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Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **55 (2009)**

Heft 3-4

PDF erstellt am: **26.04.2024**

Persistenter Link: https://doi.org/10.5169/seals-110107

L'Enseignement Mathématique (2) 55 (2009), 329–357

ON A THEOREM OF RENÉ THOM IN GÉOMÉTRIE FINIE

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ABSTRACT. We study generalisations of the following fact: a generic compact curve in the plane intersects every straight line in a finite number of points; moreover, for each such curve, this number is bounded. Our results develop the first part of René Thom's 1968 paper on Géométrie finie ("finite geometry").

INTRODUCTION

In his article [21], Thom defines the *k-degree* $\deg_k A$ of a subset A of $\mathbf{R}^N$ to be the supremum over all affine k-planes H of the number of intersections of H with A. For example, the k-degree of an n-dimensional algebraic subset A of $\mathbf{R}^{n+k}$ is finite, at most equal to the algebraic degree of A as an affine variety, unless A contains some affine subspace of positive dimension [1]).

He then sketches a proof of the following result:

THEOREM (Thom). *Let V be a compact smooth manifold of dimension n. There exists a dense open subset $\mathcal{U}$ in $C^\infty(V, \mathbf{R}^{n+k})$ such that, for all $f \in \mathcal{U}$, the k-degree of $f(V)$ is finite.*

Oddly enough, the present paper provides, it seems, the first complete proof (and a somewhat better statement, Theorem 1.1). It is essentially an illustration of Thom's beautiful and now classical ideas founding singularity

*) Expanded version of a previous text by the second author.

[1]) In the second part of his paper, Thom sketches a deep converse, established later by W. F. Pohl [18]: for each positive integer m, a compact connected real C^4 submanifold A of $\mathbf{CP}^{n+k}$ which meets almost every complex projective k-plane in exactly m points is either complex algebraic of complex dimension n, or the image of the real projective subspace $\mathbf{RP}^{2n} \subset \mathbf{CP}^{2n} \subset \mathbf{CP}^{n+k}$ under a complex projective transformation.

theory [10]; the piece of the puzzle that was missing in [21] is a basic but apparently not so widely known version of a theorem of Tougeron going back to the same period [23] and leading to a precise local formulation of Thom's theorem, which is stated as Theorem 3.1 and proved in the first part of the present article.

In the second part, the same ideas first yield the following "dual" version of Thom's theorem (better stated as Theorem 5.1):

THEOREM. *Let B be a metrisable, separable k-dimensional manifold.*

(i) *The Whitney-open subset of $C^\infty(\mathbf{R}^{n+k}, B)$ consisting of all proper maps contains a dense open subset $\mathcal{W}$ such that, for every $g \in \mathcal{W}$, the k-degree of $g^{-1}(b)$ is finite for all $b \in B$.*

(ii) *Given a point $0 \in B$, the Whitney-open subset of $C^\infty(\mathbf{R}^{n+k}, B)$ consisting of those maps g for which $g^{-1}(0)$ is compact contains a dense open subset $\mathcal{W}_0$ such that, for every $g \in \mathcal{W}_0$, the k-degree of $g^{-1}(0)$ is finite.*

As before, this follows from a more precise local statement, Theorem 5.2. After its proof, we show that the estimates provided are sharp for $k = 1$, and then give an idea of the geometry hidden behind algebra in low dimensions.

Sections 6 and 7 deal with extensions to differential geometry, where the affine k-planes are replaced by geodesics or, more generally, by the leaves of what we call a *texture* — expressing, we hope, the essence of the problem. A slight generalisation of Thom's transversality lemma in jet spaces (in the easy case where 'transversality' means 'non-intersection') is needed, whose proof in Section 8 might introduce non-specialists to such matters.

NOTATION, CONVENTIONS AND DEFINITIONS. We consider only C^∞ metrisable, separable, finite-dimensional manifolds. Given two manifolds M, N, we denote by $J^s(M,N)$ $(s \in \mathbf{N})$ the *manifold of s-th order jets* of maps $f\colon M \to N$ and by $j^s f(x) \in J^s(M,N)$ the *s-th order jet* of f at $x \in M$. We endow $C^\infty(M,N)$ with the *Whitney C^∞ topology*, generated by the open subsets $\mathcal{U}_U := \{f : j^s f(M) \subset U\}$ when U varies among the open subsets of $J^s(M,N)$ and s in $\mathbf{N}$. It has the Baire property [13, 8].

A subset $\mathcal{F}$ of $C^\infty(M,N)$ *has codimension greater than $c \in \mathbf{N}$* when, for every (metrisable, separable) manifold Λ of dimension c, there exists a residual subset of $C^\infty(\Lambda \times M, N)$ consisting of maps $f_\bullet\colon (\lambda, x) \mapsto f_\lambda(x)$ such that every f_λ lies off $\mathcal{F}$. To put it smoothly, (Baire-)almost every smooth family $\lambda \mapsto f_\lambda \in C^\infty(M,N)$ depending on c parameters avoids $\mathcal{F}$. In that case, we shall say that $C^\infty(M,N) \setminus \mathcal{F}$ is *c-large.*

The subset $\mathcal{F}$ *has infinite codimension* when it has codimension greater than c for every c, in which case we shall say that $C^\infty(M,N) \setminus \mathcal{F}$ is *huge.* Thus, a subset of $C^\infty(M,N)$ is huge when it is c-large for every c.

I. THOM'S THEOREM

1. STATEMENT OF THE RESULT; THE LOCAL k-DEGREE

Here is the version we shall prove:

THEOREM 1.1. *Let V be a compact manifold of dimension n. There exists in $C^\infty(V,\mathbf{R}^{n+k})$ a huge open subset $\mathcal{U}$ such that, for all $f \in \mathcal{U}$, the k-degree of $f(V)$ is finite.*

The very definition of a huge subset yields a generalisation (in which openness will follow at once from our proof of the theorem):

COROLLARY 1.2. *Let V be a compact manifold of dimension n and Λ a manifold. There exists in $C^\infty(\Lambda \times V,\mathbf{R}^{n+k})$ a huge open subset consisting of maps $f_\bullet\colon (\lambda,x) \mapsto f_\lambda(x)$ such that the k-degree of $f_\lambda(V)$ is finite for all $\lambda \in \Lambda$.*

The proof of Theorem 1.1 is based upon Thom's transversality lemma, but this is not a mere affair of transversality in multijet spaces, as the degree of $f(V)$ is *not* a bounded function of $f \in \mathcal{U}$: indeed, for each integer m, there are embeddings f of the unit circle $\mathbf{S}^1$ into the plane $\mathbf{R}^2$ whose image has no degenerate flat points (and therefore, as we shall see, has finite 1-degree) and contains the part of the graph $y = \sin x$ obtained for $0 \le x \le m\pi$. Hence it meets the x-axis transversally at $m+1$ points, an open condition in $C^\infty(\mathbf{S}^1,\mathbf{R}^2)$.

The following key idea is again in Thom's article [21]: for each continuous map f of a topological space V into $\mathbf{R}^{n+k}$ and each $a \in V$, we define the *local k-degree of f at a* to be

$$\deg_{k,a}(f) := \inf_U \deg_k f(U),$$

where the infimum is taken over all open neighbourhoods U of a in V. The *local k-degree* of a subset A of $\mathbf{R}^{n+k}$ at $a \in A$ is the local k-degree at a of the inclusion map $A \hookrightarrow \mathbf{R}^{n+k}$.

LEMMA 1.3. *For each continuous map f of a* compact *space V into $\mathbf{R}^{n+k}$, the k-degree of $f(V)$ is finite if and only if the local k-degree of f at every point is. Therefore, the k-degree of a compact subset A of $\mathbf{R}^{n+k}$ is finite if and only if its local k-degree at every point is.*

Proof. If the local k-degree of f at every point is finite, then V admits a finite covering $\mathcal{F}$ by open subsets U such that $\deg_k f(U)$ is finite. For each affine k-plane $H \subset \mathbf{R}^{n+k}$, as every point of $f(V) \cap H$ belongs to some $f(U)$ with $U \in \mathcal{F}$, we have $\#(f(V) \cap H) \leq \sum_{U \in \mathcal{F}} \#(f(U) \cap H) \leq \sum_{U \in \mathcal{F}} \deg_k f(U)$, hence $\deg_k f(V) \leq \sum_{U \in \mathcal{F}} \deg_k f(U) < \infty$. The "only if" part is obvious.

Thus, Theorem 1.1 will follow if we can prove that there is a huge open subset $\mathcal{U}$ of $C^\infty(V, \mathbf{R}^{n+k})$ consisting of maps f whose local k-degree at every point a is finite.

Denoting by $\mathbf{G}(k, n+k)$ the *Grassmann manifold* of all k-planes H through the origin in $\mathbf{R}^{n+k}$ and by $p_H \colon \mathbf{R}^{n+k} \to \mathbf{R}^{n+k}/H$ the canonical projection, we clearly have

$$\deg_{k,a}(f) \leq \inf_U \sup_{H,b} \#\big((p_H \circ f)^{-1}(b) \cap U\big)\,, \tag{1.1}$$

where the infimum is taken over all open neighbourhoods U of a in V and the supremum over all $H \in \mathbf{G}(k, n+k)$ and $b \in \mathbf{R}^{n+k}/H$ (of course, equality holds when f is injective).

To make the problem amenable to transversality arguments in jet spaces, we now introduce a more algebraic bound for the right-hand side of (1.1).

2. MULTIPLICITIES

DEFINITION. Given two n-dimensional manifolds M, N, let $\mathcal{E}_a = \mathcal{E}_a(M)$ be the real algebra of all smooth germs $(M, a) \to \mathbf{R}$ and let $\mathcal{M}_a = \mathcal{M}_a(M) := \{f \in \mathcal{E}_a : f(a) = 0\}$ denote its maximal ideal. The *multiplicity* $\mu(F)$ of a smooth germ $F \colon (M, a) \to (N, b)$ is the codimension in $\mathcal{E}_a$, *as a real vector subspace*, of the ideal $\mathcal{E}_a F^* \mathcal{M}_b$ generated by the germs $F^* g := g \circ F$ with $g \in \mathcal{M}_b$. For every integer d, the *multiplicity* of the d-jet $\widehat{F}_d = j^d F(a)$ is (see below) the codimension $\mu(\widehat{F}_d)$ in $\mathcal{E}_a$ of the ideal $\mathcal{E}_a F^* \mathcal{M}_b + \mathcal{M}_a^{d+1}$ as a real vector subspace, hence

$$1 = \mu(\widehat{F}_0) \leq \cdots \leq \mu(\widehat{F}_d) \leq \mu(\widehat{F}_{d+1}) \leq \cdots \leq \mu(F)\,. \tag{2.1}$$

For each chart germ $\varphi\colon (M,a) \to (\mathbf{R}^n,0)$, the map $\varphi_*\colon f \mapsto f\circ\varphi^{-1}$ of $\mathcal{E}_a$ onto $\mathcal{E} := \mathcal{E}_0(\mathbf{R}^n)$ is an isomorphism of algebras and therefore sends the maximal ideal $\mathcal{M}_a$ onto $\mathcal{M} := \mathcal{M}_0(\mathbf{R}^n)$. Given chart germs $\varphi\colon (M,a) \to (\mathbf{R}^n,0)$ and $\psi\colon (N,b) \to (\mathbf{R}^n,0)$, it follows that, for each smooth germ $F\colon (M,a) \to (N,b)$, the smooth germ $G = \psi\circ F\circ\varphi^{-1}$ satisfies $\mu(G) = \mu(F)$ and $\mu(\widehat{G}_d) = \mu(\widehat{F}_d)$ for every d. Now,

- since the mean value formula implies that $\mathcal{M}$ is generated by the germs of the coordinate maps, the ideal $\mathcal{E}G^*\mathcal{M}$ is generated by the components of G;
- for integer d, as Taylor's formula implies that the ideal $\mathcal{M}^{d+1}$, generated by the monomials of degree $d+1$, is the set of all germs $f \in \mathcal{E}$ with $j^d f(0) = 0$, the d-th order jet $j^d f(0)$ of each $f \in \mathcal{E}$ can be identified to the image of f in $\mathcal{E}/\mathcal{M}^{d+1}$.

