Zeitschrift: Bulletin de la Société Fribourgeoise des Sciences Naturelles = Bulletin der Naturforschenden Gesellschaft Freiburg

Herausgeber: Société Fribourgeoise des Sciences Naturelles

Band: 114 (2025)

Artikel: Is time imaginary?

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DOI: https://doi.org/10.5169/seals-1092166

Is time imaginary?

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The negative metric of the 4-Space Wave equation is due to our measurement procedure. The 4-Space Wave equation can be mapped onto the 4D-LAPLACE equation by adding a quarter turn to all imaginary quaternion coordinates.

La métrique négative de l'équation d'onde du 4-Espace est due à notre procédure de mesure. En ajoutant un quart de tour à toutes les coordonnées quaternion imaginaires, l'équation d'onde du 4-Espace peut être projetée sur l'équation 4D de LAPLACE.

Die negative Metrik der Wellengleichung im 4-Raum ist eine Folge unserer Messprozedur. Durch eine Vierteldrehung aller Imaginär-Koordinaten im Quaternion-Raum kann die 4-Raum-Wellengleichung auf die 4D-LAPLACE-Gleichung projiziert werden.

Introduction

HENRI POINCARÉ demonstrated in 1905[1], that time **t** can be interpreted as an imaginary fourth spacetime coordinate[2] **ict**, and that LORENTZ transformations can then be mapped onto ordinary rotations of a four-dimensional EUCLIDEAN sphere.

This subject, just briefly touched by POINCARÉ, was further elaborated by HERMANN MINKOWSKI in 1908[3]. MINKOWSKI reformulated MAXWELL's equations as a set of symmetrical equations in the four variables (x, y, z, ict),

[1] HENRI POINCARÉ, (1905–1906), "Sur la dynamique de l'électron" [On the Dynamics of the Electron], *Rendiconti del Circolo Matematico di Palermo*, 21: 129–176.
[2] wherein $i^2 = -1$ and c is the speed of light.
[3] HERMANN MINKOWSKI, (1907–1908), *"Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern"* [The Fundamental Equations for Electromagnetic Processes in Moving Bodies], Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse: 53–111.

and he redefined vector variables for the electromagnetic quantities, showing their invariance under LORENTZ transformations. He concluded that time and space should be treated equally, and that physical events take place in a unified four-dimensional **spacetime** continuum.

Within this theory, the coordinates of a physical event in spacetime are represented as a **four-vector** (t, x, y, z). A LORENTZ boost rotates the four-vector around a particular axis in four-dimensional space, whilst its length remains constant The "rotation" in a plane spanned by a space unit vector and a time unit vector, while formally still a rotation in coordinate space, is noteworthy a LORENTZ boost in physical spacetime with real inertial coordinates. The analogy with EUCLIDEAN rotations is, however, only partial, since the radius of the sphere is actually imaginary, which turns rotations into hyperbolic rotations.

MINKOWSKI spacetime works well for handling LORENTZ boosts in two dimensions spanned by a space unit vector and a time unit vector, but it fails at handling general rotations in (ict, x, y, z), due to its insufficient distinction between the space vectors (x, y, z) themselves. Furthermore, there seems to be no physical reason for passing from real time to imaginary time[1]. The question is, thus, whether imaginary time is a physical necessity, or whether it is just an artifact of our measurement procedure.

Measurement

Measuring a quantity a implies dividing it by a reference quantity r of the same nature, in order to obtain a dimensionless number (length) L, which is amenable to treatment with mathematical tools. Related to measurement is counting. Measurement applies to continuous quantities, where an external reference is needed, whereas counting applies to discrete quantities, whose discretization already provides an internal reference.

Measurement, thus, always implies a division: $L = a/r$, in order to determine how many times the reference quantity **r** is contained in the quantity **a** to be measured.

In the case of multidimensional vector quantities $a = (a_0, a_1, \ldots, a_n)$, whose components are independent from each other, a "cartesian" approach is generally made, in which each component of **a** is divided by a corresponding component of the reference vector **r**, and the results are summed up to obtain the measured length L:

$$L = \sum \left(a_0/r_0 + a_1/r_1 + \ldots + a_n/r_n \right).$$

[1] The mathematical beauty of substituting a hyperbolic rotation by an ordinary rotation is obviously not a physical reason.

To simplify the algebraic procedure, a scale vector **s** is then defined, whose components are the reciprocals of the components of the reference vector:

$$s = \left({}^{1}\!/_{r_0}, {}^{1}\!/_{r_1}, \ldots, {}^{1}\!/_{r_n}\right).$$

The measurement result **L** is now obtained as the **inner product** of the quantity a to be measured and the scale **s** which is applied: $L = a \cdot s$.

