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Factoring the wave equation II

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Summary

In a previous article [MU23] it was shown that wave equations of the form $(\partial_t^2 - \sum_{(i=1)}^{n} \partial_i^2)F = 0$ for $1 \leqslant n \leqslant 7$ can be remapped onto the corresponding Laplace equations of dimensions 2, 4, or 8 by a variable change to complex, quaternion or octonion variables, and that the resulting LAPLACE equations can be factored into a product of two first-order transport equations in $\mathbb{C}$, $\mathbb{Q}$, or $\mathbb{O}$, and solved according to D'ALEMBERT'S [AL47] *method of characteristics*. The resulting characteristics are scaling-rotators in $\mathbb{C}$, $\mathbb{Q}$, or $\mathbb{O}$, operating in EUCLIDEAN space of positive metric, and giving rise to *spherical harmonics*. Back-transforming yields solution functions in the hyperbolic space of the wave equation, which has a negative metric. The solutions for dimensions 2 and 4 were presented in the previous article. In the present article, the solution for dimension 8 is presented in detail. The solutions for dimensions 3, 5, 6, and 7 can be found by inclusion into dimension 4 or 8, respectively.

Conformal remapping and factoring of the 8D wave equation

The 8D wave equation
$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial {x_1}^2} - \frac{\partial^2}{\partial {x_2}^2} - \frac{\partial^2}{\partial {x_3}^2} - \frac{\partial^2}{\partial {x_4}^2} - \frac{\partial^2}{\partial {x_5}^2} - \frac{\partial^2}{\partial {x_6}^2} - \frac{\partial^2}{\partial {x_7}^2}\right) E = 0$, can be remapped onto the 8D Laplace equation in Octonion space $\mathbb{O}$ by the transform:

$\{x_0 \to ct; x_1 \to i \cdot x_1; x_2 \to j \cdot x_2; x_3 \to k \cdot x_3; x_4 \to l \cdot x_4; x_5 \to m \cdot x_5; x_6 \to n \cdot x_6; x_7 \to o \cdot x_7\}$, with $i^2 = j^2 = k^2 = l^2 = m^2 = n^2 = o^2 = ijklmno = -1$.

Back-mapping is achieved by the reciprocal transform[1]:

$\{ct \to x_0; x_1 \to -i \cdot x_1; x_2 \to -j \cdot x_2; x_3 \to -k \cdot x_3; x_4 \to -l \cdot x_4; x_5 \to -m \cdot x_5; x_6 \to -n \cdot x_6; x_7 \to -o \cdot x_7\}$

The resulting 8D LAPLACE equation,
$\left(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2} + \frac{\partial^2}{\partial x_5^2} + \frac{\partial^2}{\partial x_6^2} + \frac{\partial^2}{\partial x_7^2}\right) G = 0$,
which we rewrite in shorthand as:

[1] The reciprocal of a complex number, quaternion or octonion is its conjugate complex quantity, divided by its length. For differential operators, as will be considered in the present context, the length is irrelevant.

$(\partial_0^2+\partial_1^2+\partial_2^2+\partial_3^2+\partial_4^2+\partial_5^2+\partial_6^2+\partial_7^2)G=0,$

can be factored in $\mathbb{O}$ in two ways into a product of two octonion transport equations:

$$(\partial_0^2+{\partial_1}^2+{\partial_2}^2+{\partial_3}^2+{\partial_4}^2+\partial_5^2+{\partial_6}^2+{\partial_7}^2)\,\mathrm{G}=$$
$$((\partial_0-i\partial_1-j\partial_2-k\partial_3-l\partial_4-m\partial_5-n\partial_6-o\partial_7)\,(\partial_0+i\partial_1+j\partial_2+k\partial_3+l\partial_4+m\partial_5+n\partial_6+o\partial_7))\,\mathrm{G}=0,$$
$$=(\partial_0-i\partial_1-j\partial_2-k\partial_3-l\partial_4-m\partial_5-n\partial_6-o\partial_7)\,((\partial_0+i\partial_1+j\partial_2+k\partial_3+l\partial_4+m\partial_5+n\partial_6+o\partial_7)\,\mathrm{G})=0$$
$$=(\partial_0-i\partial_1-j\partial_2-k\partial_3-l\partial_4-m\partial_5-n\partial_6-o\partial_7)\,g_2=0,$$

