(\frac{3}{2}, \frac{1}{2})(\frac{3}{2}, -\frac{1}{2})(\frac{1}{2}, -\frac{1}{2})$	—

The sum of these energies is the energy of the state $J = 3/2$, $\tau = 3/2$. When this energy is subtracted from the energies of the two states with $J = 3/2$, which is obtained from the other table, the value of the level $J = 3/2$, $\tau = 1/2$ is left.

Using the values given above for the matrix elements we obtain the following values of the energy levels:

State \ Type of force	WIGNER	MAJORANA	HEISENBERG	BARTLET
$J=\frac{3}{2}$ $\tau=\frac{3}{2}$	$\frac{5}{6}(I_0+I_2)+\frac{4}{3}I_1$	$\frac{5}{6}(I_0+I_2)-\frac{7}{3}I_1$	$\frac{5}{6}(I_0+I_2)+\frac{4}{3}I_1$	$-\frac{5}{6}(I_0+I_2)+\frac{7}{3}I_1$
$J=\frac{7}{2}$ $\tau=\frac{1}{2}$	$I_0+I_2+I_1$	$I_0+I_2-I_1$	$-\frac{1}{2}(I_0+I_2)+I_1$	$\frac{1}{2}(I_0+I_2)+I_1$
$J=\frac{5}{2}$ $\tau=\frac{1}{2}$	$I_0+I_2+I_1$	$I_0+I_2-\frac{16}{9}I_1$	$-\frac{1}{2}(I_0+I_2)+I_1$	$\frac{1}{2}(I_0+I_2)+\frac{2}{9}I_1$
$J=\frac{1}{2}$ $\tau=\frac{1}{2}$	$I_0+I_2+I_1$	$I_0+I_2-\frac{8}{3}I_1$	$-\frac{1}{2}(I_0+I_2)+I_1$	$\frac{1}{2}(I_0+I_2)-\frac{2}{3}I_1$
$J=\frac{3}{2}$ $\tau=\frac{1}{2}$	$\frac{11}{6}(I_0+I_2)-\frac{2}{3}I_1$	$\frac{11}{6}(I_0+I_2)-\frac{7}{3}I_1$	$\frac{1}{3}(I_0+I_2)-\frac{2}{3}I_1$	$-\frac{1}{3}(I_0+I_2)+\frac{4}{3}I_1$

The same results expressed in terms of the Slater integrals are:

State \ Type of force	WIGNER	MAJORANA	HEISENBERG	BARTLET
$J=\frac{3}{2}$ $\tau=\frac{3}{2}$	$3F_0-5F_2$	$-\frac{2}{3}F_0+\frac{40}{3}F_2$	$3F_0-5F_2$	$\frac{2}{3}F_0-\frac{40}{3}F_2$
$J=\frac{7}{2}$ $\tau=\frac{1}{2}$	$3F_0-3F_2$	F_0+7F_2	$-6F_2$	$2F_0-4F_2$
$J=\frac{5}{2}$ $\tau=\frac{1}{2}$	$3F_0-3F_2$	$\frac{2}{9}F_0+\frac{98}{9}F_2$	$-6F_2$	$\frac{11}{9}F_0-\frac{1}{9}F_2$
$J=\frac{3}{2}$ $\tau=\frac{1}{2}$	$3F_0-3F_2$	$-\frac{2}{3}F_0+\frac{46}{3}F_2$	$-6F_2$	$\frac{1}{3}F_0+\frac{13}{3}F_2$
$J=\frac{3}{2}$ $\tau=\frac{1}{2}$	$3F_0+7F_2$	$\frac{4}{3}F_0+\frac{46}{3}F_2$	$+4F_2$	$\frac{2}{3}F_0-\frac{22}{3}F_2$

The connection between the I_l and the F_k is given by:

$$I_0+I_2=2F_0+2F_2 \qquad F_0=\frac{5}{12}(I_0+I_2)+\frac{1}{6}I_1 \qquad F_0=F^0$$
$$\text{or}$$
$$I_1=F_0-5F_2 \qquad F_2=\frac{1}{12}(I_0+I_2)-\frac{1}{6}I_1 \qquad F_2=F^2/25.$$

(Only the combinations I_0+I_2 appears, and thus there are only two parameters I_0+I_2 and I_1 replacing F_0 and F_2.)

In the short range limit — $I_l=0$, $l>0$ (or $F^2=5F_0$), the results for Wigner forces and Majorana forces coincide, because then only space-symmetric functions contribute to the energy (the same applies for HEISENBERG and BARTLET, except for a change in sign).

