L'Enseignement Mathématique (2) 54 (2008), 213–249

FORMAL AND RIGID GEOMETRY: AN INTUITIVE INTRODUCTION AND SOME APPLICATIONS

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ABSTRACT. We give an intuitive introduction to formal and rigid geometry, and discuss some applications in algebraic and arithmetic geometry and to singularity theory, with special emphasis on recent applications to the Milnor fibration and the motivic zeta function by J. Sebag and the author.

1. INTRODUCTION

Let R be a complete discrete valuation ring, with quotient field K, and residue field k. We choose a *uniformizing parameter* π, i.e. π generates the unique maximal ideal of R. Geometers may take $R = \mathbf{C}[[t]]$, the ring of formal power series over the complex numbers, with $K = \mathbf{C}((t))$, $k = \mathbf{C}$, $\pi = t$, while number theorists might prefer to think of $R = \mathbf{Z}_p$, the ring of p-adic integers, with $K = \mathbf{Q}_p$, $k = \mathbf{F}_p$, $\pi = p$.

Very roughly, a *formal scheme over* R consists of an algebraic variety over k, together with algebraic information on an infinitesimal neighbourhood of this variety. If X is a variety over R, we can associate to X its *formal completion* $\widehat{X}$ in a natural way. It is a formal scheme over R and can be seen as an infinitesimal tubular neighbourhood of the special fibre X_0 in X. Its underlying topological space coincides with the space underlying X_0, but additional infinitesimal information is contained in the sheaf of regular functions on $\widehat{X}$.

An important aspect of the formal scheme $\widehat{X}$ is the following phenomenon. A closed point x on the scheme-theoretic generic fibre of X over K has coordinates in some finite extension of the field K and (unless X is proper over R) there is no natural way to associate to the point x a point of the special fibre X_0/k by reduction modulo π. However, by inverting π in the structure sheaf of $\widehat{X}$, we can associate a generic fibre X_η to the formal scheme $\widehat{X}$, which is a rigid variety over K. Rigid geometry provides a satisfactory theory of analytic geometry over non-archimedean fields. A point on X_η has coordinates in *the ring of integers* of some finite extension K' of K : if we denote by R' the normalization of R in K', we can canonically identify $X_\eta(K')$ with $\widehat{X}(R')$. By reduction modulo π we obtain a canonical "contraction" of the generic fibre X_η to the special fibre X_0. Roughly speaking, the formal scheme $\widehat{X}$ has the advantage that its generic and its special fibre are tightly connected; what glues them together are the R'-sections on X, where R' runs over the finite extensions of R.

A main disadvantage of rigid geometry is the artificial nature of the topology on rigid varieties: it is not a classical topology, but a Grothendieck topology. In the nineties, Berkovich developed his spectral theory of non-archimedean spaces. His spaces carry a true topology, which allows one to apply classical techniques from algebraic topology. In particular, the unit disc R becomes arcwise connected, while it is totally disconnected with respect to its π-adic topology.

In Section 3 we give a brief survey of the basic theory of formal schemes, and Section 4 is a crash course on rigid geometry. Section 5 contains the basic definitions of Berkovich's approach to non-archimedean geometry. In the final Section 6 we discuss briefly some applications of the theory, with special emphasis on the relation with arc spaces of algebraic varieties and with the Milnor fibration.

This intuitive introduction merely aims to provide some insight into the theory of formal schemes and rigid varieties. We do not provide proofs; instead, we have chosen to give a list of more thorough introductions to the different topics dealt with in this note.

2. Conventions and notation

- For any field F, we denote by F^{alg} an algebraic closure, and by F^{s} the separable closure of F in F^{alg}.
- If S is any scheme, an *S-variety* is a separated reduced scheme of finite type over S.
- For any locally ringed space (site) X, we denote the underlying topological space (site) by $|X|$.
- Throughout this note, R denotes a complete discrete valuation ring, with residue field k, and quotient field K. We fix a uniformizing parameter π,

i.e. a generator of the maximal ideal of R. For any integer $n \geq 0$, we denote by R_n the quotient ring $R/(\pi^{n+1})$. A *finite extension R' of R* is by definition the normalization of R in some finite field extension K' of K; R' is again a complete discrete valuation ring.

