Factoring the wave equation

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Wave equations of the form $(\partial_t^2 - \sum_{(i=1)}^n \partial_i^2)F = 0$ *for* $1 \le n \le 7$ *can be remapped onto the corresponding Laplace equations of dimensions 2, 4, or 8 by a variable change to complex, quaternion or octonion variables. The LAPLACE equations can be factored into a product of two first-order transport equations in* $\mathbb{C}$, $\mathbb{Q}$, *or* $\mathbb{O}$ *and solved according to the method of characteristics. The characteristics are scaling-rotators in* $\mathbb{C}$, $\mathbb{Q}$, *or* $\mathbb{O}$ *operating in EUCLIDEAN space and giving rise to spherical harmonics. Back-transforming yields solution functions in hyperbolic space of negative metric.*

Introduction

JEAN LE ROND D'ALEMBERT has shown, back in 1747, that the second-order partial differential equation $(\partial_t^2 - c^2\partial_x^2)F = 0$, which describes the standing waves on a string, is reducible in $\mathbb{R}$ to the superposition of a forward-running and a backward-running travelling wave, $f_1(x - ct)$ and $f_2(x + ct)$, respectively, wherein f_1 and f_2 are arbitrary, twice differentiable functions[1]. The arguments $(x - ct)$ and $(x + ct)$ are called the *characteristic variables* of $(\partial_t^2 - c^2\partial_x^2)F = 0$. D'ALEMBERT'S method of solving this partial differential equation is today known as the *"method of characteristics"*. It starts with a formal factoring of the partial differential operator into a product of lower-order partial differential operators. In the above case we obtain:

$$
\begin{aligned}
\left(\partial_t^2 - c^2\partial_x^2\right)\mathrm{F} &= \left(\left(\partial_t + c\cdot\partial_x\right)\left(\partial_t - c\cdot\partial_x\right)\right)\mathrm{F} = 0,\\
&= \left(\partial_t + c\cdot\partial_x\right)\left(\left(\partial_t - c\cdot\partial_x\right)\mathrm{F}\right) = \left(\partial_t + c\cdot\partial_x\right)\mathrm{f}_1 = 0 \text{ or}\\
\left(\partial_t^2 - c^2\partial_x^2\right)\mathrm{F} &= \left(\left(\partial_t - c\cdot\partial_x\right)\left(\partial_t + c\cdot\partial_x\right)\right)\mathrm{F} = 0\\
&= \left(\partial_t - c\cdot\partial_x\right)\left(\left(\partial_t + c\cdot\partial_x\right)\mathrm{F}\right) = \left(\partial_t - c\cdot\partial_x\right)\mathrm{f}_2 = 0.
\end{aligned}
$$

[1] JEAN LE ROND D'ALEMBERT (1747) *"Recherches sur la courbe que forme une corde tendue mise en vibration", Histoire de l'académie royale des sciences et belles lettres de Berlin*, vol. 3, pages 214-219.

It is easily seen that if F contains a term $f_1(x-ct)$, this term yields zero under $(\partial_t + c\cdot\partial_x)$, and that if F contains a term $f_2(x+ct)$, this term yields zero under $(\partial_t - c\cdot\partial_x)$, because of the inner derivatives. The solution is thus a superposition of arbitrary, twice differentiable functions $f_1(x-ct)$ and $f_2(x+ct)$. In other words, a superposition of forward-running and backward-running travelling waves.

The beauty of D'ALEMBERT'S *method of characteristics* is its closeness to the physical nature of the underlying phenomena which give rise to the partial differential equation in question. Mathematics acts here as a powerful analytical tool for revealing the physics behind the differential equation.