It is interesting to note that in the long range limit, when all the I_l are equal ($F^k=0$, $k>0$), the term values of the Heisenberg interaction vanish for $\tau=1/2$. This fact can be explained in a general way. With the help of the isotopic spin $\vec{\tau}$ the Heisenberg interaction can be written as:

$$\sum_{i<k}V_{ik}=\sum_{i<k}\frac{1+4(\vec{\tau}^{(i)},\vec{\tau}^{(k)})}{2}V(|\vec{r}_i-\vec{r}_k|).$$

If $V(|\vec{r}_i - \vec{r}_k|)$ has a constant value in the region where the wave-functions differ from zero, we can put it out of the integral sign when calculating the energy, and what remains is the diagonal value of the operator $\sum\limits_{i<k} [1 + 4(\vec{\tau}^{(i)}, \vec{\tau}^{(k)})]/2$ belonging to the state considered. This value can be immediately found as follows (it is an eigenvalue if the total isotopic spin τ is a quantum number of the given state):

$$T = \sum_{i<k} \frac{1}{2}\left(1 + 4(\vec{\tau}^{(i)}, \vec{\tau}^{(k)})\right) = \sum_{i<k} \frac{1}{2} + \sum_{i<k} 2(\vec{\tau}^{(i)}, \vec{\tau}^{(k)}) = \frac{n(n-1)}{4} +$$
$$+ \tau(\tau+1) - n\tau^{(i)}(\tau^{(i)}+1) = \tau(\tau+1) + \frac{n^2}{4} - n$$

$$\tau^{(i)}(\tau^{(i)}+1) = \frac{3}{4}.$$

This is the factor of F_0 or I_0 when $I_l = I_0$, in any state with given n and τ. In the above case $n = 3$ and we obtain:

$$\tau = \frac{3}{2} \qquad T = 3; \qquad \tau = \frac{1}{2} \qquad T = 0.$$

The coefficients of F_0 for Bartlet forces (and jj-coupling) can be evaluated in the same way was used by RACAH[19]). He evaluated the diagonal value of $\sum\limits_{i<k}(1 - P_\sigma^{ik})/2$, and therefore, in order to obtain the diagonal value of $\sum\limits_{i<k} P_\sigma^{ik}$, the value he obtained should be multiplied by -2 and to it $n(n-1)/4$ should be added. The result is:

$$\frac{n(n-1)}{4} - \frac{n\,j(j+1) - J(J+1)}{(2l+1)^2}.$$

§ 13. *Matrix elements of the Tensor Forces Interaction.*

The interaction energy in the case of tensor forces is:

$$V_{12} = S_{12}\,V(r) = \left[\frac{(\vec{s}^{(1)}, \vec{r})\,(\vec{s}^{(2)}, \vec{r})}{r^2} - \frac{1}{3}(\vec{s}^{(1)}, \vec{s}^{(2)})\right] V(r) \qquad \vec{r} = \vec{r}_2 - \vec{r}_1. \quad (24)$$

It appears also in atomic spectroscopy as the magnetic spin-spin interaction. Also here the development in a series of spherical harmonics is complicated and would not be practical to work with. The tensor forces play an important role in nuclear physics. They have been used to explain the quadrupole moment of the Deutron and, very extensively, in the analysis of scattering experiments. They may also give, in second approximation, a contribution to the doublet splitting.

By our method it is quite a simple matter to obtain matrix elements of the interaction (24). It consists, in fact, only of the

computation of matrix elements of spherical harmonics with $l = 2$, which can be done generally by the use of the Gaunt formula[26]), or else by direct integration. The operator S_{12} of (24) can be written as:

$$
\begin{aligned}
&\left(\vec{s}^{(1)}, \frac{\vec{r}}{r}\right)\left(\vec{s}^{(2)}, \frac{\vec{r}}{r}\right) - \frac{1}{3}(\vec{s}^{(1)}, \vec{s}^{(2)}) = (s_+^{(1)} s_-^{(2)} + s_-^{(1)} s_+^{(2)})\left(\frac{1}{4}\sin^2\vartheta - \frac{1}{6}\right) + \\
&\quad + s_z^{(1)} s_z^{(2)}\left(\cos^2\vartheta - \frac{1}{3}\right) + \frac{1}{2}(s_z^{(1)} s_-^{(2)} + s_-^{(1)} s_z^{(2)})\sin\vartheta\cos\vartheta\, e^{i\varphi} + \\
&\quad + \frac{1}{2}(s_z^{(1)} s_+^{(2)} + s_+^{(1)} s_z^{(2)})\sin\vartheta\cos\vartheta\, e^{-i\varphi} + \frac{1}{4} s_-^{(1)} s_-^{(2)} \times \\
&\quad \times \sin^2\vartheta\, e^{2i\varphi} + \frac{1}{4} s_+^{(1)} s_+^{(2)} \sin^2\vartheta\, e^{-2i\varphi} = \\
&= \sqrt{\frac{8}{45}}\left[s_z^{(1)} s_z^{(2)} - \frac{1}{4}\left(s_+^{(1)} s_-^{(2)} + s_-^{(1)} s_+^{(2)}\right)\right] Y_2^{(0)}(\vartheta\,\varphi) - \\
&\quad - \frac{1}{\sqrt{15}}(s_z^{(1)} s_-^{(2)} + s_-^{(1)} s_z^{(2)}) Y_2^{(1)}(\vartheta, \varphi) + \\
&\quad + \frac{1}{\sqrt{15}}(s_z^{(1)} s_+^{(2)} + s_+^{(1)} s_z^{(2)}) Y_2^{(-1)}(\vartheta, \varphi) + \\
&\quad + \frac{1}{\sqrt{15}} s_-^{(1)} s_-^{(2)} Y_2^{(2)}(\vartheta, \varphi) + \frac{1}{\sqrt{15}} s_+^{(1)} s_+^{(2)} Y_2^{(-2)}(\vartheta, \varphi).
\end{aligned}
\tag{25}
$$