- Once we fix a value $|\pi| \in]0,1[$, the discrete valuation v on K defines a non-archimedean absolute value $|.|$ on K, with $|z| = |\pi|^{v(z)}$ for $z \in K^*$. This absolute value induces a topology on K, called the *π-adic topology*. The ideals $\pi^n R$, $n \geq 0$, form a fundamental system of open neighbourhoods of the zero element in K. The π-adic topology is totally disconnected. The absolute value on K extends uniquely to an absolute value on K^{alg}. For any integer $m > 0$, we endow $(K^{\text{alg}})^m$ with the norm $\|z\| := \max_{i=1,\dots,m} |z_i|$.

3. Formal geometry

In this note we will only consider formal schemes which are topologically of finite type over the complete discrete valuation ring R. This case is in many respects simpler than the general one, but it serves our purposes. For a more thorough introduction to the theory of formal schemes, we refer to [23, § 10], [22, no. 182], [28], or [9].

Intuitively, a *formal scheme X_∞ over R* consists of its special fibre X_0, which is a scheme of finite type over k, endowed with a structure sheaf containing additional algebraic information on an infinitesimal neighbourhood of X_0.

3.1 Affine formal schemes

For any tuple of variables $x = (x_1, \dots, x_m)$, we define an R-algebra $R\{x\}$ as the projective limit

$$R\{x\} := \varprojlim_n R_n[x] .$$

The R-algebra $R\{x\}$ is canonically isomorphic to the algebra of convergent power series over R, i.e. the subalgebra of $R[[x]]$ consisting of the elements

$$c(x) = \sum_{i=(i_1,\dots,i_m)\in \mathbf{N}^m} \left(c_i \prod_{j=1}^{m} x_j^{i_j}\right) \in R[[x]]$$

such that $c_i \to 0$ (w.r.t. the π-adic topology on K) as $|i| = i_1 + \ldots + i_m$ tends to ∞. This means that for each $n \in \mathbf{N}$, there exists a value $i_0 \in \mathbf{N}$ such that c_i is divisible by π^n in R if $|i| > i_0$. Note that this is exactly the condition which guarantees that the images of $c(x)$ in the quotient rings $R_n[[x]]$ are actually polynomials, i.e. belong to $R_n[x]$. The algebra $R\{x\}$ can also be characterized as the sub-algebra of $R[[x]]$ consisting of those power series which converge on the closed unit disc $R^m = \{z \in K^m \mid ||z|| \leq 1\}$, since an infinite sum converges in a non-archimedean field if and only if its terms tend to zero. One can show that $R\{x\}$ is Noetherian [23, 0.(7.5.4)].

An R-algebra A is called *topologically of finite type* (*tft*) over R if it is isomorphic to an algebra of the form $R\{x_1, \ldots, x_m\}/I$, for some integer $m > 0$ and some ideal I. For any integer $n \geq 0$, we denote by A_n the quotient ring $A/(\pi^{n+1})$. It is an R_n-algebra of finite type. Then A is the limit of the projective system $(A_n)_{n\in\mathbf{N}}$, and if we endow each ring A_n with the discrete topology then A becomes a topological ring with respect to the limit topology (the π-adic topology on A). By definition, the ideals $\pi^n A$, $n > 0$, form a fundamental system of open neighbourhoods of the zero element of A.

To any *tft* R-algebra A we can associate a ringed space $\mathrm{Spf}\,A$. It is defined as the direct limit

$$\mathrm{Spf}\,A := \varinjlim_{n} \mathrm{Spec}\,A_n$$

in the category of topologically ringed spaces (where the topology on $\mathcal{O}_{\mathrm{Spec}\,A_n}$ is discrete for every n). So the structure sheaf $\mathcal{O}_{\mathrm{Spf}\,A}$ is a sheaf of topological R-algebras in a natural way. Moreover, one can show that the stalks of this structure sheaf are local rings. A *tft affine formal R-scheme* is a locally topologically ringed space in R-algebras which is isomorphic to a space of the form $\mathrm{Spf}\,A$.

