Free subgroups in groups with few relators

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FREE SUBGROUPS IN GROUPS WITH FEW RELATORS

by John S. WILSON

1. INTRODUCTION

In [11], we proved the following result:

THEOREM 1. *Let G be an abstract (resp. pro-p) group which has a presentation with n generators $x_1, \dots, x_n$ and m relators, where $m < n$, and let Y be any generating set for G. Then there are $n - m$ elements of Y that freely generate a free abstract (resp. pro-p) group.*

The Freiheitssatz proved by Magnus in [3] in 1930 is essentially the special case of Theorem 1 for abstract groups with $Y = \{x_1, \dots, x_n\}$ and $m = 1$. In [5] and [6] Romanovskiĭ proved the case of Theorem 1 in which $Y = \{x_1, \dots, x_n\}$. The proof of the general case in [11] was indirect, relying on Romanovskiĭ's result in [6]. In [9] Romanovskiĭ and the author gave a direct proof of a more general result concerning quotients of a free product of n groups, for the case of abstract groups. Our first object here is to give a much simpler proof of Theorem 1 in the abstract case and to indicate the modifications required for the case of pro-p groups. We shall also prove a result for pro-p groups that is similar in spirit to the main result of [9]; this result has the following consequence.

THEOREM 2. *Let G be a finitely generated pro-p group generated by a family $\mathcal{A}$ of n finitely generated pro-p subgroups each having $\mathbf{Z}_p$ as an image, and suppose that the kernel R of the natural map from the free pro-p product F of the groups in $\mathcal{A}$ to G is generated (as a closed normal subgroup) by m elements, where $m < n$. Let $\mathcal{B}$ be a family of subgroups of G that generate G. Then $\bigcup\{B \mid B \in \mathcal{B}\}$ contains $n - m$ elements that freely generate a free pro-p group.*

In particular, either $|\mathcal{B}| \geqslant n - m$ or some subgroup in $\mathcal{B}$ contains a non-abelian free pro-p subgroup.

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2. PROOF OF THEOREM 1

Theorem 1 is reminiscent of the Steinitz exchange lemma from linear algebra; indeed, it is a precise analogue of the statement that if V is an n-dimensional vector space over a field Q and R is a subspace of dimension at most m, then any set Y such that $R \cup Y$ spans V contains $n-m$ elements that are linearly independent modulo R. Most earlier proofs of results like Theorem 1 have relied on

(a) the above statement from linear algebra, but with V a right vector space over a skew-field Q,

(b) the Magnus embedding, and

(c) a rather complicated induction argument.

In the proof below, (c) is eliminated. We begin therefore with the ingredient (b).

Our notation for conjugates and commutators in a group G is as follows: we write $a^b = b^{-1}ab$ and $[a,b] = a^{-1}b^{-1}ab$. We shall write N' for the *derived group* of a group N; in the case of pro-p groups, N' refers of course to the *closure* of the abstract group generated by all commutators.

2.1 THE MAGNUS EMBEDDING

Let H be a group and M a right $\mathbf{Z}H$-module. It is convenient to write elements of the split extension $G = H \ltimes M$ as matrices

$$\begin{pmatrix} h & 0 \\ m & 1 \end{pmatrix} \qquad (h \in H,\ m \in M).$$

Thus matrix multiplication

$$\begin{pmatrix} h_1 & 0 \\ m_1 & 1 \end{pmatrix} \begin{pmatrix} h_2 & 0 \\ m_2 & 1 \end{pmatrix} = \begin{pmatrix} h_1h_2 & 0 \\ m_1h_2 + m_2 & 1 \end{pmatrix}$$

reflects the fact that $(h_1m_1)(h_2m_2) = (h_1h_2)(m_1^{h_2}m_2)$. We may regard M as a $\mathbf{Z}G$-module, and then the map δ taking $g \in G$ to its $(2,1)$-entry is a *derivation*, i.e. $\delta(g_1g_2) = (\delta g_1)g_2 + \delta g_2$ for all $g_1, g_2 \in G$. The *Magnus embedding* for abstract groups is the map j from F/R' in (b), (c) below.