It follows that $\mu(\widehat{G}_d)$ is determined by $\widehat{G}_d = j^d G(0)$ and, therefore, that $\mu(\widehat{F}_d)$ is determined by $\widehat{F}_d = j^d F(a)$.

The *multiplicity* $\mu_a(F)$ *of* $F \in C^\infty(M,N)$ *at* $a \in M$ is the multiplicity of the germ of F at a.

The following result will play an essential role in our arguments (as its proof is short, we give it even though it can be found, e.g., in [5]):

PROPOSITION 2.1. *For each positive integer* m *and every smooth germ* $F\colon (\mathbf{R}^n,0) \to (\mathbf{R}^n,0)$, *the inequality* $\mu(\widehat{F}_m) \le m$ *implies that* $\mu(F) = \mu(\widehat{F}_{d-1})$ *for some integer* d *with* $1 \le d \le m$. *Thus, by* (2.1), *the inequalities* $\mu(\widehat{F}_m) \le m$ *and* $\mu(F) \le m$ *are equivalent.*

Proof. For $\mu(\widehat{F}_m) \le m$, we have $1 = \mu(\widehat{F}_0) \le \cdots \le \mu(\widehat{F}_m) \le m$ by (2.1) and therefore, for some $d \in \{1,\dots,m\}$, $\mu(\widehat{F}_d) = \mu(\widehat{F}_{d-1})$ or, in other words, $\mathcal{E}F^*\mathcal{M} + \mathcal{M}^{d+1} = \mathcal{E}F^*\mathcal{M} + \mathcal{M}^d$, that is

$$\mathcal{M}^d \subset \mathcal{E}F^*\mathcal{M} + \mathcal{M}^{d+1}. \tag{2.2}$$

We claim that this implies

$$\mathcal{M}^d \subset \mathcal{E}F^*\mathcal{M}, \tag{2.3}$$

hence $\mathcal{E}F^*\mathcal{M} = \mathcal{E}F^*\mathcal{M} + \mathcal{M}^d$ and $\mu(F) = \mu(\widehat{F}_{d-1})$.

Indeed, denoting by $x_1,\dots,x_n \in \mathcal{M}$ the germs of the coordinate maps, (2.2) implies that every monomial x^α with $|\alpha| = d$ can be written $x^\alpha = g_\alpha + R_\alpha$ with $g_\alpha \in \mathcal{E}F^*\mathcal{M}$ and $R_\alpha \in \mathcal{M}^{d+1}$. Therefore, the g_α's belong to $\mathcal{M}^d$ and, by Nakayama's lemma, they generate it over $\mathcal{E}$, as the monomials x^α with $|\alpha| = d$ do and have the same projections into $\mathcal{M}^d/\mathcal{M}^{d+1}$, yielding (2.3).

Recall that a subset S of a manifold is *stratified* when it is the disjoint union of finitely many submanifolds called *strata*, the smallest codimension of which is the *codimension* of S. For example, every algebraic subset of a finite-dimensional real vector space admits a canonical stratification [25, 5, 4]. Apart from Thom's transversality lemma, the key result used in the sequel is the following variant of a theorem of Tougeron:

THEOREM (Tougeron). *For each positive integer n and every positive integer m, denoting by $J^m(n,n)$ the $2n$-codimensional vector subspace of $J^m(\mathbf{R}^n,\mathbf{R}^n)$ consisting of those $\widehat{F}_m = j^m F(0)$ such that $F(0) = 0$, the set $\Sigma^m(n)$ of all $j^m F(0) \in J^m(n,n)$ with $\mu(F) > m$ is a non-empty algebraic subset, whose codimension*

$$c_n(m) := \dim J^m(n,n) - \dim \Sigma^m(n)$$

tends to infinity when $m \to \infty$.

Proof. Proposition 2.1 implies that $\Sigma^m(n)$ is the set of those $\widehat{F}_m \in J^m(n,n)$ which satisfy $\mu(\widehat{F}_m) > m$. It contains 0 since $\operatorname{codim} \mathcal{M}^{m+1} \geq m+1$. The reason why it is algebraic is that it is the set of those $\widehat{F} = \widehat{F}_m = j^m F(0)$ in $J^m(n,n)$ such that, denoting by $F_1, \dots, F_n$ the components of F, the linear map $\widehat{F}_\flat \colon \big(j^m a_1(0), \dots, j^m a_n(0)\big) \mapsto j^m(a_1 F_1 + \cdots + a_n F_n)(0)$ of $(\mathcal{E}/\mathcal{M}^{m+1})^n$ into $\mathcal{E}/\mathcal{M}^{m+1}$ has corank greater than m. Thus, $\Sigma^m(n)$ is the inverse image under the linear map $\widehat{F} \mapsto \widehat{F}_\flat$ of the algebraic set of all linear maps $(\mathcal{E}/\mathcal{M}^{m+1})^n \to \mathcal{E}/\mathcal{M}^{m+1}$ with corank greater than m.

To see that $c_n(m) \to \infty$ when $m \to \infty$, first notice that, denoting by $\pi_\ell^m \colon J^\ell(n,n) \to J^m(n,n)$ the canonical projection $j^\ell F(0) \mapsto j^m F(0)$, one has

$$\Sigma^\ell(n) \subset (\pi_\ell^m)^{-1}\big(\Sigma^m(n)\big) \tag{2.4}$$

for all positive integers m, ℓ with $\ell \geq m$: indeed, if the smooth map germ $F \colon (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$ satisfies $\widehat{F}_m \notin \Sigma^m(n)$, i.e. $\mu(\widehat{F}_m) \leq m$, Proposition 2.1 yields $\mu(F) = \mu(\widehat{F}_m)$, hence, by (2.1), $\mu(\widehat{F}_\ell) = \mu(\widehat{F}_m) \leq m \leq \ell$, i.e. $\widehat{F}_\ell \notin \Sigma^\ell(n)$.

As π_ℓ^m is a submersion, the codimension of the right-hand side of (2.4) is $c_n(m)$. Therefore, (2.4) implies that $c_n(m)$ is a non-decreasing function of m and all we have to prove is the following

LEMMA. *For each positive integer m, there exists an integer $\ell > m$ with $c_n(\ell) > c_n(m)$.*

Given m, let $H\colon (\mathbf{R}^n,0) \to (\mathbf{R}^n,0)$ be the smooth germ given by $H(x) := (x_1^{m+1},\dots,x_n^{m+1})$. As every x^α with $|\alpha| = nm+1$ must satisfy $\alpha_j \geq m+1$ for some j, we have

$$\mathcal{M}^{nm+1} \subset \mathcal{E}H^*\mathcal{M} \subset \mathcal{M}^{m+1}, \tag{2.5}$$

hence $m < \mu(H) < \infty$. We claim that the lemma holds for $\ell = \mu(H)$.

If such were not the case, the highest dimensional stratum S^ℓ of $\Sigma^\ell(n)$ and the highest dimensional stratum S^m of $\Sigma^m(n)$ would have the same codimension $c_n(m)$, and so would $(\pi_\ell^m)^{-1}(S^m)$. Therefore it would follow from (2.4) that $S^\ell \cap (\pi_\ell^m)^{-1}(S^m)$ is a non-empty open subset of $(\pi_\ell^m)^{-1}(S^m)$.

Now, given a smooth map germ $F\colon (\mathbf{R}^n,0) \to (\mathbf{R}^n,0)$ satisfying $\widehat{F}_\ell \in S^\ell \cap (\pi_\ell^m)^{-1}(S^m)$, the second inclusion in (2.5) yields $\widehat{F}_m + s\widehat{H}_m = \widehat{F}_m$ and therefore $\widehat{F}_\ell + s\widehat{H}_\ell \in (\pi_\ell^m)^{-1}(S^m)$ for every $s \in \mathbf{R}$; if $S^\ell \cap (\pi_\ell^m)^{-1}(S^m)$ were open in $(\pi_\ell^m)^{-1}(S^m)$, it would contain $\widehat{F}_\ell + s\widehat{H}_\ell$ for all small enough s, whereas we shall now see that there are only finitely many real numbers s with $\widehat{F}_\ell + s\widehat{H}_\ell \in \Sigma^\ell(n)$.

It is enough to show that the set T of those $t \in \mathbf{R}$ which satisfy $(1-t)\widehat{F}_\ell + t\widehat{H}_\ell \in \Sigma^\ell(n)$ is finite, as $\widehat{F}_\ell + \frac{t}{1-t}\widehat{H}_\ell$ and $(1-t)\widehat{F}_\ell + t\widehat{H}_\ell$ have the same multiplicity for $t \neq 1$. Now, T is algebraic, being the inverse image of $\Sigma^\ell(n)$ under the affine map $t \mapsto (1-t)\widehat{F}_\ell + t\widehat{H}_\ell$, and it does not contain 1 since we have $\mu(\widehat{H}_\ell) \leq \mu(H) = \ell$ by (2.1); hence, it is indeed finite. □

REMARKS. Stratifications make the proof shorter than in [5], where the theorem is stated a little differently though all the ingredients are present.

If $n = 1$, both Proposition 2.1 and Tougeron's theorem are evident since $\mu(F)$ is the supremum of the integers d such that $j^{d-1}F(0) = 0$, implying that $\Sigma^m(n) = \{0\}$ and $c_n(m) = m$.

The codimension $c_n(m)$ does not seem as easy to compute in general because of moduli: the multiplicity of a jet is the codimension of its orbit for contact equivalence (*Mather's $\mathcal{K}$-equivalence*, called *V-equivalence* in [12]) but, for $n > 1$, there may exist continuous families of such orbits with the same codimension, making $c_n(m)$ smaller — less than m, to begin with.

We refer to [12], [8] or [5] for a proof of the following consequence of the Malgrange preparation theorem:

PROPOSITION 2.2. *If a smooth map F between manifolds M, N of the same positive dimension has multiplicity μ at $a \in M$, then its local degree at a is at most μ: there exists an open neighbourhood U_1 of a such that, for every $y \in N$, the subset $F^{-1}(y) \cap U_1$ contains at most μ points.*

COROLLARY 2.3. *Let $F_\bullet\colon (u,x) \to F_u(x)$ be a smooth map of $\Lambda \times V$ into B, where Λ, V, B are three manifolds with $\dim V = \dim B > 0$. If F_{u_0} has multiplicity μ at a, there exist open neighbourhoods Y of a in V and U of u_0 in Λ such that, for every $y \in B$ and every $u \in U$, the subset $F_u^{-1}(y) \cap Y$ contains at most μ points.*

Proof. Setting $\widetilde{F}(u,x) := \big(u, F_u(x)\big)$, one has $\mu_{(u_0,a)}(\widetilde{F}) = \mu_a(F_{u_0}) = \mu$: indeed, one may assume that $(\Lambda, u_0) = (\mathbf{R}^c, 0)$ and $(V,a) = (B, F_{u_0}(a)) = (\mathbf{R}^n, 0)$ since the problem is local. Setting $\widetilde{\mathcal{E}} := \mathcal{E}_0(\mathbf{R}^{c+n})$ and $\widetilde{\mathcal{M}} := \mathcal{M}_0(\mathbf{R}^{c+n})$, the ideal $\widetilde{\mathcal{E}}\widetilde{F}^*\widetilde{\mathcal{M}}$ of $\widetilde{\mathcal{E}}$ contains the ideal $\mathcal{I}$ generated by $u_1, \dots, u_c$; hence, its codimension is the dimension of $(\widetilde{\mathcal{E}}/\mathcal{I})/(\widetilde{\mathcal{E}}\widetilde{F}^*\widetilde{\mathcal{M}}/\mathcal{I})$, which is isomorphic to $\mathcal{E}/\mathcal{E}F_0^*\mathcal{M}$ as $\varphi_u(x) = \varphi_0(x) + u_1 \int_0^1 \partial_1\varphi_\bullet(tu,x)\,dt + \cdots + u_c \int_0^1 \partial_c\varphi_\bullet(tu,x)\,dt$ for all $\varphi_\bullet \in \widetilde{\mathcal{E}}$, for example the components of $F_\bullet$.