and

$$({\partial_0}^2+{\partial_1}^2+{\partial_2}^2+{\partial_3}^2+{\partial_4}^2+{\partial_5}^2+{\partial_6}^2+{\partial_7}^2)\,\mathrm{G}=$$
$$((\partial_0+i\partial_1+j\partial_2+k\partial_3+l\partial_4+m\partial_5+n\partial_6+o\partial_7)\,(\partial_0-i\partial_1-j\partial_2-k\partial_3-l\partial_4-m\partial_5-n\partial_6-o\partial_7))\,\mathrm{G}=0,$$
$$=(\partial_0+i\partial_1+j\partial_2+k\partial_3+l\partial_4+m\partial_5+n\partial_6+o\partial_7)\,((\partial_0-i\partial_1-j\partial_2-k\partial_3-l\partial_4-m\partial_5-n\partial_6-o\partial_7)\,\mathrm{G})=0$$
$$=(\partial_0+i\partial_1+j\partial_2+k\partial_3+l\partial_4+m\partial_5+n\partial_6+o\partial_7)\,g_1=0,$$

The question arises, whether the shift of parentheses is allowed, because the associative law does not hold in general octonion space. However, as was shown by EMIL ARTIN, every sub-algebra spanned by only two octonions is still associative (ARTIN'S theorem)[1]. The 8D-LAPLACE equation comprises only two different octonions, i.e. the octonion differential operator and its complex conjugate, whose imaginary part is collinear with that of the operator, and the octonion operand. The shift of parentheses is thus allowed and we can solve the 8D-LAPLACE equation as outlined.

The octonion differential operator is a rotational transport operator (any octonion being a scaling-rotation operator in $\mathbb{O}$). We have thus advantage, here too, in taking the transport equations into polar coordinates.

Formulating new transport equations in polar coordinates (r,θ)[2] **yields:**

$(\partial_0+i\partial_1+j\partial_2+k\partial_3+l\partial_4+m\partial_5+n\partial_6+o\partial_7)\,g_1=0$ yields: $(\partial_\theta+\underline{u}\cdot r\cdot\partial_r)\varphi_1=0$, which is satisfied by any twice differentiable function $\varphi_1(r\cdot e^{-\underline{u}\cdot\theta})$, and $(\partial_0-i\partial_1-j\partial_2-k\partial_3-l\partial_4-m\partial_5-n\partial_6-o\partial_7)\,g_2=0$ yields: $(\partial_\theta-\underline{u}\cdot r\cdot\partial_r)\varphi_2=0$, which is satisfied by any twice differentiable function $\varphi_2(r\cdot e^{\underline{u}\cdot\theta})$. Therein, $\underline{u}=\frac{1}{\sqrt{q_1^2+q_2^2+q_3^2+q_4^2+q_5^2+q_6^2+q_7^2}}(iq_1+jq_2+kq_3+lq_4+mq_5+nq_6+oq_7)=(iu_1+ju_2+ku_3+lu_4+mu_5+nu_6+ou_7)$ is a pure unit octonion. The characteristics in this case are the octonion expressions $(r\cdot e^{-\underline{u}\cdot\theta})$ and $(r\cdot e^{\underline{u}\cdot\theta})$. The solution is therefore a superposition of arbitrary, twice differentiable functions $\varphi_1(r\cdot e^{-\underline{u}\cdot\theta})$ and $\varphi_2(r\cdot e^{\underline{u}\cdot\theta})$. Stationary waves can result from the superposition of two opposite-rotating functions, but there is also a possibility of stationary waves resulting from the self-interference of a single rotating function.