Note that a reciprocal relationship exists between L and s: The larger the scale, the smaller are the values of L, and vice-versa. As L is a measure of a, a reciprocal relationship also exists between a (the measured quantity) and s (the scale).

Square norm :

The scale vector **s**, measured at itself, should noteworthy yield the unit 1: $s \cdot s = (s_0 s_0 + s_1 s_1 + \cdots + s_n s_n) = \sum s_i^2 = 1$. (normalization)

The quantity **a**, measured at the scale vector, should yield the measurement value L : $a \cdot s = (a_0 s_0 + a_1 s_1 + \cdots + a_n s_n) = \sum a_i s_i = L$. (measurement)

From this it follows that the squares of the components of a vector a sum up to the square of its length **L** : $a \cdot a = (a_0 a_0 + a_1 a_1 + \cdots + a_n a_n) = \sum a_i^2 = L^2$. (cartesian square norm).

Composite algebras:

The cartesian square norm also exists in the three composite (complex) division algebras, noteworthy the complex numbers, the quaternions and the octonions. Instead of an inner product of vectors, the algebraic product of the complex number c with its conjugate complex c^* is then used (with $i^2 = j^2 = k^2 = l^2 = m^2 = n^2 = o^2 = ijk = ijklmno = -1$):

$$\begin{aligned}
L^2 &= c \cdot c^* = (x_0 + ix_1) \cdot (x_0 - ix_1) = \left(x_0^2 + x_1^2\right). \\
L^2 &= q \cdot q^* = (x_0 + ix_1 + jx_2 + kx_3) \cdot (x_0 - ix_1 - jx_2 - kx_3) \\
&= \left(x_0^1 + x_1^2 + x_2^2 + x_3^2\right). \\
L^2 &= o \cdot o^* = (x_0 + ix_1 + jx_2 + kx_3 + lx_4 + mx_5 + nx_6 + ox_7) \\
&\quad \cdot (x_0 - ix_1 - jx_2 - kx_3 - lx_4 - mx_5 - nx_6 - ox_7) \\
&= \left(x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2\right).
\end{aligned}$$

The cartesian square norm is a sum of positive squares, and it characterizes EUCLIDEAN spaces of positive metric, which are required to describe physical quantities for which conservation laws apply, such as the energy, which, according to P.A.M. DIRAC, is a sum of four positive squares[1].

[1] $E^2 = \left(m_0 c^2\right)^2 + (p_x c)^2 + (p_y c)^2 + (p_z c)^2$. See MAX PLANCK (1906), "Das Prinzip der

Measurement in composite algebraic number spaces:

The "inner product" measurement formula given above holds as well in composite algebras:

$$a \cdot s = (a_0 s_0 + a_1 s_1 + \cdots + a_n s_n) = \sum a_i s_i = L.$$

However, recall that there is a reciprocal relationship between the measured quantity **a** and the measurement scale **s**. If the measured quantity **a** is in **EUCLIDEAN space**, the measurement scale s spans necessarily a **reciprocal EUCLIDEAN** space!

If therefore the measured quantity a is a quaternion ($a_0 + ia_1 + ja_2 + ka_3$), the measurement scale s must be a reciprocal unit quaternion $(s_0 - is_1 - js_2 - ks_3)$! The inversion of sign in the imaginary part is necessary to make the inner product containing only positive terms:

$$(a_0; ia_1; ja_2; ka_3) \cdot (s_0; -is_1; -js_2; -ks_3) = (a_0 s_0 + a_1 s_1 + a_2 s_2 + a_3 s_3)$$

EUCLIDEAN space versus relativistic spacetime

Conserved physical quantities must obey LAPLACE's equation in EUCLIDEAN space: $\Delta A = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) A = 0$. (under empty space conditions)

This equation can be solved by factoring the LAPLACE operator in quaternion number space and integrating the resulting first-order quatemion transport equations[1].