Let us go back to the proof of Theorem 1.1. With the notation of (1.1), Corollary 2.3 yields

LEMMA 2.4. *Given positive integers n, k and an n-dimensional manifold V, the following inequality holds for all $f \in C^\infty(V, \mathbf{R}^{n+k})$ and $a \in V$:*

$$\deg_{k,a}(f) \le \sup_{H \in \mathbf{G}(k,n+k)} \mu_a(p_H \circ f) =: \mu_{k,a}(f)\,.$$

Proof. Recall that the *Stiefel manifold* $\mathbf{St}(n, n+k)$ is the set of those $u = (u_1, \dots, u_n) \in (\mathbf{R}^{n+k})^n$ such that $u_i \cdot u_j = 1$ if $i = j$ and $u_i \cdot u_j = 0$ for $i \neq j$, where the dot stands for the standard scalar product of $\mathbf{R}^{n+k}$. For $(u,y) \in \mathbf{St}(n,n+k) \times \mathbf{R}^{n+k}$, we let $u \cdot y := (u_1 \cdot y, \dots, u_n \cdot y) \in \mathbf{R}^n$.

Given a positive integer m, we should prove that $f \in C^\infty(V, \mathbf{R}^{n+k})$ satisfies $\deg_{k,a}(f) \le m$ if we have $\mu_a(p_H \circ f) \le m$ for all $H \in \mathbf{G}(k, n+k)$ or, equivalently (taking an orthogonal basis $u \in \mathbf{St}(n, n+k)$ of $H^\perp$), $\mu_a(u \cdot f) \le m$ for all $u \in \mathbf{St}(n, n+k)$.

Then, for $u_0 \in \mathbf{St}(n, n+k)$, the hypotheses of Corollary 2.3 are satisfied with $\Lambda := \mathbf{St}(n, n+k)$, $B := \mathbf{R}^n$ and $F_u(x) := u \cdot f(x)$, $\mu := \mu_a(u_0 \cdot f) \le m$. It follows that there exist open neighbourhoods $Y_{u_0} \subset V$ and $U_{u_0} \subset \mathbf{St}(n, n+k)$ of a and u_0 respectively such that, for $(y,u) \in \mathbf{R}^n \times U_{u_0}$, the equation $u \cdot f(x) = y$ has at most m solutions $x \in Y_{u_0}$. In other words, for every $u \in U_{u_0}$, the subset $f(Y_{u_0})$ meets every affine k-plane orthogonal to the linear span of u in at most m points. Now, as $\mathbf{St}(n, n+k)$ is compact, we can choose values $u_1, \dots, u_p$ of u_0 so that $\{U_{u_1}, \dots, U_{u_p}\}$ is a covering of $\mathbf{St}(n, n+k)$. Setting $Y := Y_{u_1} \cap \cdots \cap Y_{u_p}$, we do obtain $\deg_{k,a}(f) \le \deg_k f(Y) \le m$. $\square$

3. Local version and proof of Theorem 1.1

For all integers $n \geq 1$ and $p \geq 0$, with the notation of Tougeron's theorem, we let

$$m_n(p) := \min\{m : c_n(m) > p\}\,,$$

which equals $p+1$ if $n=1$ since $c_1(m) = m$. The following local version of Theorem 1.1 provides *universal* bounds for the local degree:

THEOREM 3.1. *For all positive integers n, k and each n-dimensional manifold V, there exists an increasing sequence $(\mathcal{U}_c)_{c\in\mathrm{N}}$ of dense open subsets of $C^\infty(V, \mathbf{R}^{n+k})$ such that each $\mathcal{U}_c$ is c-large and consists of maps f whose local k-degree at every point is at most $m_n(kn+n+c)$ (in particular, if $n=1$, the local k-degree of f at every point is at most $k+2+c$). Thus, the open set $\mathcal{U} := \bigcup_c \mathcal{U}_c$ is huge and every element of $\mathcal{U}$ has finite local k-degree at every point of V.*

This implies Theorem 1.1. For compact V, by Lemma 1.3, every $f \in \mathcal{U}$ has finite k-degree.

The proof of Theorem 3.1 uses the following consequence of Tougeron's theorem:

LEMMA 3.2. *Let n, k be two positive integers. For every n-dimensional manifold V and every positive integer m, the set $\Sigma^m(V, \mathbf{R}^{n+k})$ of all $j^m f(a) \in J^m(V, \mathbf{R}^{n+k})$ with $\mu_{k,a}(f) > m$ is a closed stratified set whose codimension, being at least $c_n(m) - nk$, tends to infinity when $m \to \infty$.*

Postponing the proof of this lemma until Section 4, let us first deduce Theorem 3.1 from

THOM'S TRANSVERSALITY LEMMA IN JET SPACES (EASY CASE). *Given manifolds M, N, an integer m and a closed stratified subset Σ of $J^m(M,N)$ whose codimension is greater than the dimension of M, the set of those $f \in C^\infty(M,N)$ such that $j^m f(M) \cap \Sigma = \varnothing$ is open and dense*[2]).

As the condition $j^m f(M) \cap \Sigma = \varnothing$ reads $j^m f(M) \subset J^m(M,N) \setminus \Sigma$, openness follows from the definition of the Whitney topology.

[2]) A particular case of Lemma 7.4 hereafter, proved in Section 8.

Proof of Theorem 3.1. For each $c \in \mathbf{N}$, if $m = m_n(kn+n+c)$, we have $\operatorname{codim}\Sigma^m(V,\mathbf{R}^{n+k}) > n+c$ by Lemma 3.2 and the definition of m. Hence, for each c-dimensional manifold Λ, the set of those $f_\bullet\colon (\lambda,x) \mapsto f_\lambda(x)$ in $C^\infty(\Lambda \times V, \mathbf{R}^{n+k})$ which satisfy $j^m f_\lambda(a) \notin \Sigma^m(V,\mathbf{R}^{n+k})$ (i.e. $\mu_{k,a}(f_\lambda) \leq m$) for all (λ,a) is open and dense: this follows from the transversality lemma with $M = \Lambda \times V$ and $N = \mathbf{R}^{n+k}$, taking for Σ the set of all $j^m f_\bullet(\lambda,x)$ with $j^m f_\lambda(x) \in \Sigma^m(V,\mathbf{R}^{n+k})$ (which is closed, stratified and of codimension greater than $n+c$ since the map $j^m f_\bullet(\lambda,x) \mapsto j^m f_\lambda(x)$ is a smooth submersion).

It follows that, for each integer c, we can define $\mathcal{U}_c$ to be the set of those $f \in C^\infty(V,\mathbf{R}^{n+k})$ such that $j^m f(V) \cap \Sigma^m(V,\mathbf{R}^{n+k}) = \varnothing$ with $m = m_n(kn+n+c)$, hence $\deg_{k,a}(f) \leq \mu_{k,a}(f) \leq m_n(kn+n+c)$ for all $a \in V$ by Lemma 2.4.

REMARKS. The more general case of Thom's transversality lemma in jet spaces ([20, 13, 8, 12] and Section 8 hereafter) implies that, generically, $j^m f$ is transversal to $\Sigma^m(V,\mathbf{R}^{n+k})$ even for $n \geq c_n(m) - kn$. Thus, the closed set $(j^m f)^{-1}\big(\Sigma^m(V,\mathbf{R}^{n+k})\big)$ of those $a \in V$ which satisfy $\mu_{k,a}(f) > m$, admits the stratification whose strata are the inverse images under $j^m f$ of the strata of $\Sigma^m(V,\mathbf{R}^{n+k})$. When n and k are not too large, such stratifications can be defined explicitly in geometric terms (see the remarks at the end of Section 5).

If $n = 1$, we may [15] assume $V = \mathbf{R}$ or $\mathbf{R}/\mathbf{Z}$, and the condition $\mu_{k,a}(f) > m$ means that, for some $u \in \mathbf{S}^k$, the function $t \mapsto u \cdot f(t)$ (scalar product) has multiplicity greater than m at a, i.e. $\big(u \cdot \frac{d^j}{dt^j} f(a)\big)_{1 \leq j \leq m} = 0$.

EXAMPLE. If $n = k = 1$ and $f \in \mathcal{U}_0$, we have $\mu_{k,a}(f) \leq 3$ for all a. When the parametrised plane curve f is an immersion, which is generically the case, this does mean that it has no degenerate flat points. However, as we wish the subsets $\mathcal{U}_c$ to be as large as possible, some maps f in our set $\mathcal{U}_0$ are not immersions, e.g. $f(t) = (t^2, t^3)$.

4. PROOF OF LEMMA 3.2

Given open subsets $U \subset \mathbf{R}^n$ and $V \subset \mathbf{R}^p$, each $j^m f(x) \in J^m(U,V)$ can be written $j^m f(x) = \big(x, f(x), (D^j f(x))_{1 \leq j \leq m}\big)$; thus, denoting by $J^m(n,p)$ the space of polynomial maps $P\colon (\mathbf{R}^n,0) \to (\mathbf{R}^p,0)$ of degree at most m (which may be identified to $j^m P(0)$ as in Tougeron's theorem), setting $X^\ell := \underbrace{(X,\ldots,X)}_{\ell \text{ times}}$ and identifying $(D^j f(x))_{1 \leq j \leq m}$ to the element $X \mapsto \frac{1}{1!} D^1 f(x) X^1 + \cdots + \frac{1}{m!} D^m f(x) X^m$ of $J^m(n,p)$, we can see that $J^m(U,V) = U \times V \times J^m(n,p)$.

By definition, $\Sigma^m(V,\mathbf{R}^{n+k})$ is the set of those $j^mf(a) \in J^m(V,\mathbf{R}^{n+k})$ which satisfy $\mu_a(p_H \circ f) > m$ for some H in $\mathbf{G}(k,n+k)$; thus (taking an orthonormal basis $u \in \mathbf{St}(n,n+k)$ of the orthogonal $H^\perp$ of H), it is the projection of the set $\widetilde{\Sigma}^m(V,\mathbf{R}^{n+k})$ of all $\big(u,j^mf(a)\big) \in \mathbf{St}(n,n+k) \times J^m(V,\mathbf{R}^{n+k})$ with $\mu_a(u\cdot f) > m$.

Hence, for each chart φ of V, the image of $\Sigma^m(V,\mathbf{R}^{n+k}) \cap J^m(\operatorname{dom}\varphi,\mathbf{R}^{n+k})$ under the chart $\Phi^m_{\varphi,id}\colon j^mf(a) \mapsto j^m(f\circ\varphi^{-1})\big(\varphi(a)\big)$ of $J^m(V,\mathbf{R}^{n+k})$ is the projection into $J^m(\operatorname{Im}\varphi,\mathbf{R}^{n+k})$ of the set $\widetilde{\Sigma}^m_k(\operatorname{Im}\varphi)$ of those $\big(u,j^mf(a)\big)$ in $\mathbf{St}(n,n+k)\times J^m(\operatorname{Im}\varphi,\mathbf{R}^{n+k})$ such that the point $j^m(u\cdot f)(a)$ of $J^m(\operatorname{Im}\varphi,\mathbf{R}^n) = \operatorname{Im}\varphi \times \mathbf{R}^n \times J^m(n,n)$ lies in $\operatorname{Im}\varphi \times \mathbf{R}^n \times \Sigma^m(n)$.