In order to obtain the matrix elements in the $(j_1, j_2, m_{j_1}, m_{j_2})$-scheme we use the wave-functions A 3. S_{12} operates on the spin coordinates of the two nucleons, with respect to which the summation can then be carried out, and what remains is a sum of integrals on the r-coordinate with products of f, g, f', g' as coefficients. The angular integration can be easily done and the radial integrals can be expressed in terms of the I_l. In addition to the integrals met with before, there appears also:

$$
I_{2,10} = N_2 N_{10} \int_0^\infty V(r)\, R_2(r)\, R_{10}(r)\, dr = \sqrt{\frac{2}{5}}\left(\frac{3}{2} I_1 - \frac{5}{2} I_2\right).
$$

I_0 does not appear in these sums because S_{12} contains only spherical harmonics of order 2 [as a result of subtracting $(\vec{s}^{(1)}, \vec{s}^{(2)})/3$].

We shall give here the results for the configuration $(p_{3/2})_N^2\, p_{3/2\,P}$. The exchange integrals K are easily derived from the direct integrals J by an exchange of only the space coordinates of the two nucleons (multiplication with P_x). This means that integrals arising from symmetric functions (even Λ, like I_2, $I_{2,10}$, etc.) remain unchanged, while those which arise from antisymmetric functions (odd Λ, like I_1) are multiplied by -1. This is a result of the fact that the operator S_{12} is diagonal with respect to the total spin (of the two nucleons) S, and that its eigenvalue for the singlet state ($S = 0$) is zero. There-

fore only that part of the wave-function which is in the triplet state contributes to the matrix elements, and that part is symmetrical with respect to the two spin coordinates. Thus instead of spin and space exchange, space exchange alone is sufficient to obtain the K from the J integral. This is clearly seen in the results:

$$J\left(\frac{3}{2}, \frac{3}{2}\right) = -\frac{1}{42} I_2$$

$$J\left(\frac{3}{2}, \frac{1}{2}\right) = \frac{1}{180} I_1 - \frac{1}{84} I_2$$

$$J\left(\frac{3}{2}, -\frac{1}{2}\right) = \frac{1}{180} I_1 - \frac{5}{756} I_2 - \frac{1}{135} \sqrt{\frac{5}{2}} I_{2,10} = -\frac{1}{180} I_1 + \frac{1}{84} I_2$$

$$J\left(\frac{3}{2}, -\frac{3}{2}\right) = \frac{1}{60} I_1 - \frac{1}{252} I_2 - \frac{1}{90} \sqrt{\frac{5}{2}} I_{2,10} = \frac{1}{42} I_2$$

$$J\left(\frac{1}{2}, \frac{1}{2}\right) = -\frac{19}{1134} I_2 - \frac{4}{405} \sqrt{\frac{5}{2}} I_{2,10} = -\frac{2}{135} I_1 + \frac{1}{126} I_2$$

$$J\left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{60} I_1 - \frac{25}{2268} I_2 - \frac{1}{810} \sqrt{\frac{5}{2}} I_{2,10} = \frac{2}{135} I_1 - \frac{1}{126} I_2$$

$$K\left(\frac{3}{2}, \frac{3}{2}\right) = -\frac{1}{42} I_2$$

$$K\left(\frac{3}{2}, \frac{1}{2}\right) = -\frac{1}{180} I_1 - \frac{1}{84} I_2$$

$$K\left(\frac{3}{2}, -\frac{1}{2}\right) = -\frac{1}{180} I_1 - \frac{5}{756} I_2 - \frac{1}{135} \sqrt{\frac{5}{2}} I_{2,10} = -\frac{1}{60} I_1 + \frac{1}{84} I_2$$

$$K\left(\frac{3}{2}, -\frac{3}{2}\right) = -\frac{1}{60} I_1 - \frac{1}{252} I_2 - \frac{1}{90} \sqrt{\frac{5}{2}} I_{2,10} = -\frac{1}{30} I_1 + \frac{1}{42} I_2$$

$$K\left(\frac{1}{2}, \frac{1}{2}\right) = -\frac{2}{135} I_1 + \frac{1}{126} I_2$$

$$K\left(\frac{1}{2}, -\frac{1}{2}\right) = -\frac{1}{60} I_1 - \frac{25}{2268} I_2 - \frac{1}{810} \sqrt{\frac{5}{2}} I_{2,10} = -\frac{1}{54} I_1 - \frac{1}{126} I_2 .$$