Note that the transition morphisms $\mathrm{Spec}\,A_m \to \mathrm{Spec}\,A_n$, $m \leq n$, are nilpotent immersions and therefore homeomorphisms. Hence the underlying topological space $|\mathrm{Spf}\,A|$ of $\mathrm{Spf}\,A$ is the set of *open* prime ideals J of A (i.e. prime ideals containing π), endowed with the Zariski topology, and it is canonically homeomorphic to $|\mathrm{Spec}\,A_0|$.

So we see that $\mathrm{Spf}\,A$ is the locally topologically ringed space in R-algebras

$$(|\,\mathrm{Spec}\,A_0|, \varprojlim_{n} \mathcal{O}_{\mathrm{Spec}\,A_n}).$$

In particular, we have $\mathcal{O}_{\mathrm{Spf}\,A}(\mathrm{Spf}\,A) = A$. Whenever f is an element of A, we denote by $D(f)$ the set of open prime ideals of A which do not contain f. This is an open subset of $|\mathrm{Spf}\,A|$, and the ring of sections $\mathcal{O}_{\mathrm{Spf}\,A}(D(f))$ is the π-adic completion $A_{\{f\}}$ of the localization A_f.

A *morphism* between *tft* affine formal R-schemes is by definition a morphism of locally ringed spaces in R-algebras[1]). If $h\colon A \to B$ is a morphism of *tft* R-algebras, then h induces a direct system of morphisms of R-schemes $\operatorname{Spec} B_n \to \operatorname{Spec} A_n$ and by passage to the limit a morphism of *tft* affine formal R-schemes $\operatorname{Spf}(h)\colon \operatorname{Spf} B \to \operatorname{Spf} A$. The resulting functor Spf induces an equivalence between the opposite category of *tft* R-algebras and the category of *tft* affine formal R-schemes, just like in the algebraic scheme case.

The *special fibre* X_0 of the affine formal R-scheme $X_\infty = \operatorname{Spf} A$ is the k-scheme $X_0 = \operatorname{Spec} A_0$. As we have seen, the natural morphism of topologically locally ringed spaces $X_0 \to X_\infty$ is a homeomorphism.

EXAMPLE 1. Any finite extension R' of R is a *tft* R-algebra. The affine formal scheme $\operatorname{Spf} R'$ consists of a single point, corresponding to the maximal ideal of R', but the ring of sections on this point is the entire ring R'. So, in some sense the infinitesimal information in the topology of $\operatorname{Spec} R'$ (the generic point) is transferred to the structure sheaf of $\operatorname{Spf} R'$.

If $A = R\{x,y\}/(\pi - xy)$ and $X_\infty = \operatorname{Spf} A$ then, as a topological space, X_∞ coincides with its special fibre $X_0 = \operatorname{Spec} k[x,y]/(xy)$, but the structure sheaf of X_∞ is much "thicker" than the one of X_0. The formal R-scheme X_∞ should be seen as an infinitesimal tubular neighbourhood around X_0.

3.2 FORMAL SCHEMES

A *formal scheme X_∞ topologically of finite type (tft) over R* is a locally topologically ringed space in R-algebras which has a finite open cover by *tft* affine formal R-schemes. A *morphism* between *tft* formal R-schemes is a morphism of locally ringed spaces in R-algebras.

It is often convenient to describe X_∞ in terms of the direct system $(X_n := X_\infty \times_R R_n)_{n \geq 0}$. The locally ringed space X_n is a scheme of finite type over R_n, for any n; if $X_\infty = \operatorname{Spf} A$ then $X_n = \operatorname{Spec} A_n$. For any pair of integers $0 \leq m \leq n$, the natural map of R_n-schemes $u_{m,n}\colon X_m \to X_n$ induces an isomorphism of R_m-schemes $X_m \cong X_n \times_{R_n} R_m$. The scheme X_0 is called the *special fibre* of X_∞, and X_n is the *n-th infinitesimal neighbourhood* of X_0 in X_∞ (or "*thickening*"). The natural morphism of locally topologically ringed spaces $X_n \to X_\infty$ is a homeomorphism for each $n \geq 0$.