Now, the map $s_\varphi\colon \big(u,j^mf(a)\big) \mapsto j^m(u\cdot f)(a)$ is a submersion of the space $\mathbf{St}(n,n+k) \times J^m(\operatorname{Im}\varphi,\mathbf{R}^{n+k})$ onto $J^m(\operatorname{Im}\varphi,\mathbf{R}^n)$ whose restriction to each fibre of the projection $\mathbf{St}(n,n+k)\times J^m(\operatorname{Im}\varphi,\mathbf{R}^{n+k}) \ni \big(u,j^mf(a)\big) \mapsto \big(a,f(a)\big)$ onto $J^0(\operatorname{Im}\varphi,\mathbf{R}^{n+k})$ is the polynomial submersion $(u,P) \overset{s}{\longmapsto} u\cdot P$ of $\mathbf{St}(n,n+k) \times J^m(n,n+k)$ onto $J^m(n,n)$. Thus, $\widetilde{\Sigma}^m_k(\operatorname{Im}\varphi)$ is the product of $J^0(\operatorname{Im}\varphi,\mathbf{R}^{n+k})$ by the algebraic subset $s^{-1}\big(\Sigma^m(n)\big)$ of $(\mathbf{R}^{n+k})^n \times J^m(n,n+k)$. It follows that the trivialisations $\big(u,j^mf(a)\big) \mapsto \big(u,\Phi^m_{\varphi,id}\big(u,j^mf(a)\big)\big)$ of the fibre bundle $\mathbf{St}(n,n+k)\times J^m(V,\mathbf{R}^{n+k}) \to J^0(V,\mathbf{R}^{n+k})$ make $\widetilde{\Sigma}^m(V,\mathbf{R}^{n+k})$ into a locally trivial bundle with algebraic fibre $s^{-1}\big(\Sigma^m(n)\big)$ and the vector bundle charts $\Phi^m_{\varphi,id}$ of $J^m(V,\mathbf{R}^{n+k}) \to J^0(V,\mathbf{R}^{n+k})$ make $\Sigma^m(V,\mathbf{R}^{n+k})$ into a locally trivial bundle whose fibre is the image of $s^{-1}\big(\Sigma^m(n)\big)$ under the projection of $(\mathbf{R}^{n+k})^n \times J^m(n,n+k)$ onto $J^m(n,n+k)$, which image is semi-algebraic by the Tarski-Seidenberg theorem [4].

Therefore, the following result, which belongs to the toolkit of singularity theory [25, 14, 7], provides $\Sigma^m(V,\mathbf{R}^{n+k})$ with a stratification by smooth bundles over $J^0(V,\mathbf{R}^{n+k})$:

PROPOSITION 4.1. *Let $P \overset{\pi}{\longrightarrow} M$ be a smooth fibration whose fibre F is (smooth) semi-algebraic in a finite-dimensional vector space E, and let Σ be a subset of P with the following property: there exist a nonempty semi-algebraic subset $B \subset F$ of E, a covering $\mathcal{U}$ of M by open subsets and, for each $U \in \mathcal{U}$, a smooth trivialisation Φ_U of π over U, sending $\Sigma \cap \pi^{-1}(U)$ onto $U \times B$. Then, Σ admits a stratification by smooth submanifolds, each of which is a smooth subbundle of P.*

Proof. Call a point $a \in B$ *regular* when it has an open neighbourhood U in E such that $B \cap U$ is a C^∞ submanifold. Thus, the set of all regular points of B is an open subset of B and a C^∞ submanifold, and so is the union $\operatorname{Reg}(B)$ of its connected components of maximal dimension.

It can be proved [25, 11, 14, 6] that the submanifold $\mathrm{Reg}(B)$ is nonempty, analytic and that, denoting its dimension by $\dim B$, its complement $\mathrm{Sing}\, B := B \setminus \mathrm{Reg}(B)$ is semi-algebraic and satisfies[3]) $\dim \mathrm{Sing}\, B < \dim B$. Hence, B is the disjoint union of finitely many nonempty analytic submanifolds, namely the nonempty terms of the sequence $\mathrm{Reg}(B), \mathrm{Reg}(\mathrm{Sing}\, B), \mathrm{Reg}(\mathrm{Sing}\,\mathrm{Sing}\, B), \ldots$ This stratification[4]) is *canonical*, meaning that it is invariant by C^∞ diffeomorphisms preserving B.

For $U, U_1 \in \mathcal{U}$, the smooth diffeomorphism $\Phi_{U_1} \circ \Phi_U^{-1}$ of $(U \cap U_1) \times F$ onto itself is of the form $(x, y) \mapsto \big(x, h_x(y)\big)$. Each h_x extends to a smooth diffeomorphism between open subsets of E containing F, implying that h_x preserves each stratum of B since it preserves B. Thus, for each stratum S of B, the formulae $\Phi_U\big(\widetilde{S} \cap \pi^{-1}(U)\big) := U \times S$ define a smooth subbundle $\widetilde{S}$ of P, and the submanifolds $\widetilde{S}$ obviously form a stratification of Σ.

The codimension of $\Sigma^m(V, \mathbf{R}^{n+k})$ equals at least the codimension $c_n(m)$ of $s^{-1}\big(\Sigma^m(n)\big)$, *minus* the dimension of $\mathbf{St}(n, n+k)$. This estimate can be improved by recalling that the multiplicity of $s(u, P) = u \cdot P$ depends only on the linear span of $u_1, \ldots, u_n$, implying that the fibre of $\Sigma^m(V, \mathbf{R}^{n+k})$ is the projection of an algebraic subset of $\mathbf{G}(n, n+k) \times J^m(n, n+k)$ with codimension $c_n(m)$, hence $\mathrm{codim}\, \Sigma^m(V, \mathbf{R}^{n+k}) \geq c_n(m) - nk$.

Finally, $\Sigma^m(V, \mathbf{R}^{n+k})$ is closed since its fibre is the image of the closed subset $s^{-1}\big(\Sigma^m(n)\big)$ under the projection of $\mathbf{St}(n, n+k) \times J^m(n, n+k)$ onto $J^m(n, n+k)$. This projection is proper since $\mathbf{St}(n, n+k)$ is compact.

REMARK. The Tarski-Seidenberg theorem is necessary only if the "black box" in the proof of Theorem 1.1 is Thom's transversality lemma in jet spaces. The easy case (see Lemma 8.1 hereafter) of Thom's elementary transversality lemma [19] could have been used instead to prove that, for almost every $f_\bullet \in C^\infty(\Lambda \times V, \mathbf{R}^{n+k})$, the map $(u, \lambda, x) \mapsto j^m(u \cdot f_\lambda)(x)$ of $\mathbf{St}(n, n+k) \times \Lambda \times V$ into $J^m(V, \mathbf{R}^n)$ takes its values off $\{j^m F(a) \in J^m(V, \mathbf{R}^n) : \mu_a(F) > m\}$, which is an algebraic subbundle with fibre $\Sigma^m(n)$ of the bundle $J^m(V, \mathbf{R}^n) \to J^0(V, \mathbf{R}^n)$. Theorem 7.1 hereafter has to be proved in this fashion.

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[3]) With the usual convention that $\dim \varnothing = d$ for *every* $d \in \mathbf{Z}$.

[4]) With little effort [14], it can be refined into another canonical stratification satisfying Whitney's conditions (A) and (B), but this is not needed in the present paper.

II. Related results and notions

5. A dual version of Thom's theorem

Here are analogues of Theorems 1.1 and 3.1 for inverse images:

THEOREM 5.1. *Given a k-dimensional manifold B and a point of B, which we name 0, let $\mathcal{P}$ and $\mathcal{P}_0$ be the open subsets of $C^\infty(\mathbf{R}^{n+k}, B)$ consisting respectively of all proper maps and of those maps g for which $g^{-1}(0)$ is compact. Then:*

(i) *There is a huge open subset $\mathcal{W}_0$ of $\mathcal{P}_0$ such that, for $g \in \mathcal{W}_0$, the k-degree of $g^{-1}(0)$ is finite.*

(ii) *Similarly, there exists a huge open subset $\mathcal{W}$ of $\mathcal{P}$ such that, for $g \in \mathcal{W}$, the k-degree of $g^{-1}(b)$ is finite for all $b \in B$.*

THEOREM 5.2. *For all positive integers n, k, every k-dimensional manifold B and every point of B, which we name 0, there exist two increasing sequences $(\mathcal{V}_{0,c})_{c\in\mathrm{N}}$ and $(\mathcal{V}_c)_{c\in\mathrm{N}}$ of dense open subsets of $C^\infty(\mathbf{R}^{n+k}, B)$ such that:*

(i) *Each $\mathcal{V}_{0,c}$ is c-large and consists of maps g such that the local k-degree of $g^{-1}(0)$ at every point is at most $m_k(kn+n+c)$. In particular, if $k=1$, the local k-degree of $g^{-1}(0)$ at every point is at most $2n+1+c$.*

(ii) *Each $\mathcal{V}_c$ is c-large and consists of maps g such that, for every $b \in B$, the local k-degree of $g^{-1}(b)$ at each of its points is at most $m_k(kn+k+n+c)$. In particular, if $k=1$, the local k-degree of $g^{-1}(b)$ at every point is at most $2n+2+c$.*

Thus, the open subsets $\mathcal{V} := \bigcup_c \mathcal{V}_c$ and $\mathcal{V}_0 := \bigcup_c \mathcal{V}_{0,c}$ are huge and, for $g \in \mathcal{V}_0$ (resp. $g \in \mathcal{V}$), the subset $g^{-1}(0)$ (resp. every $g^{-1}(b)$) has finite local k-degree at every point.

REMARK. In these two statements and their analogues, the point $0 \in B$ could be replaced by a compact submanifold of codimension k in a higher-dimensional manifold, at the expense of a few additional technicalities.

Theorem 5.1 *follows from Theorem* 5.2. Indeed, for $g \in \mathcal{W}_0 := \mathcal{P}_0 \cap \mathcal{V}_0$ (resp. $g \in \mathcal{W} := \mathcal{P} \cap \mathcal{V}$), Lemma 1.3 and (i) (resp. (ii)) do imply that $g^{-1}(0)$ (resp. every $g^{-1}(b)$) has finite k-degree.

Proof of Theorem 5.2. The following analogue of Lemma 2.4 is a particular case of Lemma 7.2:

LEMMA 5.3. *Given positive integers n, k and a k-dimensional manifold B, the inequality*

$$\deg_{k,a} g^{-1}(b) \leq \sup_{H \in \mathbf{G}(k,n+k)} \mu_a(g|_{a+H}) =: \mu_{k,a}(g^{-1})$$

holds for all $g \in C^\infty(\mathbf{R}^{n+k}, B)$, $b \in B$ and $a \in g^{-1}(b)$.

Now comes the analogue of Lemma 3.2:

LEMMA 5.4. *Let n, k be two positive integers. For every k-dimensional manifold B and every positive integer m, the set $\Sigma^m(\mathbf{R}^{n+k}, B)$ of all jets $j^m g(a) \in J^m(\mathbf{R}^{n+k}, B)$ with $\mu_{k,a}(g^{-1}) > m$ is a closed stratified subset whose codimension is at least $c_k(m) - nk$ and therefore tends to infinity when $m \to \infty$. Moreover, given a point $0 \in B$, the intersection $\Sigma^m(\mathbf{R}^{n+k}, B)_0$ of $\Sigma^m(\mathbf{R}^{n+k}, B)$ with $\{j^m g(a) : g(a) = 0\}$ is a stratified set of codimension at least $c_k(m) - nk + k$.*

Proof. The set $\Sigma^m(\mathbf{R}^{n+k}, B)$ consists of all $j^m g(a) \in J^m(\mathbf{R}^{n+k}, B)$ which satisfy $\mu_a(g|_{a+H}) > m$ for some H in $\mathbf{G}(k, n+k)$. In other words (taking an orthonormal basis u of H and setting $u^*(t_1, \ldots, t_k) := \sum t_j u_j$), we see that $\Sigma^m(\mathbf{R}^{n+k}, B)$ is the projection of the set $\widetilde{\Sigma}^m(\mathbf{R}^{n+k}, B)$ of those $\big(u, j^m g(a)\big) \in \mathbf{St}(k, n+k) \times J^m(\mathbf{R}^{n+k}, B)$ such that $\mu_0\big(g \circ (a + u_*)\big) > m$. Hence, for each chart ψ of B, the chart $\Phi^m_{id,\psi} : j^m g(a) \mapsto j^m(\psi \circ g)(a)$ of $J^m(\mathbf{R}^{n+k}, B)$ sends $\Sigma^m(\mathbf{R}^{n+k}, B) \cap J^m(\mathbf{R}^{n+k}, \operatorname{dom} \psi)$ onto the projection into $J^m(\mathbf{R}^{n+k}, \operatorname{Im} \psi)$ of the set $\widetilde{\Sigma}^m_n(\operatorname{Im} \psi)$ of those $\big(u, j^m g(a)\big)$ in $\mathbf{St}(k, n+k) \times J^m(\mathbf{R}^{n+k}, \operatorname{Im} \psi)$ such that the point $j^m\big(g \circ (a + u_*)\big)(0)$ of $J^m_0(\mathbf{R}^k, \operatorname{Im} \psi) = \operatorname{Im} \psi \times J^m(k, k)$ lies in $\operatorname{Im} \psi \times \Sigma^m(k)$.