When treating the tensor forces as a perturbation on the strong spin-orbit interaction which gives rise to the jj-coupling scheme, the sum method can be used to determine the first order contributions to the energy levels. This is done in the same manner as in the case of central forces. We give here the results for an ordinary potential and a potential which is multiplied by the operator P_x (this simply replaces J by K and vice versa). The I_l in the following table are 12 times those defined above, as it is customary to define S_{12} as

$3/r^2\,(\vec{\sigma}^{(1)},\,\vec{r})\,(\vec{\sigma}^{(2)},\,\vec{r})-(\vec{\sigma}^{(1)},\,\vec{\sigma}^{(2)})$ which is $3\cdot 4 = 12$ times the operator S_{12} in (24).

State		Ordinary potential	Potential with space exchange
$J=\frac{3}{2}$	$\tau=\frac{3}{2}$	$\frac{2}{3}I_1$	$-\frac{2}{3}I_1$
$J=\frac{7}{2}$	$\tau=\frac{1}{2}$	$\frac{1}{5}I_1-\frac{3}{7}I_2$	$-\frac{1}{5}I_1-\frac{3}{7}I_2$
$J=\frac{5}{2}$	$\tau=\frac{1}{2}$	$-\frac{1}{9}I_1+\frac{5}{21}I_2$	$-\frac{23}{45}I_1+\frac{5}{21}I_2$
$J=\frac{1}{2}$	$\tau=\frac{1}{2}$	$-\frac{7}{15}I_1+I_2$	$-\frac{13}{15}I_1+I_2$
$J=\frac{3}{2}$	$\tau=\frac{1}{2}$	$\frac{2}{3}I_1$	$-\frac{16}{15}I_1$

The matrix elements of the mutual spin-orbit interaction (17) can be calculated with the help of those in the $(m_l',\,m_l)$-scheme. The wave-functions (20) are used and the matrix elements are the sums of matrix elements in the $(m_l,\,m_l')$-scheme, with coefficients which are products of f, g, f', and g'. Here it is seen, much better than in the $(m_l,\,m_l')$-scheme that the K integrals differ from the J only by the sign of the I_l arising from space antisymmetric functions. The reason for it is the same as in the case of the tensor forces. The operator $(\vec{s}^{(1)}+\vec{s}^{(2)},\,\vec{\Lambda})$ is diagonal with respect to the magnitude of the total spin S $(\vec{S}=\vec{s}^{(1)}+\vec{s}^{(2)})$ and its eigenvalue in the singlet state is zero. [In the $(m_l,\,m_l')$-scheme this fact manifests itself in the vanishing of the matrix elements when $m_{s_1}=\pm 1/2$ $m_{s_2}=\mp 1/2$]. The results are:

$$J\left(\frac{3}{2},\frac{3}{2}\right)=I_2 \qquad J\left(\frac{3}{2},\frac{1}{2}\right)=\frac{1}{6}I_1+\frac{1}{2}I_2 \qquad J\left(\frac{3}{2},-\frac{1}{2}\right)=\frac{1}{6}I_1+\frac{1}{6}I_2$$

$$J\left(\frac{3}{2},-\frac{3}{2}\right)=0 \qquad J\left(\frac{1}{2},\frac{1}{2}\right)=\frac{5}{9}I_2 \qquad\qquad J\left(\frac{1}{2},-\frac{1}{2}\right)=\frac{4}{9}I_2$$

$$K\left(\frac{3}{2},\frac{3}{2}\right)=I_2 \qquad K\left(\frac{3}{2},\frac{1}{2}\right)=-\frac{1}{6}I_1+\frac{1}{2}I_2 \qquad K\left(\frac{3}{2},-\frac{1}{2}\right)=-\frac{1}{6}I_1+\frac{1}{6}I_2$$

$$K\left(\frac{3}{2},-\frac{3}{2}\right)=0 \qquad K\left(\frac{1}{2},\frac{1}{2}\right)=\frac{5}{9}I_2 \qquad\qquad K\left(\frac{1}{2},-\frac{1}{2}\right)=\frac{4}{9}I_2.$$

From these matrix elements the first order contributions to the energy levels of the configuration $(p_{3/2})^2_N\,p_{3/2\,P}$ can be calculated with the help of the sum method. These contributions are:

τ \ J	$\frac{7}{2}$	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
$\frac{3}{2}$			$\frac{2}{3}I_1$	
$\frac{1}{2}$	$\frac{1}{2}I_1+\frac{3}{2}I_2$	$\frac{1}{2}I_1+\frac{13}{18}I_2$	$-\frac{1}{3}I_1+I_2$	$\frac{1}{2}I_1-\frac{1}{6}I_2$

§ 14. *Numerical results.*

In order to obtain numerical results we have to evaluate the I_l. We assume that the potential of the interaction between the nucleons has the radial dependence of the Yukawa potential. The general formulas for the I_l in this case were given in § 6.