LEMMA 1. *Let R be a normal subgroup of the free group F with basis $\{x_1, \dots, x_n\}$, and let $H = F/R$. Let M be a $\mathbf{Z}H$-module and $t_1, \dots, t_n \in M$.*

(a) *The assignment*

$$x_i \mapsto \begin{pmatrix} x_iR & 0 \\ t_i & 1 \end{pmatrix}$$

determines a homomorphism

$$\mu\colon F \to \begin{pmatrix} H & 0 \\ M & 1 \end{pmatrix}.$$

(b) $R' \leqslant \ker \mu \leqslant R$; *let j be the map from F/R' induced by μ.*

(c) *If M is the free $\mathbf{Z}H$-module with basis $\{t_1, \dots, t_n\}$ then j is injective.*

Proof. Assertion (a) is clear, and so is (b) since the image of R under μ is abelian. The following proof of (c), included for the reader's convenience, is due to Romanovskiĭ.

There is certainly an embedding θ of F/R' in a group of the form

$$\begin{pmatrix} H & 0 \\ N & 1 \end{pmatrix}$$

for a $\mathbf{Z}H$-module N. Indeed, we can take for N the abelian group B of all functions $b\colon H \to R/R'$, which is a right $\mathbf{Z}H$-module with action defined by $(bh)(x) = b(xh^{-1})$ for $b \in B$, $h \in H$, $x \in H$; since the split extension of B by H is the unrestricted standard *wreath product* $R/R' \ \overline{\mathrm{wr}}\ F/R$, the existence of a suitable map θ follows from the Kaloujnine–Krasner theorem ([1]; see also e.g. [10, Theorem 4.4.1]). Explicitly, θ can be defined as follows. Choose a set-theoretic section $\sigma\colon F/R \to F/R'$ to the canonical projection $q\colon F/R' \to F/R$ (that is, a function such that its composite with q is the identity map on F/R), and for each $fR' \in F/R'$ define $\delta(fR') \in B$ by

$$(\delta(fR'))(uR) = \sigma(uf^{-1}R) \cdot fR' \cdot (\sigma(uR))^{-1} \qquad \text{for all } uR \in F/R\,.$$

Simple calculations show that (with B written multiplicatively) we have $\delta(\bar{f}_1\bar{f}_2) = (\delta\bar{f}_1)^{\bar{f}_2}(\delta\bar{f}_2)$ for all $\bar{f}_1, \bar{f}_2 \in F/R'$ and also that if $\bar{f} \in R/R'$ and $\delta\bar{f}$ is the identity element of B then $\bar{f}$ is the identity element of R/R'. It follows immediately that the map θ defined by

$$\theta(fR') = \begin{pmatrix} fR & 0 \\ \delta(fR') & 1 \end{pmatrix} \in \begin{pmatrix} H & 0 \\ N & 1 \end{pmatrix}$$

is an injective homomorphism.

To prove (c) it suffices now to show that the diagram

$$\begin{array}{ccccc} F & \longrightarrow & F/R' & \xrightarrow{\ \theta\ } & \begin{pmatrix} H & 0 \\ N & 1 \end{pmatrix} \\ & & {\scriptstyle j}\searrow & & \nearrow{\scriptstyle \bar{\theta}} \\ & & & \begin{pmatrix} H & 0 \\ M & 1 \end{pmatrix} & \end{array}$$

can be completed with a map $\bar{\theta}$. Define $v_i \in N$ by

$$\theta(x_iR') = \begin{pmatrix} x_iR & 0 \\ v_i & 1 \end{pmatrix},$$

and let $\kappa\colon M \to N$ be the $\mathbf{Z}H$-module homomorphism defined by $t_i \mapsto v_i$. Then the map

$$\begin{pmatrix} h & 0 \\ m & 1 \end{pmatrix} \longmapsto \begin{pmatrix} h & 0 \\ \kappa m & 1 \end{pmatrix}$$

has the required property.