Now, the map $s_\psi : \big(u, j^m g(a)\big) \mapsto j^m\big(g \circ (a + u_*)\big)(0)$ is a submersion of $\mathbf{St}(n, n+k) \times J^m(\mathbf{R}^{n+k}, \operatorname{Im} \psi)$ onto $\operatorname{Im} \psi \times J^m(k, k)$ whose restriction to each fibre of the projection of $\mathbf{St}(n, n+k) \times J^m(\mathbf{R}^{n+k}, \operatorname{Im} \psi)$ onto $J^0(\mathbf{R}^{n+k}, \operatorname{Im} \psi)$ is the polynomial submersion $(u, P) \overset{s_1}{\longmapsto} P \circ u_*$ of $\mathbf{St}(k, n+k) \times J^m(n+k, k)$ onto $J^m(k, k)$. Thus, $\widetilde{\Sigma}^m_n(\operatorname{Im} \psi)$ is the product of $J^0(\mathbf{R}^{n+k}, \operatorname{Im} \psi)$ by the algebraic fibre $s_1^{-1}\big(\Sigma^m(k)\big) \subset (\mathbf{R}^{n+k})^k \times J^m(n+k, k)$. As in the proof of Lemma 3.2, it follows that $\Sigma^m(\mathbf{R}^{n+k}, B)$ is a sub-bundle of $J^m(\mathbf{R}^{n+k}, B) \to J^0(\mathbf{R}^{n+k}, B)$ whose fibre is the semi-algebraic projection of $s_1^{-1}\big(\Sigma^m(k)\big)$ into $J^m(n+k, k)$. Applying Proposition 4.1, we do get a stratification of $\Sigma^m(\mathbf{R}^{n+k}, B)$.

For $P \in J^m(n+k,k)$, the multiplicity $\mu_0(P \circ u_*)$ depends only on the linear span of u. Hence, the codimension of $\Sigma^m(\mathbf{R}^{n+k},B)$ (which is closed for the same reason as in the proof of Lemma 3.2) is at most the codimension $c_k(m)$ of $s_1^{-1}\big(\Sigma^m(k)\big)$ *minus* the dimension nk of $\mathbf{G}(k,n+k)$.

Proof of the theorem. As in the proof of Theorem 3.1, it follows from Thom's transversality lemma, Lemma 5.4 and the definition of the Whitney topology that we can take for $\mathcal{V}_c$ the set of those $g \in C^\infty(\mathbf{R}^{n+k},B)$ such that, setting $m := m_k(kn+n+k+c)$, the map $j^m g$ takes its values in the complement of $\Sigma^m(\mathbf{R}^{n+k},B)$, and apply Lemma 5.3 to conclude. Similarly, we can take for $\mathcal{V}_{0,c}$ the set of those $g \in C^\infty(\mathbf{R}^{n+k},B)$ such that, setting $m := m_k(kn+n+c)$, the map $j^m g$ takes its values in the complement of $\Sigma^m(\mathbf{R}^{n+k},B)_0$.

REMARKS. The choice $\mathcal{W}_0 := \mathcal{P}_0 \cap \mathcal{V}$ would be good enough for Theorem 5.1. However, a sharp bound for the generic local k-degree of a given fibre of g is an interesting additional piece of information.

For example, if $k=1$, the bound $2n+1$ for the local degree is realised for $n>1$ by the polynomial map $g\colon \mathbf{R}^{n+1} \to \mathbf{R}$ given by

$$g(x_1,\ldots,x_{n-1},y,z) = y\big(y^2 - 2\sum x_j\big)\prod(y^2 - x_j) + \sum x_j^{2n} - z:$$

- it belongs to $\mathcal{V}_0$ because its (algebraic) degree is $2n+1$ and the hypersurface $S = g^{-1}(0)$ does not contain any line (on such a line, absurdly, y should be constant as well as every x_j);
- it satisfies $\deg_{1,0} g^{-1}(0) = 2n+1$ since S has $2n+1$ intersection points with lines $(x,z) =$ constant in every neighbourhood of the origin: indeed, if we fix Z and mutually distinct positive $X_1,\ldots,X_{n-1}$ and set $G_t(Y) = Y\big(Y^2 - 2\sum X_j\big)\prod(Y^2 - X_j) + t\big(\sum X_j^{2n} - Z\big)$, then G_0 has $2n+1$ simple real roots; hence, by the implicit function theorem, for t small enough, G_t has $2n+1$ real roots $Y_j(t)$ depending analytically on t; thus, for $\varepsilon > 0$ small enough, the polynomial $g(\varepsilon^2 X, y, \varepsilon^{4n} Z) = \varepsilon^{2n+1} G_{\varepsilon^{2n-1}}(\varepsilon^{-1} y)$ in y has the $2n+1$ real roots $\varepsilon Y_j(\varepsilon^{2n-1})$.

Even for $n+k \geq c_n(m) - kn$ or $c_n(m) - kn + k$, the map $j^m g$ is transversal to $\Sigma^m(\mathbf{R}^{n+k},B)$ or $\Sigma^m(\mathbf{R}^{n+k},B)_0$, yielding a stratification of the whole of $\mathbf{R}^{n+k}$ or $g^{-1}(0)$. For example, if $k=1$, we may assume $B = \mathbf{R}$ or $\mathbf{R}/\mathbf{Z}$ and the condition $\mu_{1,a}(g^{-1}) > m$ means exactly that, for some $u \in \mathbf{S}^n$, the function $t \mapsto g(a+tu)$ has multiplicity greater than m at 0, i.e. $D^j g(a) u^j = 0$ for $1 \leq j \leq m$. Thus, the conditions $g(a) = 0$ and $\mu_{1,a}(g^{-1}) > m$ mean that $\rho_g^m(a,u) := (D^j g(a) u^j)_{0 \leq j \leq m} = 0$ for some $u \in \mathbf{S}^n$, and Thom's elementary transversality lemma can be used to prove that the set $\mathcal{V}^{(m)}$ of those g such that $0 \in \mathbf{R}^{m+1}$ is a regular value of ρ_g^m is open and dense (this follows from Lemma 8.2 hereafter); for $m = 2n+1$, a regular value is a non-value since $\dim(\mathbf{R}^{n+1} \times \mathbf{S}^n) < m+1$, hence $\mathcal{V}^{(2n+1)} = \mathcal{V}_{0,0}$. For $m \leq 2n$ and $g \in \mathcal{V}^{(m)}$, the subset $\widetilde{S}_m := (\rho_g^m)^{-1}(0)$ is a $(2n-m)$-dimensional submanifold of $\mathbf{R}^{n+1} \times \mathbf{S}^n$, whose projection into $\mathbf{R}^{n+1}$ is the set S_m of those a in the hypersurface $S := g^{-1}(0)$ at which $\mu_{1,a}(g^{-1}) > m$. If 0 is a regular value of g (i.e. if g lies in the open and dense subset $\mathcal{V}^{(0)}$), then S is smooth and S_m is the set of those a at which S has contact of order at least m with some affine line, hence $S_1 = S_0 = S$.

EXAMPLE. If $n = 2$ and $k = 1$, then, generically, S is a smooth surface in $\mathbf{R}^3$ in which S_4 is a set of isolated points and S_3 is a (singular) curve, called the *flecnodal curve*. The surface $\widetilde{S}_2$ just introduced can project badly, as S_2 consists of those $a \in S$ at which there exists an asymptotic direction and therefore splits into three parts: the open set $S_{2,0}$ of those $a \in S$ at which there are two simple asymptotic directions (*hyperbolic points*), the *parabolic curve* $S_{2,1}$, consisting of those points at which there is one double asymptotic direction, and a set $S_{2,2}$ of isolated *flat points*, at which every tangent direction is asymptotic (implying that $\widetilde{S}_2$ is very badly projected).

The following figures correspond to $g(x,y,z) := \frac{1}{5}y^5 + x(-y^3 - y^2 + y + x) - z$, which can easily be shown to satisfy all the above transversality conditions. In that case, $S_4 = \{0\}$ and it is an exercise to verify that $\deg_{1,0} g^{-1}(0) = 5$.

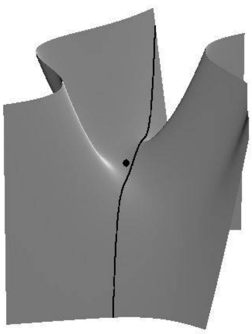

FIGURE 1

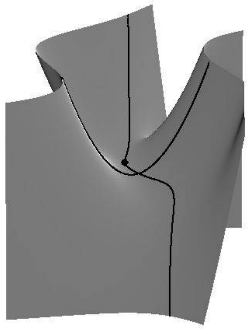

FIGURE 2

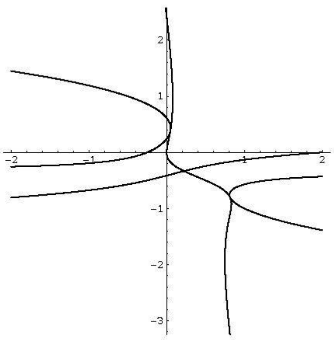

FIGURE 3

Figure 1 represents the surface S near the origin, marked as a dot; the curve is a section of S by a vertical plane close to $\{x = 0\}$. Figure 2 shows the flecnodal curve S_3. Figure 3 shows the flecnodal and parabolic curves, projected in the (x, y)-plane: there are two "godrons" [22, 24], i.e. parabolic points on the flecnodal curve. See also [1, 9, 17].

Of course, such explicit stratifications become more and more intricate when the (co)dimension increases, though the theory does imply the existence of *some* good stratification(s).

6. A Riemannian extension

If $k = 1$, since the lines in $\mathbf{R}^{n+1}$ are its Euclidean geodesics, one can try to replace $\mathbf{R}^{n+1}$ by a Riemannian manifold and lines by geodesics. As geodesics have an unfortunate tendency *not* to be properly embedded (think of the irrational lines in the flat 2-torus $\mathbf{T}^2$), the definition of the local degree has to be modified:

Definition. Let W be a manifold endowed with a linear connection, e.g. Riemannian. For each continuous map f of a topological space V into W and each $a \in V$, the *local 1-degree of f at a* is

$$\deg_{1,a} f := \inf_{U, U_1} \sup_L \#\big(f(U) \cap L\big),$$

where the infimum is taken over all open neighbourhoods U of a in V and all open neighbourhoods U_1 of $f(a)$ in W, and the supremum is over all — *connected* — geodesics L *of* U_1. The *local 1-degree of a subset A of W at $a \in A$* is the local 1-degree at a of the inclusion map $A \hookrightarrow W$.

We can now state a generalisation of Theorem 5.2 for $k = 1$, a corollary of Theorem 7.1 in the sequel:

Theorem 6.1. *Under the hypotheses of the definition, if W has dimension $n+1$, $n > 0$, then, for every smooth curve B and every point of B, which we name 0, there exist two increasing sequences $(\mathcal{V}_{0,c})_{c \in \mathrm{N}}$ and $(\mathcal{V}_c)_{c \in \mathrm{N}}$ of dense open subsets of $C^\infty(W, B)$ such that:*

(i) *Each $\mathcal{V}_{0,c}$ is c-large and consists of maps g such that the local 1-degree of $g^{-1}(0)$ at every point is at most $2n + 1 + c$.*

(ii) *Each $\mathcal{V}_c$ is c-large and consists of maps g such that, for every $b \in B$, the local 1-degree of $g^{-1}(b)$ at each of its points is at most $2n + 2 + c$.*

In particular, the open subsets $\mathcal{V} := \bigcup_c \mathcal{V}_c$ and $\mathcal{V}_0 := \bigcup_c \mathcal{V}_{0,c}$ are huge and, for $g \in \mathcal{V}_0$ (resp. $g \in \mathcal{V}$), the subset $g^{-1}(0)$ (resp. every $g^{-1}(b)$) has finite local 1-degree at every point.

REMARKS. An easy compactness argument shows that for $g \in \mathcal{V}_0$ (resp. $g \in \mathcal{V}$), if $g^{-1}(0)$ (resp. $g^{-1}(b)$) is compact, then it intersects each properly embedded geodesic L of W in a finite number of points.