The LAPLACE operator can be thought as being an inner product of the Nabla operator with itself in a 4-dimensional ordinary space:

$$\Delta A = \nabla \cdot \nabla A = \left(\left(\frac{1}{c}\frac{\partial}{\partial t}; \frac{\partial}{\partial x_1}; \frac{\partial}{\partial x_2}; \frac{\partial}{\partial x_3}\right) \cdot \left(\frac{1}{c}\frac{\partial}{\partial t}; \frac{\partial}{\partial x_1}; \frac{\partial}{\partial x_2}; \frac{\partial}{\partial x_3}\right)\right) A = 0,$$

or, else, as the inner product of the vectorized quaternion Nabla operator with its quaternion reciprocal[2].

$$\Delta A = \nabla \cdot \nabla^* A = \left(\left(\frac{1}{c}\frac{\partial}{\partial t}; i\frac{\partial}{\partial x_1}; j\frac{\partial}{\partial x_2}; k\frac{\partial}{\partial x_3}\right) \cdot \left(\frac{1}{c}\frac{\partial}{\partial t}; -i\frac{\partial}{\partial x_1}; -j\frac{\partial}{\partial x_2}; -k\frac{\partial}{\partial x_3}\right)\right) A = 0.$$

Relativität und die Grundgleichungen der Mechanik". *Verhandlungen der Deutschen Physikalischen Gesellschaft.* 8 (7): 136-141; Walter Gordon (1926). *"The Compton effect according to Schrödinger's theory".* Z. Phys. 40. 117-133 ; Paul Dirac (1928). "The Quantum Theory of the Electron". Proc. R. Soc. Lond. A. 117 (778): 610-624.

[1] EDGAR MÜLLER, *„Factoring the wave equation"*, Bull. Soc. Frib. Sc. Nat. - Vol. 112 (2023), 106-113.

[2] The reciprocal of a quaternion is its complex conjugate, divided by its length. Given that the LAPLACE equation adds up to zero, we multiply it here, for the sake of simplicity, by its length.

We are still staying in EUCLIDEAN Space, where physical conservation laws apply.

On the other hand, **electrodynamics and Special Relativity obey the 4-Space Wave equation**: $\Delta A = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}\right) A = 0$, which has a negative metric and spans a hyperbolic space. There is no factoring available for the 4-Space Wave operator, unless by recurring to the matrix method of P.A.M. DIRAC[1].

The 4-Space Wave equation operator can be thought of as an inner product of a quaternion Nabla operator with itself:

$$(\nabla \cdot \nabla)\mathrm{A} = \Delta A = \left(\left(\frac{1}{c}\frac{\partial}{\partial t}; i\frac{\partial}{\partial x_1}; j\frac{\partial}{\partial x_2}; k\frac{\partial}{\partial x_3}\right) \cdot \left(\frac{1}{c}\frac{\partial}{\partial t}; i\frac{\partial}{\partial x_1}; j\frac{\partial}{\partial x_2}; k\frac{\partial}{\partial x_3}\right)\right) A = 0;$$

or, after multiplication by -1 , as the inner product of a complex Nabla operator with itself:

$$(\nabla \cdot \nabla)\mathrm{A} = \Delta A = \left(\left(\frac{1}{ic}\frac{\partial}{\partial t}; \frac{\partial}{\partial x_1}; \frac{\partial}{\partial x_2}; \frac{\partial}{\partial x_3}\right) \cdot \left(\frac{1}{ic}\frac{\partial}{\partial t}; \frac{\partial}{\partial x_1}; \frac{\partial}{\partial x_2}; \frac{\partial}{\partial x_3}\right)\right) A = 0;$$

This latter was the basis for POINCARÉs postulating **hyperbolic space-time** with an **imaginary time coordinate ict** and a **negative metric** $(-,+,+,+)$.

For mapping the hyperbolic 4-Space onto the EUCLIDEAN space of the LAPLACE equation, we must thus add an inversion to the quaternion's imaginary parts. This is best done by applying a quarter turn to all imaginary components of the quaternion coordinates: ($x_0 \to x_0; x_1 \to ix_1; x_2 \to jx_2; x_3 \to kx_3$). The negative metric of the 4-Space wave equation is herewith changed into the positive metric of the LAPLACE equation.

In POINCARÉs approach, there is just a single quarter turn applied to the time coordinate; however, the complex number space is not sufficiently large for solving the resulting 4D-LAPLACE equation. This means that the approach only allows a description involving one time-coordinate and one space coordinate, i.e. solving the 2D-LAPLACE equation.

Conclusion

The negative metric of the 4-Space Wave equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}\right) \boldsymbol{A} = 0$$

[1] P.A.M. DIRAC. The quantum theory of the electron, Proc. Roy. Soc. Lond. A., Vol. 117, is. 778, 610–624 (1928).

is due to our measurement procedure. The 4-Space Wave equation can be mapped onto the 4D-Laplace equation by adding a quarter turn to all imaginary quaternion coordinates. Following Poincaré's imaginary time approach in the complex number space, only a two-dimensional problem can be solved, however.