LEMMA 2. *Let $\delta\colon H \to W$ be a derivation from a group H to a right H-module W. If $H = \langle Z\rangle$ then the subset δH lies in the $\mathbf{Z}H$-submodule W_1 generated by δZ.*

Proof. If δh_1, $\delta h_2 \in W_1$ then $\delta(h_1h_2^{-1}) = (\delta h_1)h_2^{-1} - (\delta h_2)h_2^{-1} \in W_1$.

2.2 Embedding of group rings in skew-fields

We recall that a group G is called *orderable* if it has a total order $\leqslant$ such that if a, $b \in G$ and $a \leqslant b$ then $xay \leqslant xby$ for all $x, y \in G$; the pair $(G, \leqslant)$ is then an *ordered group*. It is well known and easily checked that if $G = H \ltimes A$ is a split extension of ordered groups $(H, \leqslant_H)$, $(A, \leqslant_A)$, and if $1 \leqslant_A a \in A$ and $h \in H$ imply $1 \leqslant_A a^h$, then G becomes an ordered group with respect to the order defined as follows: $h_1 a_1 \leqslant h_2 a_2$ if and only if either $h_1 <_H h_2$, or $h_1 = h_2$ and $a_1 \leqslant_A a_2$. The following lemma is also no doubt well known.

Lemma 3. *Each group G has a unique normal subgroup K minimal such that G/K is orderable.*

Proof. Let $(K_\lambda)_{\lambda \in \Lambda}$ be the set of kernels of maps from G to orderable groups and set $K = \bigcap K_\lambda$. We fix an order on each group G/K_λ, and we may take the set Λ to be well ordered. Now we can define an order on G/K by writing $aK < bK$ if for some $\mu \in \Lambda$ we have $aK_\mu < bK_\mu$ and $aK_\lambda = bK_\lambda$ for all $\lambda < \mu$.

An *ordered skew-field* is a skew-field Q together with an order $\leqslant$ such that both Q under addition and the set $\{h \in Q \mid h > 0\}$ under multiplication are ordered groups with respect to $\leqslant$; denote the latter group by $U_+(Q)$.

We need the following result proved by B.H. Neumann [4].

Proposition 1. *Let H be an ordered group. Then $\mathbf{Z}H$ can be embedded in an ordered skew-field Q in such a way that the order on Q induces an embedding of H (as an ordered group) in $U_+(Q)$.*

A standard candidate for Q is the skew-field of formal expressions $q = \sum_{h \in H} \lambda_h h$ with $\lambda_h \in \mathbf{Q}$ for all $h \in H$ and with support $\{h \in H \mid \lambda_h \neq 0\}$ inversely well-ordered; then $U_+(Q)$ is the set of elements q such that $\lambda_m > 0$, where $m \in H$ is the greatest element of the support of q. For the details we refer to Neumann [4], or [2, §14 and Corollary 18.6]. (In fact Neumann works with the ring of formal expressions with well-ordered support, and his embedding of H in $U_+(Q)$ is order-reversing; an order-preserving embedding is obtained by composing the inversion map on H with this embedding.)

LEMMA 4. *Let H, Q be as above and let V be a finite-dimensional right vector space over Q; thus V is naturally a $\mathbf{Z}H$-module. Then the split extension $H \ltimes V$ is orderable.*

Proof. We may regard V as the space $Q^{(n)}$ of n-tuples of elements of Q. We define an order on V by writing $(x_1, \dots, x_n) < (y_1, \dots, y_n)$ if $y_i - x_i > 0$ for the first non-zero $y_i - x_i$. Thus if $0 < v \in V$ and $h \in H$ then $vh > 0$, and so the split extension is orderable from above.