For fixed g, this number can be unbounded when L varies, even when every geodesic is properly embedded: if W is the standard flat cylinder $\mathbf{T}\times\mathbf{R}$, the geodesics which are embedded lines through (θ,x) can spiral for quite a while near the closed geodesic $\mathbf{T}\times\{x\}$. To get an analogue of Theorem 5.1, we need a stronger hypothesis, obvious from the definition of the local degree:

THEOREM 6.2. *Under the hypotheses and with the notation of Theorem* 6.1, *assume that there exists a positive integer d with the following property: every point a of W has a neighbourhood basis $\mathcal{N}_a$ consisting of open subsets U_1 such that, for every geodesic L of W, the intersection $L\cap U_1$ is the union of at most d geodesics of U_1. Then:*

(i) *Let $\mathcal{P}_0$ be the open subset of $C^\infty(W,B)$ consisting of those g for which $g^{-1}(0)$ is compact. For all g in the huge open subset $\mathcal{W}_0 := \mathcal{V}_0\cap\mathcal{P}_0$ of $\mathcal{P}_0$, the 1-degree of $g^{-1}(0)$ is finite.*

(ii) *Let $\mathcal{P}$ be the open subset of $C^\infty(W,B)$ consisting of proper maps. For all g in the huge open subset $\mathcal{W} := \mathcal{V}\cap\mathcal{P}$ of $\mathcal{P}$, the 1-degree of $g^{-1}(b)$ is finite for every $b\in B$.*

This follows at once from Theorem 6.1 and the following analogue of Lemma 1.3, a particular case of Lemma 7.5 hereafter:

LEMMA 6.3. *Under the hypotheses of Theorem* 6.2, *the 1-degree of a compact subset A of W is finite if and only if its local 1-degree at every point is.*

EXAMPLES. The hypotheses of Theorem 6.2 are satisfied when W is the standard round sphere [5]) or a smooth, simply connected, complete Riemannian manifold with everywhere non-positive curvature. In both cases — as W is diffeomorphic to $\mathbf{R}^{n+1}$ in the second situation — every compact hypersurface of W is the set of zeros of some $g\in\mathcal{P}_0$. Thus, *almost every compact smooth hypersurface of W has finite 1-degree.*

[5]) Or, more generally, a Riemannian manifold all of whose geodesics are closed; this is a consequence of Wadsley's theorem: see [3], paragraphs 0.39–0.40 page 9, Theorem A-2 page 214, and Theorem A-32 page 220.

The reason why we did not take $d = 1$ in the hypothesis of Theorem 6.2 is clear in the case of a three dimensional *lens space*, for example: it is obtained as a quotient manifold of $\mathbf{S}^3$ by a cyclic subgroup $\mathbf{Z}_d$ of $\mathbf{SO}(4)$ acting without fixed point. Most closed geodesics are of length 2π. Before closing, they wind d times around exceptional geodesics of length $2\pi/d$.

7. Textures

Definition. Given positive integers n, k, a *k-texture of corank κ* on a manifold W of dimension $n+k$ is a pair $(\mathcal{L},p)$ consisting of:

- a smooth k-dimensional foliation $\mathcal{L}$ of a $(\kappa+k)$-dimensional manifold X (we denote by L_x the leaf of $\mathcal{L}$ containing $x \in X$);
- a proper smooth map $p\colon X \to W$ such that, for every leaf L of $\mathcal{L}$, denoting by ι_L the inclusion map (which is an injective immersion), the composed map $p \circ \iota_L$ is an immersion.

The manifold X is the *total space* of the texture.

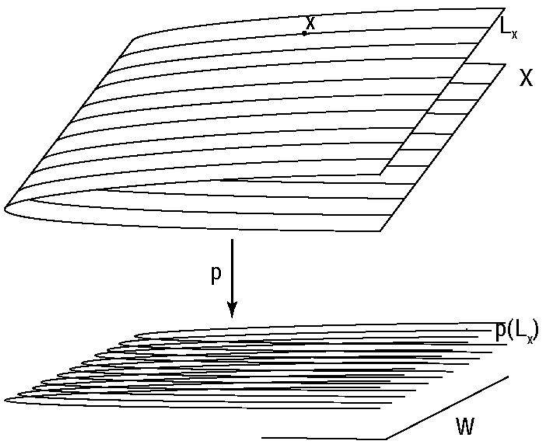

Figure 4

The role of the k-planes in Theorem 5.2 and of the geodesics in Theorem 6.1 will be played by the k-dimensional immersed manifolds $p(L)$ ("leaves" of the texture). Locally, the leaves through $a \in W$ are parametrised by the fibre $p^{-1}(a)$.

The local $(\mathcal{L},p)$-degree. Let $(\mathcal{L},p)$ be a k-texture on a manifold W. For each continuous map f of a topological space V into W and each $a \in V$, the *local $(\mathcal{L},p)$-degree of f at a* is

$$\deg_{\mathcal{L},p,a} f := \inf_{U,U_1} \sup_L \#\big(f(U) \cap p(L)\big)\,,$$

where the infimum is over all open neighbourhoods U of a in V and U_1 of $f(a)$ in W, and the supremum is over all leaves L of the foliation of $p^{-1}(U_1)$ induced by $\mathcal{L}$. The *local $(\mathcal{L},p)$-degree of a subset A of W at $a \in A$* is the local $(\mathcal{L},p)$-degree at a of the inclusion map $A \hookrightarrow W$.

FIRST EXAMPLES [6]. The local k-degree in $\mathbf{R}^{n+k}$ is the local $(\mathcal{L},p)$-degree if p is the canonical projection of $X := \mathbf{R}^{n+k} \times \mathbf{G}(k,n+k)$ onto $\mathbf{R}^{n+k}$ and, for $(a,H) \in X$, the leaf $L_{(a,H)}$ is $(a+H) \times \{H\}$.

Similarly, in Section 6, the local 1-degree is the local $(\mathcal{L},p)$-degree defined as follows: if the connection on W is the Levi-Civita connection of a Riemannian metric, p is the projection of the unit tangent sphere bundle $X = STW$ onto W and $\mathcal{L}$ the foliation of X whose leaves are the orbits of the geodesic flow; in the case of a general linear connection, p is the projection of the projectivised bundle $X = PTW$ onto W and the leaves of $\mathcal{L}$ are the integral curves of the "geodesic line field".

Hence, the following result generalises both Theorem 5.2 and Theorem 6.1:

THEOREM 7.1. *Given positive integers n,k, a k-texture $(\mathcal{L},p)$ of corank κ on an $(n+k)$-dimensional manifold W, a k-dimensional manifold B and a point of B, named 0, there exist two increasing sequences $(\mathcal{V}_{0,c})_{c\in\mathrm{N}}$ and $(\mathcal{V}_c)_{c\in\mathrm{N}}$ of dense open subsets of $C^\infty(W,B)$ with the following properties:*

(i) *Each $\mathcal{V}_{0,c}$ is c-large and consists of maps g such that the local $(\mathcal{L},p)$-degree of $g^{-1}(0)$ at every point is at most $m_k(\kappa+c)$. In particular, if $k=1$, the local k-degree of $g^{-1}(0)$ at every point is at most $\kappa+1+c$.*

(ii) *Each $\mathcal{V}_c$ is c-large and consists of maps g such that, for every $b \in B$, the local $(\mathcal{L},p)$-degree of $g^{-1}(b)$ at each of its points is at most $m_k(\kappa+k+c)$. In particular, if $k=1$, the local k-degree of $g^{-1}(b)$ at every point is at most $\kappa+2+c$.*

Thus, the open subsets $\mathcal{V} := \bigcup_c \mathcal{V}_c$ and $\mathcal{V}_0 := \bigcup_c \mathcal{V}_{0,c}$ are huge and, for $g \in \mathcal{V}_0$ (resp. $g \in \mathcal{V}$), the subset $g^{-1}(0)$ (resp. every $g^{-1}(b)$) has finite local $(\mathcal{L},p)$-degree at every point.

The proof is along the same lines as before, except that Thom's transversality lemma *cannot* be applied in the jet spaces $J^m(W,B)$ in general (see the remark following Lemma 7.4). Here is the analogue of Lemma 5.3:

[6]) More examples are given after the definition of a clean texture.

LEMMA 7.2. *Under the hypotheses of Theorem* 7.1, *the inequality*

$$\deg_{\mathcal{L},p,a} g^{-1}(b) \leq \sup_{x \in p^{-1}(a)} \mu_x(g \circ p \circ \iota_{L_x}) =: \mu_{\mathcal{L},p,a}(g^{-1})$$

holds for all $b \in B$, $g \in C^\infty(W,B)$ *and* $a \in g^{-1}(b)$.

Proof. We endow W with a Riemannian metric and each leaf L of $\mathcal{L}$ with the Riemannian metric induced by the immersion $p|_L$, namely $\|v\|_x := \|T_x p(v)\|_{p(x)}$, $v \in T_x L$. The resulting distance on L is denoted by $d_{\mathcal{L}}$.

Given $a \in W$, since p is proper, there exist a neighbourhood Ω_a of a in W and a positive number R_a such that, for each $x \in p^{-1}(\Omega_a)$, the restriction of $p \circ \iota_{L_x}$ to the open ball $B_{\mathcal{L}}(x, R_a)$ of L_x with centre x and radius R_a for $d_{\mathcal{L}}$ is an (isometric) embedding. For each positive integer m, setting $b := g(a)$, we should prove that the inequality $\deg_{\mathcal{L},p,a} g^{-1}(b) \leq m$ holds if we have $\mu_x(g \circ p \circ \iota_{L_x}) \leq m$ for every $x \in p^{-1}(a)$.

Then, for $u_0 \in p^{-1}(a)$, choosing a plaque family $\varphi_{u_0}\colon (X, u_0) \to \mathbf{R}^\kappa \times \mathbf{R}^k$ of $\mathcal{L}$, the hypotheses of Corollary 2.3 are satisfied for open subsets Λ of X and V of $\mathbf{R}^k$ satisfying $\varphi_{u_0}(\Lambda) + \{0\} \times V \subset \operatorname{Im} \varphi_{u_0}$, with $a := 0$, $B := B$, $F_u(y) := g \circ p \circ \varphi_{u_0}^{-1}\big(\varphi_{u_0}(u) + (0, y)\big)$ and therefore $\mu := \mu_{u_0}(g \circ p \circ \iota_{L_{u_0}}) \leq m$. Hence, there are open subsets $\widetilde{U}_{u_0} \ni u_0$ of X (in Λ) and $Y_{u_0} \ni 0$ of $\mathbf{R}^k$ (in V) such that, for each $u \in \widetilde{U}_{u_0}$, the equation $g \circ p \circ \varphi_{u_0}^{-1}\big(\varphi_{u_0}(u) + (0, y)\big) = b$ has at most m solutions $y \in Y_{u_0}$.

Now, we may assume that $\widetilde{U}_{u_0}$ is included in $p^{-1}(\Omega_a)$ and that there exists a positive number $\delta_{u_0} \leq R_a$ such that, for every $u \in \widetilde{U}_{u_0}$, the open ball $B_{\mathcal{L}}(u, \delta_{u_0})$ is contained in $\varphi_{u_0}^{-1}\big(\varphi_{u_0}(u) + \{0\} \times Y_{u_0}\big)$. Thus, for each $u \in \widetilde{U}_{u_0}$, the equation $g \circ p(x) = b$ has at most m solutions in the open ball $B_{\mathcal{L}}(u, \delta_{u_0})$ of L_u.

Choose values $u_1, \ldots, u_\ell \in p^{-1}(a)$ of u_0 so that $\{\widetilde{U}_{u_1}, \ldots, \widetilde{U}_{u_\ell}\}$ is a covering of $p^{-1}(a)$. As p is proper (and therefore closed), the subset $V_1 := W \smallsetminus p\big(X \smallsetminus (\widetilde{U}_{u_1} \cup \cdots \cup \widetilde{U}_{u_\ell})\big)$ is an open neighbourhood of a. Moreover, we have $p^{-1}(V_1) \subset \widetilde{U}_{u_1} \cup \cdots \cup \widetilde{U}_{u_\ell}$.