The value of $\sqrt{\nu}$, which measures the extension of the wavefunctions, can be roughly determined from the Li^7-Be^7 Coulomb energy difference. This gives: $1/\sqrt{\nu} \sim 2 \cdot 4 \times 10^{-13}$ cm.

The value of r_0 of the Yukawa potential can be taken from the analysis of scattering experiments; it is between $1 \cdot 1$ and $1 \cdot 4 \times 10^{-13}$ cm. This determines roughly $\lambda \sim 0 \cdot 5$, $\mu \sim 1$. For $\lambda = 0 \cdot 5$, $\mu = 1$ the following values are obtained:

$$I_0 = 0 \cdot 137 \ V \qquad I_1 = 0 \cdot 040 \ V \qquad I_2 = 0 \cdot 014 \ V.$$

For the tensor forces we take the same radial dependence as for the central interactions, with the same results for the I_l.

We shall now show that there are certain values of the interaction parameters for which the four excited states (or three of them) nearly coincide. Such a possibility can only mean that the interpretation in terms of the jj-coupling scheme could have been consistent with the existence of a low-lying compound level, if this were experimentally verified.

State	Exchange operator $(1 + P_x)/2$			Exchange operator $(1 + P_x)(\frac{4}{3} + \frac{1}{3}P_\sigma)/2$		
	$\gamma = 0$	$\gamma = 3$	$\gamma = 4$	$\gamma = 0$	$\gamma = 6$	$\gamma = 8$
$J = \frac{3}{2} \ \tau = \frac{3}{2}$	0·68	1·28 (1·98)	1·48 (2·18)	0·68	1·88 (2·55)	2·28 (2·98)
$J = \frac{7}{2}$	1·00	1·30 (1·80)	1·40 (1·90)	1·50	2·10 (2·60)	2·30 (2·80)
$J = \frac{5}{2}$	0·88	1·26 (1·76)	1·38 (1·88)	1·30	2·05 (2·55)	2·30 (2·80)
$J = \frac{1}{2}$	0·75	1·23 (1·73)	1·39 (1·89)	1·10	2·06 (2·56)	2·38 (2·88)
$J = \frac{3}{2} \ \tau = \frac{1}{2}$	1·38	2·34 (2·04)	2·66 (2·36)	1·86	3·80 (3·50)	4·40 (4·10)

For the case in which an exchange operator $(1 + P_x)/2$ multiplies the potential, the coincidence of the higher levels can be achieved by adding a tensor force with an exchange operator $-P_x$. If we take the exchange operator of the central forces to be

$$\frac{1}{2}(1 + P_x)\left(\frac{4}{3} + \frac{1}{3}P_\sigma\right) = \frac{1}{2}\left(\frac{4}{3}W + \frac{4}{3}M - \frac{1}{3}H + \frac{1}{3}B\right)$$

where W, M, H, and B are the exchange operators of the Wigner, Majorana, Heisenberg, and Bartlet interactions respectively, a co-

incidence can also be attained with the same form of the tensor forces. The results are given in the preceding table where the energy values are expressed in units of $V/7$ and V is the constant of the Yukawa potential in the case of central forces. The corresponding constant for tensor forces is γV.

If V is equal to 7 MeV the above energies are in units of MeV. The spacing between the ground state and the excited (compound) level is then of the right order of magnitude but bigger than the observed 0·48 MeV. If the contribution of the mutual spin-orbit interaction is added, this difference decreases. The term values so corrected are given in the above table in parentheses.

Obviously there are enough parameters to explain the coincidence and the level spacing. The above calculation points out that with reasonable values of them agreement can be achieved, but as long as more exact values of the interaction parameters (magnitude and range of the potential, exchange character, etc.) cannot be fixed with the help of other experiments, no decisive conclusion can be drawn.