[1]This is evident from the following consideration: an octonion has a real part and a 7-dimensional imaginary part, pointing somewhere into imaginary space. Considered per se it is just a complex number. Two different octonions can thus only have two different imaginary parts, pointing into two different imaginary directions. Hence two octonions can only span a quaternion space, which is associative.

[2]The differential operators in polar coordinates are $(1/r\,\partial_\theta+i\partial_r)$ and $(1/r\,\partial_\theta-i\partial_r)$, respectively. In the following, for the sake of simplicity, we multiply them by r.

Self-interference of the solution functions:

Here again, the factors $e^{-\underline{u}\cdot\theta}$ and $e^{\underline{u}\cdot\theta}$ are rotating pointers, with a periodicity in θ of 2π. The solution functions φ_1 and φ_2 will therefore interfere with themselves after 2π or a multiple thereof, and eventually reinforce or wipe out what they had written before. Stationary solutions, i.e. solutions having a stable mean value, can thus exist if the solution functions φ_1 and φ_2 are periodic, with periodicities of $n \cdot 2\pi$, wherein $n = 0, 1, 2, 3, \ldots$, i.e. a non-negative integer.

The complete polar representation of an octonion comprises a radius and 7 angles $(r, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7)$[1]

$$
\begin{aligned}
q &= (q_0 + iq_1 + jq_2 + kq_3 + lq_4 + mq_5 + nq_6 + oq_7) \\
&= \|q\| \left(\cos(\theta_1) + \underline{u} \cdot \sin(\theta_1)\right) \\
&= r \begin{pmatrix}
\cos(\theta_1) \\
+i \cdot \sin(\theta_1) \cdot \cos(\theta_2) \\
+j \cdot \sin(\theta_1) \cdot \sin(\theta_2) \cdot \cos(\theta_3) \\
+k \cdot \sin(\theta_1) \cdot \sin(\theta_2) \cdot \sin(\theta_3) \cdot \cos(\theta_4) \\
+l \cdot \sin(\theta_1) \cdot \sin(\theta_2) \cdot \sin(\theta_3) \cdot \sin(\theta_4) \cdot \cos(\theta_5) \\
+m \cdot \sin(\theta_1) \cdot \sin(\theta_2) \cdot \sin(\theta_3) \cdot \sin(\theta_4) \cdot \sin(\theta_5) \cdot \cos(\theta_6) \\
+n \cdot \sin(\theta_1) \cdot \sin(\theta_2) \cdot \sin(\theta_3) \cdot \sin(\theta_4) \cdot \sin(\theta_5) \cdot \sin(\theta_6) \cdot \cos(\theta_7) \\
+o \cdot \sin(\theta_1) \cdot \sin(\theta_2) \cdot \sin(\theta_3) \cdot \sin(\theta_4) \cdot \sin(\theta_5) \cdot \sin(\theta_6) \cdot \sin(\theta_7)
\end{pmatrix}
\end{aligned}
$$

with $r = \|q\| = \sqrt{q \cdot q^*} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2 + q_4^2 + q_5^2 + q_6^2 + q_7^2}$;

$$\theta_1 = \operatorname{arcos}\left(\frac{q_0}{\|q\|}\right);$$

$$\theta_2 = \operatorname{arcos}(u_1);$$

$$\theta_3 = \operatorname{arcos}\left(\frac{u_2}{\sqrt{(u_2^2 + u_3^2 + u_4^2 + u_5^2 + u_6^2 + u_7^2)}}\right);$$

$$\theta_4 = \operatorname{arcos}\left(\frac{u_3}{\sqrt{(u_3^2 + u_4^2 + u_5^2 + u_6^2 + u_7^2)}}\right);$$

$$\theta_5 = \operatorname{arcos}\left(\frac{u_4}{\sqrt{(u_4^2 + u_5^2 + u_6^2 + u_7^2)}}\right);$$

$$\theta_6 = \operatorname{arcos}\left(\frac{u_5}{\sqrt{(u_5^2 + u_6^2 + u_7^2)}}\right); \text{ and}$$

$$\theta_7 = \operatorname{arcos}\left(\frac{u_6}{\sqrt{(u_6^2 + u_7^2)}}\right).$$