2.3 PROOF OF THE THEOREM: ABSTRACT CASE

Let G be as in the statement of Theorem 1, and let F be free with basis $\{x_1, \dots, x_n\}$. Thus the kernel R of the obvious map from F to G can be generated as a normal subgroup by elements $r_1, \dots, r_m$, where $m < n$. Lemma 3 guarantees the existence of a smallest normal subgroup S of F with $R \leqslant S$ and F/S orderable. Write $\overline{G} = F/S$.

Let Q be an ordered skew-field containing $\mathbf{Z}\overline{G}$ as in Proposition 1. Let V be the right vector space over Q with basis $\{t_1, \dots, t_n\}$, and let M be the $\mathbf{Z}\overline{G}$-submodule generated by $t_1, \dots, t_n$; thus M is a free $\mathbf{Z}\overline{G}$-module with basis $\{t_1, \dots, t_n\}$. Define

$$\theta\colon F \to \begin{pmatrix} \overline{G} & 0 \\ M & 1 \end{pmatrix} \quad \text{by} \quad x_i \mapsto \begin{pmatrix} x_i S & 0 \\ t_i & 1 \end{pmatrix}$$

and

$$\delta\colon F \to M \quad \text{by} \quad \theta f = \begin{pmatrix} fS & 0 \\ \delta f & 1 \end{pmatrix}.$$

Let U be the subspace of V spanned by $\{\delta r_1, \dots, \delta r_m\}$, and write $W = V/U$, $r = \dim W$; so $r \geqslant n - m$. Let $\bar{\delta}$ be the map $f \mapsto U + \delta f$. Thus the set $\{\bar{\delta} x_1, \dots, \bar{\delta} x_n\}$ spans W.

Consider the map

$$\varphi\colon \begin{pmatrix} \overline{G} & 0 \\ M & 1 \end{pmatrix} \to \begin{pmatrix} \overline{G} & 0 \\ (M+U)/U & 1 \end{pmatrix},$$

and let $\psi = \varphi\theta$. By Lemma 4, the codomain of ψ is orderable, and so $F/\ker\psi$ is orderable. But $\ker\psi \leqslant S$ and $r_1, \dots, r_m \in \ker\psi$, and hence $\ker\psi = S$. Therefore ψ induces an injective map

$$j\colon \overline{G} \rightarrowtail \begin{pmatrix} \overline{G} & 0 \\ W & 1 \end{pmatrix}.$$

Now let $Y \subseteq F$ generate F modulo R. Since $R \leqslant \ker\psi$ we have $\bar{\delta} R = 0$, and therefore, since $\bar{\delta}$, like δ, is a derivation, $\bar{\delta} Y$ spans W by Lemma 2;

let $\{\bar{\delta}y_1, \dots, \bar{\delta}y_r\} \subseteq \bar{\delta}Y$ be a basis. In particular, $\bar{\delta}y_1, \dots, \bar{\delta}y_r$ generate a free $\mathbf{Z}\overline{G}$-submodule of W.

Let E be the free group with basis $\{y_1, \dots, y_r\}$, and define $\alpha\colon E \to \overline{G}$ by $y_i \mapsto y_i S$. Let $N = \ker \alpha$. By Lemma 1, the map

$$\beta\colon y_i \mapsto \begin{pmatrix} y_i S & 0 \\ \bar{\delta}y_i & 1 \end{pmatrix}$$

has kernel N'. But $\beta = j\alpha$ and j is injective, and hence $N = N'$. Since N is also a subgroup of a free group, and hence free, we must have $N = 1$. Therefore the subgroup $\langle y_1, \dots, y_r \rangle$ of F is free modulo S, and so free modulo R.

The reader will notice that the proof above gives a stronger result than Theorem 1: with the hypotheses of the theorem there is a homomorphism from G to an orderable group P such that $n - m$ elements of Y map to a basis of a free subgroup of P. The reader will also notice that there is no need to introduce M in the above proof. The reason for doing so will appear in the next section.