The method of *characteristics* does, however, no longer work out for higher-dimensional wave equations, such as those occurring in electrodynamics, quantum physics, etc., because the implied partial differential operator is not reducible in $\mathbb{R}$. Examples of 4-dimensional wave equations are:

$$\Box E = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial {x_1}^2} - \frac{\partial^2}{\partial {x_2}^2} - \frac{\partial^2}{\partial {x_3}^2}\right) E = 0 \text{ (Electromagnetic E-wave in free space)}$$

$$\Box A = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial {x_1}^2} - \frac{\partial^2}{\partial {x_2}^2} - \frac{\partial^2}{\partial {x_3}^2}\right) A = \mu_0 J \text{ (Fundamental equation of electrodynamics)}$$

$$\Box \psi = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial {x_1}^2} - \frac{\partial^2}{\partial {x_2}^2} - \frac{\partial^2}{\partial {x_3}^2}\right) \psi = -m^2\psi \text{ (KLEIN-GORDON equation)}$$

Comment on the Fundamental equation of electrodynamics : In natural coordinates; $A = (\varphi, (A_1, A_2, A_3))$ is the 4-potential, composed of scalar potential φ and vector potential A, and $J = (\rho, (J_1, J_2, J_3))$ is the 4-current density, composed of charge density ρ and current density J.

Comment on the KLEIN-GORDON equation : In *natural coordinates*; the KLEIN-GORDON equation has been factored by PAUL A.M. DIRAC in a forced way with the help of appropriate 4×4 matrices: PAUL A.M. DIRAC, (1928). "The Quantum Theory of the Electron". *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences.* 117 (778): 610–624.

Conformal remapping and factoring of the 2D wave equation

Wave equations of dimensions 2, 4, or 8, and of the intermediate dimensions by inclusion, can be conformally remapped onto the LAPLACE equation, by a change to complex, quaternion, or octonion variables:

Under the change to complex variables $\{t \to t; x \to i\cdot x\}$, the 2D wave equation $\left(\partial_t^2 - c^2\partial_x^2\right) \mathrm{F} = 0$ transforms noteworthy into $\left(\partial_t^2 + c^2\partial_x^2\right) \mathrm{G} = 0$, i.e. the LAPLACE equation.

The LAPLACE equation is reducible in $\mathbb{C}$, yielding

$$
\begin{aligned}
\left(\partial_t^2 + c^2\partial_x^2\right) \mathrm{G} &= \left(\left(\partial_t + i\cdot c\cdot \partial_x\right)\left(\partial_t - i\cdot c\cdot \partial_x\right)\right)\mathrm{G} = 0\\
&= \left(\partial_t + i\cdot c\cdot \partial_x\right)\left(\left(\partial_t - i\cdot c\cdot \partial_x\right)\mathrm{G}\right) = \left(\partial_t + i\cdot c\cdot \partial_x\right) g_1 = 0, \text{ and}\\
\left(\partial_t^2 + c^2\partial_x^2\right) \mathrm{G} &= \left(\left(\partial_t - i\cdot c\cdot \partial_x\right)\left(\partial_t + i\cdot c\cdot \partial_x\right)\right)\mathrm{G} = 0\\
&= \left(\partial_t - i\cdot c\cdot \partial_x\right)\left(\left(\partial_t + i\cdot c\cdot \partial_x\right) G\right) = \left(\partial_t - i\cdot c\cdot \partial_x\right) g_2 = 0
\end{aligned}
$$

Therein, the differential operators are complex quantities, which act as rotational transport operators onto a complex function G, or g_1 or g_2. Given the rotational nature of these transport operators, we have advantage in taking the transport equations into polar coordinates.

Formulating new transport equations in polar coordinates (r, θ)[1] **yields:**

For $\left(\partial_t + i\cdot c\cdot \partial_x\right) g_1 = 0 :\ (\partial_\theta + i\cdot r\cdot \partial_r)\varphi_1 = 0$

which is satisfied by any twice differentiable function $\varphi_1\left(r\cdot e^{-i\cdot\theta}\right)$, and

For $\left(\partial_t - i\cdot c\cdot \partial_x\right) g_2 = 0 :\ (\partial_\theta - i\cdot r\cdot \partial_r)\varphi_2 = 0$

which is satisfied by any twice differentiable function $\varphi_2\left(r\cdot e^{i\cdot\theta}\right)$, again because of the inner derivatives.