[1]) Such a morphism is automatically continuous with respect to the topology on the structure sheaves, since it maps π to itself; because the topology is the π-adic one, it is determined by the R-algebra structure. This is specific to so-called *R-adic formal schemes* and does not hold for more general formal R-schemes.

Conversely, if $(X_n)_{n\geq 0}$ is a direct system of R-schemes of finite type such that $\pi^{n+1}=0$ on X_n and such that the transition morphism $u_{m,n}\colon X_m \to X_n$ induces an isomorphism of R_m-schemes $X_m \cong X_n \times_{R_n} R_m$ for each $0 \leq m \leq n$, then this direct system determines a *tft* formal R-scheme X_∞ by putting

$$X_\infty := \varinjlim_n X_n$$

as a locally topologically ringed space in R-algebras.

In the same way, giving a morphism $f\colon X_\infty \to Y_\infty$ between *tft* formal R-schemes amounts to giving a series of morphisms $(f_n\colon X_n \to Y_n)_{n\geq 0}$, where f_n is a morphism of R_n-schemes and all the squares

$$\begin{array}{ccc} X_m & \longrightarrow & X_n \\ {\scriptstyle f_m}\downarrow & & \downarrow{\scriptstyle f_n} \\ Y_m & \longrightarrow & Y_n \end{array}$$

commute. In other words, a morphism of *tft* formal R-schemes consists of a compatible system of morphisms between all the infinitesimal neighbourhoods of the special fibres.

The formal scheme X_∞ is called *separated* if the scheme X_n is separated for each n. In fact, this will be the case as soon as the special fibre X_0 is separated. We will work in the category of separated formal schemes topologically of finite type over R; we shall call these objects *stft formal R-schemes*.

An *stft* formal scheme X_∞ over R is flat if its structure sheaf has no π-torsion. A typical example of a non-flat *stft* formal R-scheme is one with an irreducible component concentrated in the special fibre. A flat *stft* formal R-scheme can be thought of as a continuous family of schemes over the infinitesimal disc $\mathrm{Spf}\, R$. Any *stft* formal R-scheme has a maximal flat closed formal subscheme, obtained by killing π-torsion.

3.3 Coherent modules

Let A be a *tft* R-algebra. An A-module N is *coherent* if and only if it is finitely generated. Any such module N defines a sheaf of modules on $\mathrm{Spf}\, A$ in the usual way. A coherent sheaf of modules $\mathcal{N}$ on an *stft* formal R-scheme X_∞ is obtained by gluing coherent modules on affine open formal subschemes.

A more convenient description is the following: the category of coherent sheaves $\mathcal{N}$ on X_∞ is equivalent to the category of direct systems $(\mathcal{N}_n)_{n\geq 0}$, where $\mathcal{N}_n$ is a coherent sheaf on the scheme X_n and the $\mathcal{O}_{X_n}$-linear transition map $v_{m,n}\colon \mathcal{N}_m \to \mathcal{N}_n$ induces an isomorphism of coherent $\mathcal{O}_{X_m}$-modules $\mathcal{N}_m \cong u_{m,n}^{*}\mathcal{N}_n$ for any pair $m \leq n$. Morphisms between such systems are defined in the obvious way.