The *spherical harmonics* in the solutions of the 8D Laplace's equation form therefore a 7-dimensional manifold, because, beside the principal rotation in θ_1 which we are contemplating here, there are subordinated rotations in $\theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7$, which also lead to self-interference. This seven-dimensional manifold gives rise to quantum numbers in particle physics, which characterize each stationary state.

Back-transformation of the variables

In view of back-transforming the variables, we first rewrite the solution

[1]The former angle θ is now called θ_1.

functions' arguments as a CARTESIAN vector representation:

$$\varphi_1\left(r\cdot e^{-\underline{u}\cdot\theta}\right)=\varphi_1\begin{pmatrix} r\cdot\cos\theta_1\\ -i\cdot u_1\cdot r\cdot\sin\theta_1\\ -j\cdot u_2\cdot r\cdot\sin\theta_1\\ -k\cdot u_3\cdot r\cdot\sin\theta_1\\ -l\cdot u_4\cdot r\cdot\sin\theta_1\\ -m\cdot u_5\cdot r\cdot\sin\theta_1\\ -n\cdot u_6\cdot r\cdot\sin\theta_1\\ -o\cdot u_7\cdot r\cdot\sin\theta_1 \end{pmatrix}$$

$$\varphi_2\left(r\cdot e^{\underline{u}\cdot\theta}\right)=\varphi_2\begin{pmatrix} r\cdot\cos\theta_1\\ i\cdot u_1\cdot r\cdot\sin\theta_1\\ j\cdot u_2\cdot r\cdot\sin\theta_1\\ k\cdot u_3\cdot r\cdot\sin\theta_1\\ l\cdot u_4\cdot r\cdot\sin\theta_1\\ m\cdot u_5\cdot r\cdot\sin\theta_1\\ n\cdot u_6\cdot r\cdot\sin\theta_1\\ o\cdot u_7\cdot r\cdot\sin\theta_1 \end{pmatrix}$$

The CARTESIAN coordinates span here, as do the polar coordinates, the whole 8-dimensional space $\mathbb{R}^8$ of the function's arguments.

Applying the trigonometric identities $\cos(\alpha)=\cosh(\sqrt{-1}\cdot\alpha)$ and $i\cdot\sin(\alpha)=j\cdot\sin(\alpha)=k\cdot\sin(\alpha)=l\cdot\sin(\alpha)=m\cdot\sin(\alpha)=n\cdot\sin(\alpha)=o\cdot\sin(\alpha)=\sinh(\sqrt{-1}\cdot\alpha)$ yields for the arguments of the solution functions:

$$\varphi_1\begin{pmatrix} r\cdot\cosh\left(\sqrt{-1}\cdot\theta_1\right)\\ -u_1\cdot r\cdot\sinh\left(\sqrt{-1}\cdot\theta_1\right)\\ -u_2\cdot r\cdot\sinh\left(\sqrt{-1}\cdot\theta_1\right)\\ -u_3\cdot r\cdot\sinh\left(\sqrt{-1}\cdot\theta_1\right)\\ -u_4\cdot r\cdot\sinh\left(\sqrt{-1}\cdot\theta_1\right)\\ -u_5\cdot r\cdot\sinh\left(\sqrt{-1}\cdot\theta_1\right)\\ -u_6\cdot r\cdot\sinh\left(\sqrt{-1}\cdot\theta_1\right)\\ -u_7\cdot r\cdot\sinh\left(\sqrt{-1}\cdot\theta_1\right) \end{pmatrix}$$