The *characteristics* are in this case the complex scaling-rotators $\left(r\cdot e^{-i\cdot\theta}\right)$ and $\left(r\cdot e^{i\cdot\theta}\right)$, which rotate in opposite sense.

The solution is therefore a superposition of arbitrary, twice differentiable functions $\varphi_1\left(r\cdot e^{-i\cdot\theta}\right)$ and $\varphi_2\left(r\cdot e^{i\cdot\theta}\right)$. Stationary waves can result from the superposition of two opposite-rotating functions, but there is also a possibility of stationary waves resulting from the self-interference of a single rotating function.

Self-interference of the solution functions:

The factors $\left(r\cdot e^{-i\cdot\theta}\right)$ and $\left(r\cdot e^{i\cdot\theta}\right)$ are noteworthy rotating pointers, having a periodicity in θ of 2π. The solution functions φ_1 and φ_2 will therefore interfere with themselves after 2π or a multiple thereof, and eventually reinforce or wipe out what they had written before. Stationary solutions, i.e. solutions having a stable mean value at each point, can thus also exist if the solution functions φ_1 and φ_2 are periodic, with periodicities of $n\cdot 2\pi$, wherein $n = 0, 1, 2, 3, \ldots$, i.e. a non-negative integer. The fact of self-interference gives insight into the appearance of *spherical harmonics* in the solutions of the 2D LAPLACE equation; self-interference is a necessary consequence of the rotational transport operators, which lock non-transitory forward- and/or backward-rotating solution functions to periodicities of integer multiples of 2π.

[1]The differential operators in polar coordinates are $(\frac{1}{r}\partial_\theta + i\partial_r)$ and $(\frac{1}{r}\partial_\theta - i\partial_r)$, respectively. In the following, for the sake of simplicity, we multiply them by r.

Back-transformation of the variables

In view of back-transforming the variables, we rewrite the solution functions' arguments as a CARTESIAN vector representation:

$$\varphi_1\left(r\cdot e^{-i\cdot\theta}\right)=\varphi_1(x=r\cdot\cos\theta;\quad y=-i\cdot r\cdot\sin\theta)$$
$$\varphi_2\left(r\cdot e^{i\cdot\theta}\right)=\varphi_2(x=r\cdot\cos\theta;\quad y=+i\cdot r\cdot\sin\theta)$$

The CARTESIAN coordinates (x,y) span here, as do the polar coordinates (r,θ), the whole plane $\mathbb{R}^2$ of the function's arguments.

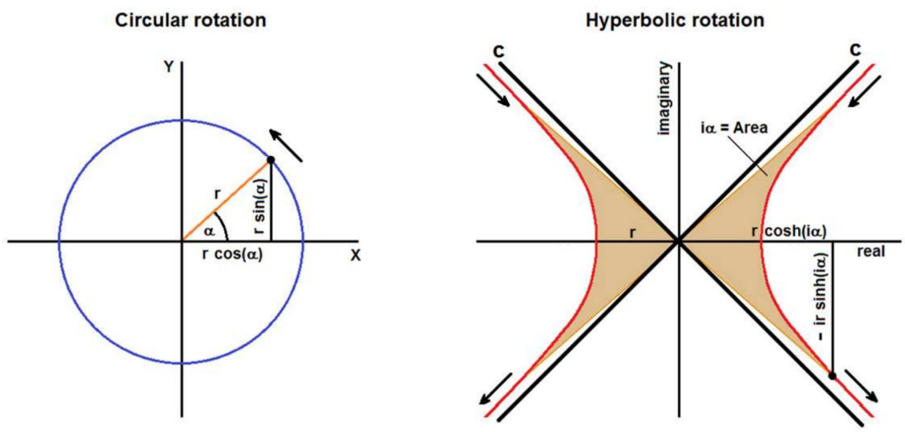
Applying now the trigonometric identities $\cos(\alpha)=\cosh(\sqrt{-1}\cdot\alpha)$ and $i\cdot\sin(\alpha)=\sinh(\sqrt{-1}\cdot\alpha)$ yields:

$$\varphi_1(x=r\cdot\cosh(\sqrt{-1}\cdot\theta);\quad y=-r\cdot\sinh(\sqrt{-1}\cdot\theta))$$
$$\varphi_2(x=r\cdot\cosh(\sqrt{-1}\cdot\theta);\quad y=+r\cdot\sinh(\sqrt{-1}\cdot\theta))$$

We now apply the back-transformation $(x\to x; y\to -iy)$ to the variables, and obtain:

$$g_1(x=r\cdot\cosh(\sqrt{-1}\cdot\theta);\quad y=+i\cdot r\cdot\sinh(\sqrt{-1}\cdot\theta))$$
$$g_2(x=r\cdot\cosh(\sqrt{-1}\cdot\theta);\quad y=-i\cdot r\cdot\sinh(\sqrt{-1}\cdot\theta))$$

The function's arguments are now in a hyperbolic space of negative metric $(+1,-1)$, describing a hyperbolic rotation, which is confined into merely two quadrants of the $\mathbb{R}^2$ plane, with the restriction $(-x\leq y\leq +x)$.



Instead of remaining bound to a circle of radius r, the function's arguments in the hyperbolic space run always into the infinite, regardless of r. With respect to the linear EUCLIDEAN space of the LAPLACE equation, the wave equation spans a reciprocal, hyperbolic space.

Conformal remapping and factoring of the 4D wave equation

The 4D wave equation $\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial {x_1}^2}-\frac{\partial^2}{\partial {x_2}^2}-\frac{\partial^2}{\partial {x_s}^2}\right)E=0$, describing an electromagnetic wave in free space, can be remapped into quaternion space $\mathbb{Q}$ by the transform: $\{\mathrm{ct}\rightarrow \mathrm{x}_0;\mathrm{x}_1\rightarrow i\cdot \mathrm{x}_1;\mathrm{x}_2 j\cdot \mathrm{x}_2;\mathrm{x}_3\rightarrow \mathrm{k}\cdot \mathrm{x}_3\}$.

We therewith obtain the 4D LAPLACE equation, which we rewrite in shorthand as:

$$\left(\frac{\partial^2}{\partial x_0^2}+\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)\mathrm{G}=\left(\partial_0^2+\partial_1^2+\partial_2^2+\partial_3^2\right)\mathrm{G}=0$$

This latter is reducible in $\mathbb{Q}$ as the product of two quaternion transport equations:

$$\begin{aligned}\left(\partial_0^2+\partial_1^2+{\partial_2}^2+\partial_3^2\right)\mathrm{G}&=\left(\left(\partial_0-i\partial_1-j\partial_2-k\partial_3\right)\left(\partial_0+i\partial_1+j\partial_2+k\partial_3\right)\right)\mathrm{G}=0\\&=\left(\partial_0+i\partial_1+j\partial_2+k\partial_3\right)\left(\left(\partial_0-i\partial_1-j\partial_2-k\partial_3\right)\mathrm{G}\right)=0\\&=\left(\partial_0+i\partial_1+j\partial_2+k\partial_3\right)g_1=0,\ \text{and}\\\left(\partial_0^2+\partial_1^2+{\partial_2}^2+\partial_3^2\right)\mathrm{G}&=\left(\left(\partial_0+i\partial_1+j\partial_2+k\partial_3\right)\left(\partial_0-i\partial_1-j\partial_2-k\partial_3\right)\right)\mathrm{G}=0\\&=\left(\partial_0-i\partial_1-j\partial_2-k\partial_3\right)\left(\left(\partial_0+i\partial_1+j\partial_2+k\partial_3\right)\mathrm{G}\right)=0\\&=\left(\partial_0-i\partial_1-j\partial_2-k\partial_3\right)g_2=0\end{aligned}$$

Here, too, the quaternion differential operators are rotational transport operators (any quaternion being a scaling-rotation operator in $\mathbb{Q}$). We have thus advantage, again, in taking the transport equations into polar coordinates.