3.4 The completion functor

Let X be any Noetherian scheme and $\mathcal{J}$ a coherent ideal sheaf on X, and denote by $V(\mathcal{J})$ the closed subscheme of X defined by $\mathcal{J}$. The *$\mathcal{J}$-adic completion* $\widehat{X/\mathcal{J}}$ of X is the limit of the direct system of schemes $(V(\mathcal{J}^n))_{n>0}$ in the category of topologically ringed spaces (where $\mathcal{O}_{V(\mathcal{J}^n)}$ carries the discrete topology). This is, in fact, a formal scheme, but in general not of the kind we have defined before; we include the construction here for later use. If $h\colon Y \to X$ is a morphism of Noetherian schemes, and if we denote by $\mathcal{K}$ the inverse image $\mathcal{J}\mathcal{O}_Y$ of $\mathcal{J}$ on Y, then h defines a direct system of morphisms of schemes $V(\mathcal{K}^n) \to V(\mathcal{J}^n)$ and by passage to the limit a morphism of topologically locally ringed spaces $\widehat{Y/\mathcal{K}} \to \widehat{X/\mathcal{J}}$, called the *$\mathcal{J}$-adic completion* of h.

If X is a separated R-scheme of finite type and $\mathcal{J}$ is the ideal generated by π, then the $\mathcal{J}$-adic completion of X is the limit of the direct system $(X_n = X \times_R R_n)_{n\geq 0}$, and this is an *stft* formal R-scheme, which we denote simply by $\widehat{X}$. It is called the *formal (π-adic) completion* of the R-scheme X. Its special fibre X_0 is canonically isomorphic to the fibre of X over the closed point of $\operatorname{Spec} R$. The formal scheme $\widehat{X}$ is flat if and only if X is flat over R. Intuitively, $\widehat{X}$ should be seen as the infinitesimal tubular neighbourhood of X_0 in X. As a topological space, it coincides with X_0, but additional infinitesimal information is contained in the structure sheaf.

Example 2. The formal completion of $X = \operatorname{Spec} R[x_1,\ldots,x_n]/(f_1,\ldots,f_\ell)$ is simply $\widehat{X} = \operatorname{Spf} R\{x_1,\ldots,x_n\}/(f_1,\ldots,f_\ell)$.

By the above construction, a morphism of separated R-schemes of finite type $f\colon X \to Y$ induces a morphism of formal R-schemes $\widehat{f}\colon \widehat{X} \to \widehat{Y}$ between the formal π-adic completions of X and Y. We get a completion functor

$$\widehat{\ }\colon (\mathit{sft\text{--}Sch}/R) \to (\mathit{stft\text{--}For}/R) : X \mapsto \widehat{X},$$

where $(\mathit{sft\text{--}Sch}/R)$ denotes the category of separated R-schemes of finite type, and $(\mathit{stft\text{--}For}/R)$ denotes the category of separated formal schemes topologically of finite type over R.

For a general pair of separated R-schemes of finite type X,Y, the completion map

$$C_{X,Y}\colon Hom_{(sft-Sch/R)}(X,Y)\to Hom_{(stft-For/R)}(\widehat{X},\widehat{Y}):f\mapsto\widehat{f}$$

is injective, but not bijective. It is a bijection, however, if X is proper over R: this is a corollary of Grothendieck's Existence Theorem; see [24, 5.4.1]. In particular, the completion map induces a bijection between R'-sections of X and R'-sections of $\widehat{X}$ (i.e. morphisms of formal R-schemes $\mathrm{Spf}\,R'\to\widehat{X}$), for any finite extension R' of the complete discrete valuation ring R. Indeed, $\mathrm{Spec}\,R'$ is a finite, hence proper R-scheme, and its formal π-adic completion is $\mathrm{Spf}\,R'$.

EXAMPLE 3. If $X=\mathrm{Spec}\,B$, with B an R-algebra of finite type, and $Y=\mathrm{Spec}\,R[z]$, then

$$Hom_{(sft-Sch/R)}(X,Y)=B\,.$$

On the other hand, if we denote by $\widehat{B}$ the π-adic completion of B, then $\widehat{X}=\mathrm{Spf}\,\widehat{B}$ and $\widehat{Y}=\mathrm{Spf}\,R\{z\}$, and we find

$$Hom_{(stft-For/R)}(\widehat{X},\widehat{Y})=\widehat{B}\,.$$

The completion map $C_{X,Y}$ is given by the natural injection $B\to\widehat{B}$; it is not surjective in general, but it is surjective if B is finite over R.