2.4 Modifications for the pro-p case

The arguments of Section 2.3 apply without essential change in the pro-p case; all subgroups are now understood to be closed, all maps continuous, and modules are modules for the *completed group ring* $\mathbf{Z}_p[[G]]$ of G over $\mathbf{Z}_p$. For information about pro-p groups and their completed group rings we refer the reader to [10]. Instead of appealing to the Kaloujnine–Krasner theorem to embed an extension in a split extenson, we may use the following well-known result.

Lemma 5. *Let A be a (closed) abelian normal subgroup of a pro-p group G and let $H = G/A$. Then G can be embedded in a pro-p group $H \ltimes B$ with B abelian, in such a way that the composite of the embedding and the map $H \ltimes B \to H$ is the quotient map $G \to H$.*

Proof. Let $(N_\lambda)_{\lambda \in \Lambda}$ be a family of open normal subgroups with $\bigcap N_\lambda = 1$. The Kaloujnine–Krasner theorem for finite groups gives embeddings

$$j_\lambda\colon G/N_\lambda \to G/AN_\lambda \ltimes B_\lambda$$

with each B_λ an abelian p-group, and we consider the subgroup of the Cartesian product $\mathrm{Cr}\,(G/AN_\lambda \ltimes B_\lambda)$ generated by the abelian normal subgroup $\mathrm{Cr}\,B_\lambda$ and the image of G under the map $g \mapsto (j_\lambda(gN_\lambda))$.

We can no longer use ordered groups as in Section 2.3, because, for example, we need to ensure that $U \cap M$ is closed in the $\mathbf{Z}_p[[G]]$-module M. Instead we need to use a deep result of Romanovskiĭ [6].

A filtration

$$A = A_{(1)} \geqslant \cdots \geqslant A_{(i)} \geqslant \cdots$$

of normal subgroups of a profinite with $\bigcap A_{(i)} = 1$ is called *convergent* if each neighbourhood of 1 contains some subgroup $A_{(i)}$. Write $\mathcal{N}$ for the class of all finitely generated pro-p groups having a convergent filtration with torsion-free central factors. If G is any finitely generated pro-p group then G has a unique minimal normal subgroup K such that $G/K \in \mathcal{N}$, namely the intersection of the kernels of all maps from G to torsion-free nilpotent pro-p groups.

PROPOSITION 2 (cf. [6, Proposition 7]). *Let H be a pro-p group in $\mathcal{N}$ and let L be the completed group ring $\mathbf{Z}_p[[H]]$ of H. Then there exist a filtration $(H_i)_{i\geqslant 1}$ with torsion-free central factors and a skew-field $Q \geqslant L$ such that the following holds: if $n \geqslant 1$ and U is a subspace of the vector space $Q^{(n)}$, then*

(i) *$U \cap L^{(n)}$ is closed in $L^{(n)}$, and*

(ii) *the $\mathbf{Z}_p$-module $M = L^{(n)}/(U \cap L^{(n)})$ has a filtration $(M_j)_{j\geqslant 1}$ of closed submodules such that $[M_j, H_i] \leqslant M_{i+j}$ and M_j/M_{j+1} is a torsion-free group for all i, j; moreover*

(iii) *$(H_i M_i)_{i\geqslant 1}$ is a filtration of $H \ltimes M$ with torsion-free central factors, and so $H \ltimes M \in \mathcal{N}$.*

In the proof of Theorem 1 for pro-p groups, we take S/R to be the intersection of the kernels of all maps from F/R to torsion-free nilpotent pro-p groups; thus $F/S \in \mathcal{N}$ and S is the smallest normal subgroup containing R with this property. Define ψ as in the proof in Section 2.3. It follows from Proposition 2 that the codomain of ψ is a pro-p group and is in $\mathcal{N}$. The rest of the proof from Section 2.3 now applies without any change.

3. IMAGES OF FREE PRODUCTS OF PRO-p GROUPS

3.1 THE MAGNUS EMBEDDING FOR FREE PRO-p PRODUCTS

The Magnus embedding used in Section 2 has been modified by Shmel′kin and Romanovskiĭ to the case of free products of groups. Everything that we require can be deduced from the following special case of Romanovskiĭ [7, Theorem 3].