§ 15. *Splitting of the $(p_N^2 \, p_P)^{2\,2}P$ ground state due to spin-orbit interaction.*

As already noted the new experiments favour the assignment $J = 1/2$ for the first excited level of Li^7. It became also clear that the level is quite sharp and there are no other levels in its close neighbourhood[8]). The explanation in terms of the LS-coupling scheme seems therefore plausible. There is also some evidence that the ground state of Li^6 is $^3S_1$ which means that in that nucleus LS-coupling exists. The configuration $p_N^2 \, p_P$ in the LS-coupling scheme was treated by WIGNER & FEENBERG[27][28]), and by HUND[29]). For Wigner and Majorana forces the lowest state is a ^2P with $\tau = 1/2$. According to this the two lowest levels of Li^7 should be interpreted as the two components of this doublet with $J = 3/2$ and $J = 1/2$. With this interpretation the problem still exists where are the other levels of this configuration, specially the next ^2F ($\tau = 1/2$) which should not be as high as 5 MeV (for reasonable values of the parameters). We shall calculate the splitting of the ground ^2P, arising from the mutual spin-orbit interaction of the type introduced by CASE and PAIS, as it is well known that the Thomas interaction and the magnetic spin-orbit interaction give too small a contribution (a splitting can arise also from tensor forces, but only in the second approximation).

The energy of the ground ^2P is for Wigner and Majorana forces:

$$3\,F_0 + 18\,F_2 = \frac{11}{4}\,(I_0 + I_2) - \frac{5}{2}\,I_1$$

(in the notation of HUND it is $3\,A + 2\,B$ where $A = F_0 + 4\,F_2$ and $B = 3\,F_2$). This energy is not degenerate (some of the energy values of higher states are twice and thrice degenerate), and thus it is possible to find the wave-function of the ground state by direct diagonalization of the energy matrix in the scheme $(m_{l_1},\, m_{l_2};\, m_{l_3})$. The energy is not degenerate also in the short range limit, so that we may calculate the energy matrix in this limit, which makes the calculation simpler.

The wave-function of the ground ^2P (or the ground ^{22}P in the notation $^{(2\,\iota+1)(2\,S+1)}L_J$) in the state $M_J = 3/2$ is a linear combination of those wave-functions $(m_{l_1}^{\pm},\, m_{l_2}^{\pm};\, m_{l_3}^{\pm})$ for which

$$M_S = m_{s_1} + m_{s_2} + m_{s_3} = +\frac{1}{2} \quad \text{and} \quad M_L = m_{l_1} + m_{l_2} + m_{l_3} = +1.$$

There are 8 such functions [in the notation $(m_{l_1}^{m_{s_1}},\, m_{l_2}^{m_{s_2}};\, m_{l_3}^{m_{s_3}})$ 1 and 2 are the neutrons numbers and 3 is the proton number] and they are listed below:

$\psi_1 = (1^+,\, -1^+;\, 1^-)$ $\psi_4 = (1^-,\, -1^+;\, 1^+)$ $\psi_7 = (1^-,\, 0^+;\, 0^+)$

$\psi_2 = (1^+,\, 0^+;\, 0^-)$ $\psi_5 = (1^+,\, 1^-;\, -1^+)$ $\psi_8 = (0^+,\, 0^-;\, 1^+)$

$\psi_3 = (1^+,\, -1^-;\, 1^+)$ $\psi_6 = (1^+,\, 0^-;\, 0^+)$

With these functions the energy matrix in the short range limit, where only I_0 is different from zero, can be built and the diagonalization carried out.

The energy matrix in units of $1/4\,I_0$.

4	−1						
−1	4						
		6	−2	2	−1	0	−1
		−2	6	−2	0	−1	1
		2	−2	6	−1	1	0
		−1	0	−1	5	−1	1
		0	−1	1	−1	5	−1
		−1	1	0	1	−1	5

The matrix is already reduced and the value of $11/4\,I_0$ of the energy of the ground ^2P is an eigenvalue of the 6-row matrix. The corresponding wave-function is a linear combination of the 6 functions ψ_3, ψ_4, ψ_5, ψ_6, ψ_7, ψ_8 the coefficients of which are the components of the eigenvector of the 6-row matrix which belongs to the eigenvalue 11. The linear equations are easily solved with the following results for the coefficients (after normalization to unity):

$$\psi\left(^{22}P_{3/2}^{g};\ M_J = \frac{3}{2}\right) = \frac{2}{\sqrt{15}}\,(1^+, -1^-; 1^+) - \frac{2}{\sqrt{15}}\,(1^-, -1^+; 1^+) +$$

$$+ \frac{2}{\sqrt{15}}\,(1^+, 1^-, -1^+) - \frac{1}{\sqrt{15}}\,(1^+, 0^-; 0^+) + \frac{1}{\sqrt{15}}\,(1^-, 0^+; 0^+) -$$

$$- \frac{1}{\sqrt{15}}\,(0^+, 0^-; 1^+) .$$

The diagonal matrix elements of the mutual spin-orbit interaction which belongs to this wave-function can be now calculated with the help of the matrix elements in the (m_l, m_i)-scheme. In order to obtain the total splitting this function should be multiplied by the wave-functions of the 4 s-nucleons and so the contribution of the interaction of the p- with the s-nucleons will be added to the interaction of the 3 p-nucleons themselves. The calculation is performed in the same way as in cases B and C above. The results are:

Energy of the sp-interaction	Energy of the pp-interaction	Total
$\frac{1}{15} \cdot \frac{15}{2}\,I_1 = \frac{1}{2}I_1$	$\frac{1}{15} \cdot 5\,I_2 = \frac{1}{3}I_2$	$\frac{1}{2}I_1 + \frac{1}{3}I_2$

This is the energy of the state $^{22}P_{3/2}$:

$$E\left(^{22}P_{3/2}^{g}\right) = \frac{1}{2}\,I + \frac{1}{3}\,I$$

the Landé interval rule yields

$$E\left(^{22}P_{1/2}^{g}\right) = -\,I - \frac{2}{3}\,I$$

so that the splitting is:

$$\Delta E = \frac{3}{2}\,I_1 + I_2 .$$

The integral I_1 was already calculated with the potential given by CASE and PAIS. The value of $\sqrt{\nu}$ can be roughly fixed by the determination of the Coulomb energy difference between Li^7 and

Be[7] in terms of it. This gives for the splitting the order of magnitude of ~ 1 MeV. The smaller level spacing in Be[7] can be explained by considering the Coulomb expansion. It is not possible to make more exact statements as the value of I_1 is very sensitive to the values of the parameters. Although this should not be considered as an exact determination of the energies, it is interesting to see that the right order of magnitude is obtained.

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<div align="center">APPENDIX.</div>

<div align="center">*Expansion of the wave-functions of two nucleons.*</div>

<div align="center">**ss.**</div>

$$(0, 0) = \psi_0^0(R)\, \psi_0^0(r) \tag{A 1}$$

<div align="center">**ps.**</div>

$$\left.\begin{aligned}
(\pm 1, 0)_s &= \frac{1}{\sqrt{2}}\{(\pm 1, 0) + (0, \pm 1)\} = \psi_1^{\pm 1}(R)\, \psi_0^0(r)\\
(0, 0)_s &= \psi_1^0(R)\, \psi_0^0(r)\\
(\pm 1, 0)_a &= \frac{1}{\sqrt{2}}\{(\pm 1, 0) - (0, \pm 1)\} = -\, \psi_1^{\pm 1}(r)\, \psi_0^0(R)\\
(0, 0)_a &= -\, \psi_1^0(r)\, \psi_0^0(R)
\end{aligned}\right\} \tag{A 2}$$

<div align="center">**pp.**</div>

$$\left.\begin{aligned}
(\pm 1, \pm 1) &= \frac{1}{\sqrt{2}}\left(\psi_2^{\pm 2}(R)\, \psi_0^0(r) - \psi_2^{\pm 2}(r)\, \psi_0^0(R)\right)\\
(\pm 1, 0)_s &= \frac{1}{\sqrt{2}}\left(\psi_2^{\pm 1}(R)\, \psi_0^0(r) - \psi_2^{\pm 1}(r)\, \psi_0^0(R)\right)\\
(\pm 1, 0)_a &= \frac{1}{\sqrt{2}}\left(\psi_1^{\pm 1}(R)\, \psi_1^0(r) - \psi_1^{\pm 1}(r)\, \psi_1^0(R)\right)\\
(0, 0) &= \frac{1}{\sqrt{3}}\left(\psi_2^0(R)\, \psi_0^0(r) - \psi_2^0(r)\, \psi_0^0(R)\right) -\\
&\quad - \frac{1}{\sqrt{6}}\left(\psi_{10}^0(R)\, \psi_0^0(r) - \psi_{10}^0(r)\, \psi_0^0(R)\right)\\
(\pm 1, \mp 1)_s &= \frac{1}{\sqrt{6}}\left(\psi_2^0(R)\, \psi_0^0(r) - \psi_2^0(r)\, \psi_0^0(R)\right) +\\
&\quad + \frac{1}{\sqrt{3}}\left(\psi_{10}^0(R)\, \psi_0^0(r) - \psi_{10}^0(r)\, \psi_0^0(R)\right)\\
(\pm 1, \mp 1)_a &= \frac{1}{\sqrt{2}}\left(\psi_1^{\pm 1}(R)\, \psi_1^{\mp 1}(r) - \psi_1^{\pm 1}(r)\, \psi_1^{\mp 1}(R)\right)
\end{aligned}\right\} \tag{A 3}$$

ds.

$$(m, 0)_s \quad = \frac{1}{\sqrt{2}} \left(\psi_2^m(R)\, \psi_0^0(r) + \psi_2^m(r)\, \psi_0^0(R) \right)$$
$$m = -2, -1, 0, 1, 2.$$
$$(\pm 2, 0)_a = \psi_1^{\pm 1}(R)\, \psi_1^{\pm 1}(r)$$
$$(\pm 1, 0)_a = \frac{1}{\sqrt{2}} \left(\psi_1^{\pm 1}(R)\, \psi_1^0(r) + \psi_1^{\pm 1}(r)\, \psi_1^0(R) \right)$$
$$(0, 0)_a \quad = \frac{1}{\sqrt{2}} \left(\psi_1^1(R)\, \psi_1^{-1}(r) + \psi_1^1(r)\, \psi_1^{-1}(R) \right)$$

A 4

dp.