$$\varphi_2\begin{pmatrix} r\cdot\cosh\left(\sqrt{-1}\cdot\theta_1\right)\\ u_1\cdot r\cdot\sinh\left(\sqrt{-1}\cdot\theta_1\right)\\ u_2\cdot r\cdot\sinh\left(\sqrt{-1}\cdot\theta_1\right)\\ u_3\cdot r\cdot\sinh\left(\sqrt{-1}\cdot\theta_1\right)\\ u_4\cdot r\cdot\sinh\left(\sqrt{-1}\cdot\theta_1\right)\\ u_5\cdot r\cdot\sinh\left(\sqrt{-1}\cdot\theta_1\right)\\ u_6\cdot r\cdot\sinh\left(\sqrt{-1}\cdot\theta_1\right)\\ u_7\cdot r\cdot\sinh\left(\sqrt{-1}\cdot\theta_1\right) \end{pmatrix}$$

We now apply the back-transformation $\{\mathrm{x}_0\to\mathrm{ct};\ \mathrm{x}_1\to-\mathrm{i}\cdot\mathrm{x}_1;\mathrm{x}_2\to-\mathrm{j}\cdot\mathrm{x}_2;\mathrm{x}_3\to-\mathrm{k}\cdot\mathrm{x}_3\}$, and obtain:

$$g_1 \begin{pmatrix} ct \cdot r \cdot \cosh\left(\sqrt{-1} \cdot \theta_1\right) \\ i \cdot u_1 \cdot r \cdot \sinh\left(\sqrt{-1} \cdot \theta_1\right) \\ j \cdot u_2 \cdot r \cdot \sinh\left(\sqrt{-1} \cdot \theta_1\right) \\ k \cdot u_3 \cdot r \cdot \sinh\left(\sqrt{-1} \cdot \theta_1\right) \\ l \cdot u_4 \cdot r \cdot \sinh\left(\sqrt{-1} \cdot \theta_1\right) \\ m \cdot u_5 \cdot r \cdot \sinh\left(\sqrt{-1} \cdot \theta_1\right) \\ n \cdot u_6 \cdot r \cdot \sinh\left(\sqrt{-1} \cdot \theta_1\right) \\ o \cdot u_7 \cdot r \cdot \sinh\left(\sqrt{-1} \cdot \theta_1\right) \end{pmatrix} ; \text{ and}$$

$$g_2 \begin{pmatrix} ct \cdot r \cdot \cosh\left(\sqrt{-1} \cdot \theta_1\right) \\ -i \cdot u_1 \cdot r \cdot \sinh\left(\sqrt{-1} \cdot \theta_1\right) \\ -j \cdot u_2 \cdot r \cdot \sinh\left(\sqrt{-1} \cdot \theta_1\right) \\ -k \cdot u_3 \cdot r \cdot \sinh\left(\sqrt{-1} \cdot \theta_1\right) \\ -l \cdot u_4 \cdot r \cdot \sinh\left(\sqrt{-1} \cdot \theta_1\right) \\ -m \cdot u_5 \cdot r \cdot \sinh\left(\sqrt{-1} \cdot \theta_1\right) \\ -n \cdot u_6 \cdot r \cdot \sinh\left(\sqrt{-1} \cdot \theta_1\right) \\ -o \cdot u_7 \cdot r \cdot \sinh\left(\sqrt{-1} \cdot \theta_1\right) \end{pmatrix} .$$

The solution function's arguments are now again in a hyperbolic space, corresponding to the metric $(+1, -1, -1, -1, -1, -1, -1, -1)$ of the 8 -space. Regardless of r, the function's arguments run always into the infinite. With respect to the linear EUCLIDEAN space of the 8D-LAPLACE equation, the 8D-Wave equation spans a reciprocal, hyperbolic space, and has not only travelling-wave-solutions, but also stationary solutions, reflecting stationary states of particle physics.

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