LEMMA 6. *Let F be the free pro-p product of the pro-p groups $A_1, \ldots, A_n$ and let $H = F/R$, where R is a (closed) normal subgroup such that $A_i \cap R = 1$ for $i = 1, \ldots, n$. Let T be the free right $\mathbf{Z}_p[[H]]$-module with basis $\{t_1, \ldots, t_n\}$. Let*

$$\mu\colon F \longrightarrow \begin{pmatrix} H & 0 \\ T & 1 \end{pmatrix}$$

be the homomorphism defined on the free factors A_i of F by

$$a \mapsto \begin{pmatrix} aR & 0 \\ t_i(a-1) & 1 \end{pmatrix} \quad \text{for } a \in A_i\,.$$

Then $\ker \mu = R'$.

As observed in [8, Lemma 5], Lemma 6 may be modified as follows.

LEMMA 7. *The conclusion of Lemma 6 remains true if the hypothesis on T is replaced by the requirement that $\{t_2, \ldots, t_n\}$ is a basis of T and $t_1 = 0$.*

Proof. This follows from the formula

$$\begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} aR & 0 \\ t_i(a-1) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix} = \begin{pmatrix} aR & 0 \\ (t_i - t_1)(a-1) & 1 \end{pmatrix}.$$

3.2 DERIVATIONS TO RIGHT VECTOR SPACES

We prove the following result concerning derivations from pro-p groups G to right vector spaces V over skew-fields containing $\mathbf{Z}_p[[G]]$. The derivations under consideration are understood to be continuous regarded as maps into finitely generated $\mathbf{Z}_p[[G]]$-submodules of V; a derivation $\delta\colon G \to V$ is *inner* if there exists some $v \in V$ such that $\delta g = v(g-1)$ for all $g \in G$.

PROPOSITION 3. *Suppose that G is a finitely generated pro-p group such that $\mathbf{Z}_p[[G]]$ can be embedded in a skew-field Q, and suppose that G is generated by subgroups A and B. Let δ be a derivation from G to a right vector space V over Q. If the restrictions $\delta|_A$, $\delta|_B$ are both inner derivations, then either G is the free pro-p product of A, B or δ is inner.*

Proof. By hypothesis, there are m_A, $m_B \in V$ such that $\delta|_A$, $\delta|_B$ are the maps $a \mapsto m_A(a-1)$, $b \mapsto m_B(b-1)$. Let M be the $\mathbf{Z}_p[[G]]$-module generated by $m_B - m_A$, let F be the free pro-p product of A, B, and N the kernel of the map $q\colon F \to G$ extending the identity maps on A, B.

Suppose that δ is not inner; then $m_A \neq m_B$ and the map $\gamma\colon g \mapsto \delta g - m_A(g-1)$ is a non-zero derivation. By Lemma 7 the (continuous) homomorphism

$$\mu\colon F \to \begin{pmatrix} F/N & 0 \\ M & 1 \end{pmatrix}$$

defined on $A \cup B$ by

$$a \mapsto \begin{pmatrix} aN & 0 \\ 0 & 1 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} bN & 0 \\ (m_B - m_A)(b-1) & 1 \end{pmatrix}$$

has kernel N'. Define $\tilde{\gamma}\colon F \to V$ by

$$\mu f = \begin{pmatrix} fN & 0 \\ \tilde{\gamma} f & 1 \end{pmatrix}.$$

Then $\tilde{\gamma}$ and γq are (continuous) derivations from F that agree on $A \cup B$, and so they are equal. However for $n \in N$ we have $\tilde{\gamma} n = \gamma q n = 0$, and so $\mu n = 1$. Thus $N = N'$, and since N is a pro-p group we have $N = 1$, as required.