Formulating new transport equations in polar coordinates (r,θ) **yields:**

For $\left(\partial_0+i\partial_1+j\partial_2+k\partial_3\right)g_1=0:\ \left(\partial_\theta+\underline{u}\cdot r\cdot\partial_r\right)\varphi_1=0$,
which is satisfied by any twice differentiable function $\varphi_1\left(r\cdot e^{-\underline{u}\cdot\theta}\right)$, and
For $\left(\partial_0-i\partial_1-j\partial_2-k\partial_3\right)g_2=0:\ \left(\partial_\theta-\underline{u}\cdot r\cdot\partial_r\right)\varphi_2=0$,
which is satisfied by any twice differentiable function $\varphi_2\left(r\cdot e^{\underline{u}\cdot\theta}\right)$.

Therein, $\underline{u}=\frac{1}{\sqrt{\left(q_1^2+q_2^2+q_8^2\right)}}\left(iq_1+jq_2+kq_3\right)=\left(iu_1+ju_2+ku_3\right)$ is a pure unit quaternion.

The *characteristics* in this case are the quaternion expressions $\left(r\cdot e^{-\underline{u}\cdot\theta}\right)$ and $\left(r\cdot e^{\underline{u}\cdot\theta}\right)$.

The solution is therefore a superposition of arbitrary, twice differentiable functions $\varphi_1\left(r\cdot e^{-\underline{u}\cdot\theta}\right)$ and $\varphi_2\left(r\cdot e^{\underline{u}\cdot\theta}\right)$. Stationary waves can result from the superposition of two opposite-rotating functions, but there is also a possibility of stationary waves resulting from the self-interference of a single rotating function.

Self-interference of the solution functions:

Here again, the factors $e^{-\underline{u}\cdot\theta}$ and $e^{\underline{u}\cdot\theta}$ are rotating pointers, with a periodicity in θ of 2π. The solution functions φ_1 and φ_2 will therefore interfere with themselves after 2π or a multiple thereof, and eventually reinforce or wipe out what they had written before. Stationary solutions, i.e. solutions having a stable mean value, can thus exist if the solution functions φ_1 and φ_2 are periodic, with periodicities of $n + 2\pi$, wherein $n = 0, 1, 2, 3, \ldots$, i.e. a non-negative integer.

The complete polar representation of a quaternion comprises a radius and 3 angles (r, θ, χ, ξ) :

$$
\begin{aligned}
q &= (q_0 + iq_1 + jq_2 + kq_3) \\
&= \|q\|(\cos(\theta) + \underline{u}\cdot\sin(\theta)) \\
&= r\left(\begin{array}{c} \cos(\theta) \\ +i\cdot\sin(\theta)\cdot\cos(\chi) \\ +j\cdot\sin(\theta)\cdot\sin(\chi)\cdot\cos(\xi) \\ +k\cdot\sin(\theta)\cdot\sin(\chi)\cdot\sin(\xi) \end{array}\right), \\
\text{with } r &= \|q\| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}; \\
\theta &= \operatorname{arcos}\left(\frac{q_0}{\|q\|}\right); \\
\chi &= \operatorname{arcos}(u_1); \\
\text{and } \xi &= \operatorname{arcos}\left(\frac{u_2}{\sqrt{(u_2^2 + u_8^2)}}\right).
\end{aligned}
$$

The *spherical harmonics* in the solutions of the 4D LAPLACE'S equation form therefore a 3-dimensional manifold, because, beside the principal rotation in θ which we are contemplating here, there are subordinated rotations in χ and in ξ, which also lead to self-interference. This three-dimensional manifold gives rise to the *principal*, the *angular* and the *magnetic* quantum numbers of atomic physics, where each stationary state of an electron orbit is uniquely characterized by its combination of the three quantum numbers.