Only the symmetric functions are listed, the antisymmetric functions are obtained from them by interchanging R and r.

$$(\pm 2, \pm 1)_s = \frac{\sqrt{3}}{2}\, \psi_3^{\pm 3}(R)\, \psi_0^0(r) - \frac{1}{2}\, \psi_1^{\pm 1}(R)\, \psi_2^{\pm 2}(r)$$

$$(\pm 2, 0)_s \quad = \frac{1}{2}\, \psi_3^{\pm 2}(R)\, \psi_0^0(r) + \frac{1}{2}\, \psi_1^0(R)\, \psi_2^{\pm 2}(r) -$$
$$- \frac{1}{\sqrt{2}}\, \psi_1^{\pm 1}(R)\, \psi_2^{\pm 1}(r)$$

$$(\pm 1, \pm 1)_s = \frac{1}{\sqrt{2}}\, \psi_3^{\pm 2}(R)\, \psi_0^0(r) - \frac{1}{\sqrt{2}}\, \psi_1^0(R)\, \psi_2^{\pm 2}(r)$$

$$(\pm 2, \mp 1)_s = \frac{1}{2\sqrt{5}}\, \psi_3^{\pm 1}(R)\, \psi_0^0(r) - \frac{1}{\sqrt{3}}\, \psi_1^{\pm 1}(R)\, \psi_{10}^0(r) +$$
$$+ \frac{1}{\sqrt{5}}\, \psi_{11}^{\pm 1}(R)\, \psi_0^0(r) - \frac{1}{\sqrt{6}}\, \psi_1^{\pm 1}(R)\, \psi_2^0(r) +$$
$$+ \frac{1}{2}\, \psi_1^{\mp 1}(R)\, \psi_2^{\pm 2}(r)$$

$$(\pm 1, 0)_s \quad = \sqrt{\frac{2}{5}}\, \psi_3^{\pm 1}(R)\, \psi_0^0(r) + \frac{1}{\sqrt{6}}\, \psi_1^{\pm 1}(R)\, \psi_{10}^0(r) -$$
$$- \frac{1}{\sqrt{10}}\, \psi_{11}^{\pm 1}(R)\, \psi_0^0(r) - \frac{1}{\sqrt{3}}\, \psi_1^{\pm 1}(R)\, \psi_2^0(r)$$

A 5

$$(0, \pm 1)_s \quad = \sqrt{\frac{3}{10}}\, \psi_3^{\pm 1}(R)\, \psi_0^0(r) - \frac{\sqrt{2}}{6}\, \psi_1^{\pm 1}(R)\, \psi_{10}^0(r) +$$
$$+ \frac{1}{\sqrt{30}}\, \psi_{11}^{\pm 1}(R)\, \psi_0^0(r) + \frac{1}{3}\, \psi_1^{\pm 1}(R)\, \psi_2^0(r) -$$
$$- \frac{1}{\sqrt{6}}\, \psi_1^{\mp 1}(R)\, \psi_2^{\pm 2}(r) - \frac{1}{\sqrt{3}}\, \psi_1^0(R)\, \psi_2^{\pm 1}(r)$$

*

$$(\pm 1, \mp 1)_s = \frac{1}{2} \sqrt{\frac{3}{5}}\, \psi_3^0(R)\, \psi_0^0(r) - \frac{1}{\sqrt{6}}\, \psi_1^0(R)\, \psi_{10}^0(r) +$$

$$+ \frac{1}{\sqrt{10}}\, \psi_{11}^0(R)\, \psi_0^0(r) + \frac{1}{2}\, \psi_1^{-1}(R)\, \psi_2^1(r) -$$

$$- \frac{1}{2\sqrt{3}}\, \psi_1^0(R)\, \psi_2^0(r) - \frac{1}{2}\, \psi_1^1(R)\, \psi_2^{-1}(r)$$

$$(0, 0)_s \quad = \frac{3}{2\sqrt{5}}\, \psi_3^0(R)\, \psi_0^0(r) + \frac{\sqrt{2}}{3}\, \psi_1^0(R)\, \psi_{10}^0(r) -$$

$$- \sqrt{\frac{2}{15}}\, \psi_{11}^0(R)\, \psi_0^0(r) - \frac{1}{2\sqrt{3}}\, \psi_1^{-1}(R)\, \psi_2^1(r) -$$

$$- \frac{1}{6}\, \psi_1^0(R)\, \psi_2^0(r) - \frac{1}{2\sqrt{3}}\, \psi_1^1(R)\, \psi_2^{-1}(r) \,.$$

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