3.3 DI-GROUPS

In order to state and prove the next result concisely, we make a definition, concerning circumstances under which certain *derivations* are guaranteed to be *inner*. We say that a finitely generated pro-p group G is a DI-*group* if its completed group ring $\mathbf{Z}_p[[G]]$ can be embedded in a skew-field and if whenever Q is a skew-field containing $\mathbf{Z}_p[[G]]$ and $\delta\colon G \to V$ is a derivation to a finite-dimensional space over Q then δ is inner. Again, our derivations are continuous maps into finitely generated $\mathbf{Z}_p[[G]]$-submodules.

Clearly $\mathbf{Z}_p$ is a DI-group, and, by Proposition 3, any pro-p group that is generated by two DI-subgroups either is the free pro-p product of the two subgroups or is again a DI-group.

THEOREM 3. *Let F be the free pro-p product of a family $\mathcal{A}$ of n finitely generated pro-p groups each having $\mathbf{Z}_p$ as an image, and let R be a normal subgroup of F generated (as a normal subgroup) by m elements of F, where $m < n$. Let S be the intersection of all normal subgroups N of F with $R \leqslant N$ and F/N torsion-free nilpotent.*

Write $\overline{G} = F/S$, and for $A \in \mathcal{A}$ write $\overline{A}$ for the image of A in $\overline{G}$. Let $\mathcal{B}$ be a family of DI*-subgroups of $\overline{G}$, set $J = \langle B \mid B \in \mathcal{B} \rangle$, and suppose that for each A in $\mathcal{A}$ with $\overline{A} \neq 1$, the subgroups $\overline{A}$ and J do not generate their free product in $\overline{G}$. Then $|\mathcal{B}| \geqslant n - m$, and there are $n - m$ members of $\mathcal{B}$ that generate in $\overline{G}$ their free product.*

Theorem 3 implies the result stated as Theorem 2 in the Introduction. Assume the hypotheses of Theorem 2 and define S, $\overline{G}$ as in Theorem 3. Let $\mathcal{B}_1$ be the family of all procyclic subgroups of groups in $\mathcal{B}$ and let $\overline{\mathcal{B}}_1$ be the family of non-trivial images of members of $\mathcal{B}_1$ in $\overline{G}$; since $\overline{G}$ is torsion-free, $\overline{\mathcal{B}}_1$ consists of DI-subgroups. By Theorem 3 there are $n - m$ members of $\overline{\mathcal{B}}_1$ that freely generate a free pro-p subgroup of $\overline{G}$, and thus their pre-images in $\mathcal{B}_1$ freely generate a free pro-p subgroup of G. Theorem 2 follows.

3.4 Proof of Theorem 3

Assume the hypotheses of the theorem. Write $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, where $\mathcal{A}_1$ contains all subgroups A with non-trivial images in $\overline{G}$ and $\mathcal{A}_2 = \mathcal{A} \setminus \mathcal{A}_1$. We can replace all groups A from $\mathcal{A}_1$ by their images in $\overline{G}$ and also identify them with their images in $\overline{G}$. Let Q be a skew-field containing $\mathbf{Z}_p[[\overline{G}]]$ with the properties given by Proposition 2. By hypothesis, for each $A \in \mathcal{A}_2$ there is a non-zero continuous homomorphism ν_A from A to the additive group of Q. Let V be the right vector space over Q with basis $\{t_A \mid A \in \mathcal{A}\}$ and let M be the $\mathbf{Z}_p[[G]]$-submodule with basis $\{t_A \mid A \in \mathcal{A}\}$. Define a group homomorphism

$$\theta\colon F \to \begin{pmatrix} \overline{G} & 0 \\ M & 1 \end{pmatrix}$$

by specifying its restriction $\theta|_A$ to the free factors as follows:

$$a \mapsto \begin{pmatrix} a & 0 \\ t_A(a-1) & 1 \end{pmatrix}, \qquad \text{for } a \in A \in \mathcal{A}_1,$$

$$a \mapsto \begin{pmatrix} 1 & 0 \\ \nu_A(a)\, t_A & 1 \end{pmatrix}, \qquad \text{for } a \in A \in \mathcal{A}_2.$$

Since the subspace of V spanned by the bottom left-hand entries of the images of the elements of F contains all elements t_A, it is equal to V.