Back-transformation of the variables

In view of back-transforming the variables, we first rewrite the solution functions' arguments as a CARTESIAN vector representation:

$$\varphi_1\left(r\cdot e^{-\underline{u}\cdot\theta}\right)=\varphi_1\begin{pmatrix} r\cdot\cos\theta \\ -i\cdot u_1\cdot r\cdot\sin\theta \\ -j\cdot u_2\cdot r\cdot\sin\theta \\ -k\cdot u_3\cdot r\cdot\sin\theta \end{pmatrix}$$

$$\varphi_2\left(r\cdot e^{\underline{u}\cdot\theta}\right)=\varphi_2\begin{pmatrix} r\cdot\cos\theta \\ i\cdot u_1\cdot r\cdot\sin\theta \\ j\cdot u_2\cdot r\cdot\sin\theta \\ k\cdot u_3\cdot r\cdot\sin\theta \end{pmatrix}$$

The CARTESIAN coordinates span here, as do the polar coordinates (r,θ), the whole 4-dimensional space $\mathbb{R}^4$ of the function's arguments.

Applying the trigonometric identities $\cos(\alpha)=\cosh(\sqrt{-1}\cdot\alpha)$ and $i\cdot\sin(\alpha)=j\cdot\sin(\alpha)=k\cdot\sin(\alpha)=\sinh(\sqrt{-1}\cdot\alpha)$ yields for the arguments of the solution functions:

$$\varphi_1\begin{pmatrix} r\cdot\cosh(\sqrt{-1}\cdot\theta) \\ -u_1\cdot r\cdot\sinh(\sqrt{-1}\cdot\theta) \\ -u_2\cdot r\cdot\sinh(\sqrt{-1}\cdot\theta) \\ -u_3\cdot r\cdot\sinh(\sqrt{-1}\cdot\theta) \end{pmatrix} \text{ and}$$

$$\varphi_2\begin{pmatrix} r\cdot\cosh(\sqrt{-1}\cdot\theta) \\ u_1\cdot r\cdot\sinh(\sqrt{-1}\cdot\theta) \\ u_2\cdot r\cdot\sinh(\sqrt{-1}\cdot\theta) \\ u_3\cdot r\cdot\sinh(\sqrt{-1}\cdot\theta) \end{pmatrix}$$

We now apply the back-transformation $\{x_0\to ct; x_1\to -i\cdot x_1; x_2\to -j\cdot x_2; x_3\to -k\cdot x_3\}$, and obtain:

$$g_1\begin{pmatrix} ct\cdot r\cdot\cosh(\sqrt{-1}\cdot\theta) \\ i\cdot u_1\cdot r\cdot\sinh(\sqrt{-1}\cdot\theta) \\ j\cdot u_2\cdot r\cdot\sinh(\sqrt{-1}\cdot\theta) \\ k\cdot u_3\cdot r\cdot\sinh(\sqrt{-1}\cdot\theta) \end{pmatrix} \text{ and}$$

$$g_2\begin{pmatrix} ct\cdot r\cdot\cosh(\sqrt{-1}\cdot\theta) \\ -i\cdot u_1\cdot r\cdot\sinh(\sqrt{-1}\cdot\theta) \\ -j\cdot u_2\cdot r\cdot\sinh(\sqrt{-1}\cdot\theta) \\ -k\cdot u_3\cdot r\cdot\sinh(\sqrt{-1}\cdot\theta) \end{pmatrix}.$$

The solution function's arguments are now again in a hyperbolic space, corresponding to the metric $(+1, -1, -1, -1)$ of the 4-space of Special Relativity. Regardless of r, the function's arguments run always into the infinite. With respect to the linear EUCLIDEAN space of the LAPLACE equation, the wave equation of the free electromagnetic wave spans a reciprocal, hyperbolic space. We are thus living in a reciprocal space, and has not only travelling-wave-solutions, but also stationary solutions.