Thus, for each $u \in p^{-1}(V_1)$, setting $\delta := \min\{\delta_{u_1}, \ldots, \delta_{u_\ell}\}$, the equation $g \circ p(x) = b$ has at most m solutions in the open ball $B_{\mathcal{L}}(u, \delta)$ of L_u. Now, if U_1 is a small enough open neighbourhood of a in V_1, then, for each $u \in p^{-1}(U_1)$, the leaf of the foliation of $p^{-1}(U_1)$ induced by $\mathcal{L}$ is contained in $B_{\mathcal{L}}(u, \delta)$. Taking $U = U_1 \cap g^{-1}(b)$ in the definition of the local degree, we do obtain $\deg_{\mathcal{L},p,a} g^{-1}(b) \leq m$. $\square$

The following statement will play the role of Lemma 5.4:

LEMMA 7.3. *Under the hypotheses of Theorem 7.1, for each positive integer m, the set $J^m(\mathcal{L},B)$ of all $j^m f(x)$ with $f\colon (L_x,x)\to B$ is a smooth manifold, in which the subset $\Sigma^m(\mathcal{L},B)$ of all $j^m f(x)$ with $\mu_x(f)>m$ is a closed stratified subset whose codimension is $c_k(m)$ and therefore tends to infinity when $m\to\infty$. Moreover, $\Sigma^m(\mathcal{L},B)_0 := \Sigma^m(\mathcal{L},B)\cap\{j^m f(x) : f(x)=0\}$ is a stratified set of codimension $c_k(m)+k$.*

Proof. We shall see that $J^m(\mathcal{L},B)$ is a smooth fibre bundle over $J^0(\mathcal{L},B)=X\times B$ with projection $\pi\colon j^m f(a)\mapsto \big(a,f(a)\big)$ and fibre $J^m(k,k)$, admitting $\Sigma^m(\mathcal{L},B)$ as a sub-bundle with fibre $\Sigma^m(k)$, hence Lemma 7.3 by Proposition 4.1.

An atlas $(\Phi^m_{\varphi,\psi})$ of smooth fibre bundle can be defined as follows: recall that a *plaque family* of the foliation $\mathcal{L}$ is a local chart φ of X with values in $\mathbf{R}^\kappa\times\mathbf{R}^k$ such that the leaves of the foliation of $\operatorname{dom}\varphi$ induced by $\mathcal{L}$ are sent onto the intersections of $\operatorname{Im}\varphi$ with the vertical k-planes $\{b\}\times\mathbf{R}^k$. For each such φ and each chart ψ of B, the chart $\Phi^m_{\varphi,\psi}$ is the diffeomorphism of $\pi^{-1}(\operatorname{dom}\varphi\times\operatorname{dom}\psi)$ onto $\operatorname{Im}\varphi\times\operatorname{Im}\psi\times J^m(k,k)$ given by

$$\Phi^m_{\varphi,\psi}\left(j^m f\big(\varphi^{-1}(b,c)\big)\right) := \big(b, j^m\big(\psi\circ f\circ(\varphi^{-1})_b\big)(c)\big)\,,$$

where $(\varphi^{-1})_b(y) := \varphi^{-1}(b,y)$. As multiplicities are invariant by coordinate changes, each $\Phi^m_{\varphi,\psi}$ clearly sends $\Sigma^m(\mathcal{L},B)\cap\pi^{-1}(\operatorname{dom}\varphi\times\operatorname{dom}\psi)$ onto $\operatorname{Im}\varphi\times\operatorname{Im}\psi\times\Sigma^m(k)$, so that we just have to check that $(\Phi^m_{\varphi,\psi})$ is indeed an atlas of algebraic fibre bundle.

Given plaque families φ,φ_1 of $\mathcal{L}$ and charts ψ,ψ_1 of B, the map $\varphi_1\circ\varphi^{-1}$ is of the form $(b,c)\mapsto\big(\theta(b),\chi_b(c)\big)$; for $(b,c)\in\varphi(\operatorname{dom}\varphi\cap\operatorname{dom}\varphi_1)$, the transition map $\Phi^m_{\varphi_1,\psi_1}\circ(\Phi^m_{\varphi,\psi})^{-1}$, restricted to the fibre of (b,c), induces the polynomial automorphism

$$\big(D^j f(c)\big)_{1\le j\le m}\mapsto\big(D^j(\psi_1\circ\psi^{-1}\circ f\circ\chi_b^{-1})\big(\chi_b(c)\big)\big)_{1\le j\le m}$$

of $J^m(k,k)$, proving Lemma 7.3.

The following lemma, proved in Section 8, yields Thom's transversality lemma in jet spaces (easy case) when $p=id_M$ and $\mathcal{L}=\{M\}$:

LEMMA 7.4. *Given a texture $(\mathcal{L},p)$ on a manifold M, a manifold N, an integer m and a closed stratified subset Σ of $J^m(\mathcal{L},N)$ whose codimension is greater than the dimension of the total space T of the texture, the set of those $f\in C^\infty(M,N)$ which satisfy $j^m(f\circ p\circ\iota_{L_t})(t)\notin\Sigma$ for all $t\in T$ is open and dense.*

Proof of openness. Write each $j^m f(x) \in J^m(M,N)$ under the form $j^m_x f$ when it is viewed as an element of the fibre of x for the "source" projection $J^m(M,N) \to M$. The set $p^*J^m(M,N)$ of all pairs $\left(t, j^m_{p(t)} f\right)$ with $t \in T$ and $f\colon \left(M, p(t)\right) \to N$ is a smooth fibre bundle over T for the projection $\left(t, j^m_{p(t)} f\right) \mapsto t$, whose fibre is the manifold $J^m_0(\mathbf{R}^d, N)$ of all $j^m f(0) \in J^m(\mathbf{R}^d, N)$, where $d := \dim M$: to each chart φ of M is associated its trivialisation $\left(t, j^m_{p(t)} f\right) \mapsto \left(t, j^m_0\left(f \circ \varphi^{-1} \circ \tau_{-\varphi(p(t))}\right)\right)$ over $p^{-1}(\operatorname{dom}\varphi)$, where $\tau_a(v) := v - a$.

Each $j^m(f \circ p \circ \iota_{L_t})(t)$ with $t \in T$ and $f\colon \left(N, p(t)\right) \to B$ is determined by $\left(t, j^m_{p(t)} f\right)$, and the map $\widehat{\rho}\colon \left(t, j^m_{p(t)} f\right) \mapsto j^m(f \circ p \circ \iota_{L_t})(t)$ of $p^*J^m(M,N)$ into $J^m(\mathcal{L}, N)$ is continuous. Now,

– as Σ is closed, so is $\widehat{\rho}^{\,-1}(\Sigma)$;
– as p is proper, so is the map $\widetilde{p}\colon \left(t, j^m_{p(t)} f\right) \mapsto j^m f\left(p(t)\right)$ of $p^*J^m(M,N)$ into $J^m(M,N)$.

Hence, the image of $\widehat{\rho}^{\,-1}(\Sigma)$ under $\widetilde{p}$ is a closed subset C. It does follow that the set of those $f \in C^\infty(M,N)$ which satisfy $j^m f(t) \in J^m(M,N) \smallsetminus C$ for all t is open.

REMARK. When $\mathcal{L}$ is not analytic, it does not seem possible to stratify C and deduce Theorem 7.1 from Thom's transversality lemma in jet spaces, as $\widehat{\rho}$ is only C^∞.

PROOF OF THEOREM 7.1. We take for $\mathcal{V}_c$ the set of those $g \in C^\infty(W,B)$ such that $j^m(g \circ p \circ \iota_{L_x})(x) \notin \Sigma^m(\mathcal{L}, B)$ for all $x \in X$, with $m = m_k(\kappa + k + c)$:

- if $c = 0$, the hypotheses of Lemma 7.4 are then satisfied for $M = W$, $N = B$ and $\Sigma = \Sigma^m(\mathcal{L}, B)$, implying that $\mathcal{V}_0$ is open and dense;
- for $c > 0$, if Λ is a c-dimensional manifold, the hypotheses of Lemma 7.4 are satisfied for $M = \Lambda \times W$, $N = B$, $T = \Lambda \times X$, $p(\lambda, x) := \left(\lambda, p(x)\right)$, taking for new $\mathcal{L}$ the foliation of T whose leaves are the subsets $\Lambda \times L$ with $L \in \mathcal{L}$ and for Σ the set of all $j^m f_\bullet(\lambda, x)$ with $j^m f_\lambda(x) \in \Sigma^m(\mathcal{L}, B)$, where $f_\bullet$ denotes a map germ $\left(\Lambda \times L_x, (\lambda, x)\right) \to B$ [note that $j^m f_\bullet(\lambda, x) \mapsto j^m f_\lambda(x)$ is a submersion $J^m(\mathcal{L}, B) \to J^m(\mathcal{L}, B)$]; this guarantees that $\mathcal{V}_c$ is c-large.

Hence, every $g \in \mathcal{V}_c$ satisfies $\mu_{\mathcal{L},p,a}(g^{-1}) \leq m_k(\kappa + k + c)$ for all a and we conclude using Lemma 7.2. Similarly, we can take for $\mathcal{V}_{0,c}$ the set of those g such that $j^m(g \circ p \circ \iota_{L_x})(x) \notin \Sigma^m(\mathcal{L}, B)$ for all $x \in X$, with $m = m_k(\kappa + c)$.

THE $(\mathcal{L}, p)$-DEGREE. Given a texture $(\mathcal{L}, p)$ on a manifold W, the *$(\mathcal{L}, p)$-degree* $\deg_{\mathcal{L},p} A$ of a subset A of W is $\sup_L \#(A \cap p(L))$, where the supremum is taken over all leaves L of the foliation $\mathcal{L}$.

CLEAN TEXTURES. A texture $(\mathcal{L}, p)$ on W is *clean* when there exists an integer d with the following property: every point a of W has a neighbourhood basis $\mathcal{N}_a$ consisting of open subsets U_1 such that, for every leaf L of $\mathcal{L}$, the number of connected components of $L \cap p^{-1}(U_1)$ is at most d.

EXAMPLES. The hypothesis of Theorem 6.2 amounts to assuming that the geodesic texture is clean.

Given positive integers ℓ, n, k, a natural clean texture on $W := \mathbf{G}(\ell, \ell+n+k)$ is as follows: X is the manifold of all $(H, K) \in \mathbf{G}(\ell, \ell+n+k) \times \mathbf{G}(\ell+k, \ell+n+k)$ with $H \subset K$, the projection p is just $(H, K) \mapsto H$, and the leaf through (H_0, K_0) is the set of all $(H, K_0) \in X$. If $\ell = 1$, one gets the projective version of Thom's (clean) original affine situation, as the leaves of the texture are the projective k-planes in $\mathbf{P}^{n+k}$.

If we view $\mathbf{G}(\ell, \ell + n + k)$ as the homogeneous space G/H, where $G = \mathbf{O}(\ell + n + k)$ and H is the subgroup $\mathbf{O}(\ell, n + k)$ consisting of those $g \in G$ which preserve $\mathbf{R}^\ell \times \{0\}$, this suggests a larger class of examples where G is a Lie group, H, K are two closed subgroups with $H/(H \cap K)$ compact, $W = G/H$ and $X \subset G/H \times G/K$ is the image — diffeomorphic to $G/(H \cap K)$ — of G under the canonical projection $\pi = (\pi_H, \pi_K)$, with $L_{\pi(g)} := \pi(gK)$; thus, the leaves of the texture are the subsets $\pi_H(gK)$.

As shown by Figure 4 (p. 347), there are many non-homogeneous examples in which p is not a fibration.

LEMMA 7.5. *Let $(\mathcal{L}, p)$ be a clean texture on a manifold W. For each continuous map f of a* compact *space V into W, the $(\mathcal{L}, p)$-degree of $f(V)$ is finite if and only if the local $(\mathcal{L}, p)$-degree of f at every point is. Therefore, the $(\mathcal{L}, p)$-degree of a compact subset A of W is finite if and only if its local $(\mathcal{L}, p)$-degree at every point is.*

Proof. Taking $U_1 \in \mathcal{N}_{f(a)}$ in the definition of the local $(\mathcal{L}, p)$-degree, we see that the local $(\mathcal{L}, p)$-degree of a continuous map $f \colon V \to W$ at $a \in V$ is finite if and only if there exists an open neighbourhood U of a in V such that the $(\mathcal{L}, p)$-degree of $f(U)$ is finite. We conclude as in the proof of Lemma 1.3.