Let R be generated as a normal subgroup of F by $r_1, \dots, r_m$. The images θr_i have the form

$$\begin{pmatrix} 1 & 0 \\ u_i & 1 \end{pmatrix}$$

and so they all lie in the subgroup

$$\begin{pmatrix} 1 & 0 \\ U \cap M & 1 \end{pmatrix},$$

where U is the subspace of V spanned by $\{u_1, \dots, u_m\}$. Write $W = V/U$. Then the kernel K of the map

$$\psi\colon F \to \begin{pmatrix} \overline{G} & 0 \\ W & 1 \end{pmatrix}$$

induced by θ contains R. Moreover K consists of the elements of S whose images under θ have bottom left entry in $U \cap M$. It follows from Proposition 2 that $U \cap M$ is closed in M and that $\overline{G} \ltimes (M/(U \cap M)) \in \mathcal{N}$; therefore $F/K \in \mathcal{N}$, and by the definition of S we conclude that $K = S$ and that θ induces an injective map

$$j\colon \overline{G} \to \begin{pmatrix} \overline{G} & 0 \\ W & 1 \end{pmatrix}.$$

By construction we have

$$jg = \begin{pmatrix} g & 0 \\ \delta g & 1 \end{pmatrix},$$

where $\delta\colon \overline{G} \to W$ is a derivation.

We note that $t_A \in U$ for each $A \in \mathcal{A}_2$; this follows since $A \leqslant S = K$, which maps under θ to the group of matrices with bottom left entry in U.

Set $\dim W = r$; thus $r \geqslant n - m$. Since all groups in $\mathcal{B}$ are DI-groups, the restriction maps $\delta|_B$ have the form $b \mapsto s_B(b-1)$ for some elements $s_B \in W$. Let U_1/U be the subspace of W spanned by $\{s_B \mid B \in \mathcal{B}\}$. Fix $A \in \mathcal{A}_1$, set $L = \langle J, A \rangle$ and consider the composite $\bar{\delta}$ of the restriction $\delta|_L$ and the map $W = V/U \to W/U_1$. Since L is not the free product of J, A and since $\bar{\delta}|_J = 0$ and $\bar{\delta}|_A$ is an inner derivation, Proposition 3 implies that $\bar{\delta} = 0$. From the definition of δ it now follows that $t_A \in U_1$. Since this holds for all $A \in \mathcal{A}_1$, we conclude that U_1 contains $\{t_A \mid A \in \mathcal{A}\}$ and hence equals V. Therefore W is spanned by $\{s_B \mid B \in \mathcal{B}\}$. Choose $\mathcal{B}_0 \subseteq \mathcal{B}$ such that $\{s_B \mid B \in \mathcal{B}_0\}$ is a basis of V.

We claim that the subgroups in $\mathcal{B}_0$ generate their free pro-p product in $\overline{G}$. Write E for the free product of the groups $B \in \mathcal{B}_0$ and consider the homomorphism $\alpha\colon E \to \langle B \mid B \in \mathcal{B}_0 \rangle \leqslant \overline{G}$. Let $N = \ker \alpha$. We have $B \cap N = 1$ for each $B \in \mathcal{B}_0$ and

$$j\alpha b = \begin{pmatrix} b & 0 \\ s_B(b-1) & 1 \end{pmatrix} \quad \text{for } b \in B \in \mathcal{B}_0\,.$$

By Lemma 4 we have $\ker j\alpha = N'$, and hence $N = N'$ since j is injective. Since N is a pro-p group it follows that $N = 1$, so that α is injective. This concludes the proof of Theorem 3.

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John S. Wilson

University College
Oxford OX1 4BH
United Kingdom
e-mail: wilsonjs@maths.ox.ac.uk