Lemma 7.5 and Theorem 7.1 yield an extension of Theorems 5.1 and 6.2:

THEOREM 7.6. *If the texture of Theorem* 7.1 *is clean, then, denoting by* $\mathcal{P}$ *and* $\mathcal{P}_0$ *the open subsets of* $C^\infty(W,B)$ *consisting respectively of proper maps and of those* g *for which* $g^{-1}(0)$ *is compact:*

(i) *For all* g *in the huge open subset* $\mathcal{W}_0 := \mathcal{V}_0 \cap \mathcal{P}_0$ *of* $\mathcal{P}_0$, *the* $(\mathcal{L},p)$*-degree of* $g^{-1}(0)$ *is finite.*

(ii) *For all* g *in the huge open subset* $\mathcal{W} := \mathcal{V} \cap \mathcal{P}$ *of* $\mathcal{P}$, *the* $(\mathcal{L},p)$*-degree of* $g^{-1}(b)$ *is finite for every* $b \in B$.

8. PROOF OF LEMMA 7.4

The following result is (the easy case of) Lemma 3.2 in [13], extracted from Morlet [16] and expressing the essence of Thom's original proof [19, 20]:

LEMMA 8.1. *Let* S *be a submanifold (possibly with boundary) of a manifold* P. *Let* $\mathcal{F}$ *be a topological space and* $j\colon \mathcal{F} \to C^\infty(T,P)$ *a mapping, where* T *is a manifold whose dimension is less than the codimension of* S. *Suppose that for each* $f \in \mathcal{F}$ *there exists a continuous mapping* $\sigma\colon E \to \mathcal{F}$, *where* E *is a manifold and* $f \in \sigma(E)$, *such that the induced mapping* $\widetilde{\sigma}\colon E \times T \to P$ *defined by* $\widetilde{\sigma}(e,t) = j\big(\sigma(e)\big)(t)$ *is* C^1 *and a submersion at every point of* $\widetilde{\sigma}^{-1}(S)$. *Then* $\{g \in \mathcal{F} : j(g)(T) \cap S = \varnothing\}$ *is dense in* $\mathcal{F}$.

Proof. Given $f \in \mathcal{F}$, take σ and $\widetilde{\sigma}$ as in the hypothesis of the lemma. Then, $\widetilde{S} := \widetilde{\sigma}^{-1}(S)$ is a C^1 submanifold of $E \times T$ whose codimension $\operatorname{codim} S$ is greater than $\dim T$, hence $\dim \widetilde{S} < \dim E$.

Clearly, $\{e \in E : j \circ \sigma(e)(T) \cap S \neq \varnothing\}$ is the image of $\widetilde{S}$ under the projection $\pi\colon E \times T \to E$ and we can apply to $\pi|_{\widetilde{S}}$ the very easy case of Sard's theorem: the image of a locally Lipschitzian map of a manifold into a higher-dimensional manifold has Lebesgue measure 0, implying that $\{e \in E : j \circ \sigma(e)(T) \cap S \neq \varnothing\}$ has Lebesgue measure 0; in particular, its complement $D := \{e \in E : j \circ \sigma(e)(T) \cap S = \varnothing\}$ is dense. As σ is continuous, for every open subset $\mathcal{U} \ni f$ of $\mathcal{F}$, the nonempty open subset $\sigma^{-1}(\mathcal{U})$ of E contains some $e \in D$, hence $\sigma(e) \in \mathcal{U} \cap \{g \in \mathcal{F} : j(g)(T) \cap S = \varnothing\}$. $\square$

End of the proof of Lemma 7.4. What follows is essentially the proof of Proposition 3.3 in [13]. As openness has already been established, setting $j(f)(t) := j^m(f \circ p \circ \iota_{L_t})(t)$ for $f \in C^\infty(M,N)$ and $t \in T$, we should prove that

$$\mathcal{T}_\Sigma := \{f \in C^\infty(M,N) : j(f)(T) \cap \Sigma = \varnothing\}$$

is dense. Each stratum V of Σ is the union of an at most countable set $\mathcal{S}_V$ of compact submanifolds with boundary S whose image under the projection $J^m(\mathcal{L},N) \to T \times N$ lies in the product $p^{-1}(\operatorname{dom}\varphi_S) \times \operatorname{dom}\psi_S$ for local charts φ_S of M and ψ_S of N with values in $\mathbf{R}^d$ and $\mathbf{R}^q$ respectively. Each $\mathcal{T}_S$ with $S \in \mathcal{S} := \bigcup_V \mathcal{S}_V$ is open since S is closed (see the proof of openness after Lemma 7.4). As Σ is the union of all $S \in \mathcal{S}$, the subset $\mathcal{T}_\Sigma$ is the intersection of all the open subsets $\mathcal{T}_S$ with $S \in \mathcal{S}$. Therefore, since $\mathcal{S}$ is countable and $C^\infty(M,N)$ has the Baire property, all we have to prove is that $\mathcal{T}_S$ is dense in $C^\infty(M,N)$ for every $S \in \mathcal{S}$.

To that effect, setting $\varphi := \varphi_S$ and $\psi := \psi_S$, we shall apply Lemma 8.1 with $\mathcal{F} := C^\infty(M,N)$, $P := J^m(\mathcal{L},N)$ and, still, $j(f)(t) := j^m(f \circ p \circ \iota_{L_t})(t)$. Let $\eta \in C^\infty(\operatorname{dom}\psi, [0,1])$ be a compactly supported function equal to 1 in a neighbourhood of the image S_2 of S under the projection of $J^m(\mathcal{L},N)$ onto N, and let $\theta \in C^\infty(\operatorname{dom}\varphi, [0,1])$ be a compactly supported function equal to 1 in a neighbourhood of the image S_1 under p of the image of S under the projection of $J^m(\mathcal{L},N)$ onto T.

Denoting by E_0 the space of all polynomial maps $\mathbf{R}^d \to \mathbf{R}^q$ of degree at most m, the space E in Lemma 8.1 will be the open neighbourhood of 0 in E_0 consisting of those e such that $\psi\big(f(x)\big) + \theta(x)\,\eta\big(f(x)\big)\, e\big(\varphi(x)\big)$

(a) lies in $\operatorname{Im}\psi$ for all $x \in \operatorname{supp}\theta \cap f^{-1}(\operatorname{supp}\eta)$,

and (b) lies off $\psi(S_2)$ for all $x \in S_1 \cap f^{-1}\big(\operatorname{supp}(1-\eta) \cap \operatorname{supp}\eta\big)$.

The mapping σ is well-defined — because of (a) — by

$$\sigma(e)(x) = \begin{cases} \psi^{-1}\Big(\psi\big(f(x)\big) + \theta(x)\eta\big(f(x)\big)e\big(\varphi(x)\big)\Big), & \text{if } \big(x, f(x)\big) \in \operatorname{dom}\varphi \times \operatorname{dom}\psi \\ f(x) & \text{otherwise.} \end{cases}$$

Continuity is easy to prove [13, 12, 8].

To check that $\tilde{\sigma}\colon (e,t) \mapsto j^m\big(\sigma(e) \circ p \circ \iota_{L_t}\big)(t)$ is a submersion at every point (e_0, t_0) with $j^m\big(\sigma(e_0) \circ p \circ \iota_{L_{t_0}}\big)(t_0) \in S$, we should prove that the mapping $e \mapsto j^m_{t_0}\big(\sigma(e) \circ p \circ \iota_{L_{t_0}}\big)$ into the fibre $J^m_{t_0}(\mathcal{L},N)$ is a submersion at e_0.

As $x_0 := p(t_0) \in S_1$ and $\sigma(e_0)(x_0) \in S_2$, we must have $f(x_0) \in \operatorname{dom}\psi$ (otherwise, $f(x_0) = \sigma(e_0)(x_0) \in S_2 \subset \operatorname{dom}\psi$, a contradiction) and therefore, by (b), $f(x_0) \notin \operatorname{supp}(1-\eta) \cap \operatorname{supp}\eta$. As $\eta\big(f(x_0)\big) = 0$ would yield the contradiction $f(x_0) = \sigma(e_0)(x_0) \in S_2 \subset \eta^{-1}(1)$, the point $f(x_0)$ must lie in the interior of $\{\eta = 1\}$; since x_0 lies in the interior of $\{\theta = 1\}$, it follows that $j^m_{t_0}\big(\sigma(e) \circ p \circ \iota_{L_{t_0}}\big) = j^m_{t_0}\Big(\big(\psi^{-1} \circ (\psi \circ f + e \circ \varphi)\big) \circ p \circ \iota_{L_{t_0}}\Big)$ for all $e \in E$.

This implies our result: indeed, $p \circ \iota_{L_{t_0}}$ is a local diffeomorphism of L_{t_0} onto a submanifold $V \ni x_0$ and the affine map $e \mapsto j^m_{\varphi(x_0)}\big((\psi \circ f \circ \varphi^{-1} + e)|_{\varphi(V)}\big)$ of E_0 into $J^m_{\varphi(x_0)}\big(\varphi(V), \mathbf{R}^q\big)$ is a submersion, the underlying linear map $e \mapsto j^m_{\varphi(x_0)}\big(e|_{\varphi(V)}\big)$ being clearly onto.

If we inject the general Sard theorem into the previous proof via the full Lemma 3.2 in [13], we get the following generalisation of Lemma 7.4:

LEMMA 8.2. *Given a texture $(\mathcal{L}, p)$ on a manifold M with total space T, a manifold N, an integer m and a stratified subset Σ of $J^m(\mathcal{L}, N)$, the set of those $f \in C^\infty(M, N)$ such that the map $T \ni t \mapsto j^m(f \circ p \circ \iota_{L_t})(t)$ is transversal to Σ is residual.*

The particular case where $(\mathcal{L}, p) = (\{M\}, id_M)$ is

THOM'S TRANSVERSALITY LEMMA IN JET SPACES. *Given manifolds M, N, an integer m and a stratified subset Σ of $J^m(M, N)$, the set of those $f \in C^\infty(M, N)$ such that $j^m f$ is transversal to Σ is residual.*

ACKNOWLEDGEMENTS. This article owes everything to conversations of the first author with René Thom and of the second author with Marcel Berger[7]), Alain Chenciner[8]), Jean-Jacques Risler[9]) and Bernard Teissier. We also thank André Bellaïche, Michel Coste, Paul Gauduchon, Étienne Ghys, Sabir Gusein-Zade, Harold Levine, Santiago López de Medrano, Ricardo Uribe-Vargas and Marina Ville. The pictures are due to Yves Martinez-Maure and Electre Petropoulos.

NOTE DES RÉDACTEURS

Le texte qui suit est extrait d'une lettre de René Thom, datée de Strasbourg, le 14 décembre 1959, adressée à André Haefliger à l'IAS, Princeton, où celui-ci entamait un séjour de deux ans comme assistant de Whitney. On voit que Thom développait déjà à cette époque les idées de son article de 1969 [21].

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[7]) Who informed him of the existence of Thom's paper and communicated to him pages on géométrie finie extracted from his forthcoming book *Géométrie vivante* [2].

[8]) Chenciner not only gave some very useful advice on at least three drafts of the present work: his encouragements and questions also led to the idea of textures.

[9]) Whose help with Tougeron's theorem was especially precious.

EXTRAIT D'UNE LETTRE DE RENÉ THOM À ANDRÉ HAEFLIGER

« Pas grand-chose de neuf de mon côté; j'ai écrit récemment un petit article de caractère semi-pédagogique sur la théorie des enveloppes (considérée comme application de la théorie des singularités). Je m'occupe toujours de la conjecture faible; je suis intéressé en ce moment par la détermination de l'"ordre local" d'une variété plongée (i.e. le nombre maximum de points en lesquels elle est localement coupée par un plan de dimension complémentaire). Il me semble probable que toute application différentiable dont le graphe est d'ordre local fini est "algébroïde", topologiquement équivalente à une application polynomiale. Ceci impliquerait que toute application analytique réelle est localement algébroïde; qu'en pensez-vous ? »

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(Reçu le 25 novembre 2008)

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