If X is a separated R-scheme of finite type, and $\mathcal{N}$ is a coherent sheaf of $\mathcal{O}_X$-modules, then $\mathcal{N}$ induces a direct system $(\mathcal{N}_n)_{n\geq 0}$, where $\mathcal{N}_n$ is the pull-back of $\mathcal{N}$ to X_n. This system defines a coherent sheaf of modules $\widehat{\mathcal{N}}$ on $\widehat{X}$. If X is proper over R, it follows from Grothendieck's Existence Theorem that the functor $\mathcal{N}\to\widehat{\mathcal{N}}$ is an equivalence between the category of coherent $\mathcal{O}_X$-modules and the category of coherent $\mathcal{O}_{\widehat{X}}$-modules [24, 5.1.6]. Moreover, there is a canonical isomorphism $H^q(\widehat{X},\widehat{\mathcal{N}})\cong H^q(X,\mathcal{N})$ for each coherent $\mathcal{O}_X$-module $\mathcal{N}$ and each $q\geq 0$.

If an *stft* formal R-scheme Y_∞ is isomorphic to the π-adic completion $\widehat{Y}$ of a separated R-scheme Y of finite type, we call the formal scheme Y_∞ *algebraizable*, with algebraic model Y. The following theorem is the main criterion for recognizing algebraizable formal schemes [24, 5.4.5]: if Y_0 is proper over k, and $\mathcal{L}$ is an invertible $\mathcal{O}_{Y_\infty}$-bundle such that the pull-back $\mathcal{L}_0$ of $\mathcal{L}$ to Y_0 is ample, then Y_∞ is algebraizable. Moreover, the algebraic model Y for Y_∞ is unique up to canonical isomorphism, there exists a unique line bundle $\mathcal{M}$ on Y with $\mathcal{L}=\widehat{\mathcal{M}}$, and $\mathcal{M}$ is ample. For an example of a proper formal $\mathbf{C}[[t]]$-scheme which is not algebraizable, see [28, 5.24(b)].

3.5 Formal blow-ups

Let X_∞ be a flat *stft* formal R-scheme, and let $\mathcal{I}$ be a coherent ideal sheaf on X_∞ such that $\mathcal{I}$ contains a power of the uniformizing parameter π. We can define the *formal blow-up of* X_∞ *at the centre* $\mathcal{I}$ as follows [13, § 2]: if $X_\infty = \mathrm{Spf}\,A$ is affine, and I is the ideal of global sections of $\mathcal{I}$ on X_∞, then the formal blow-up of X_∞ at $\mathcal{I}$ is the $\pi\mathcal{O}_{\mathrm{Spec}\,A}$-adic completion of the blow-up of $\mathrm{Spec}\,A$ at I. The general case is obtained by glueing.

The formal blow-up of X_∞ at $\mathcal{I}$ is again a flat *stft* formal R-scheme, and the composition of two formal blow-ups is again a formal blow-up [13, 2.1+2.5]. If X is a separated R-scheme of finite type and $\mathcal{I}$ is a coherent ideal sheaf on X containing a power of π, then the formal blow-up of $\widehat{X}$ at $\widehat{\mathcal{I}}$ is canonically isomorphic to the π-adic completion of the blow-up of X at $\mathcal{I}$.

4. Rigid geometry

In this note, we'll be able to cover only the basics of rigid geometry. We refer the reader to the books [10, 20] and the research papers [8, 13, 39, 42] for a more thorough introduction. A nice survey on Tate's approach to rigid geometry can be found in [30].

4.1 Analytic geometry over non-archimedean fields

Let L be a non-archimedean field (i.e. a field which is complete with respect to an absolute value that satisfies the ultrametric property); we assume that the absolute value on L is non-trivial. For instance, if K is our complete discretely valued field, then we can turn K into a non-archimedean field by fixing a value $|\pi| \in]0,1[$ and putting $|x| = |\pi|^{v(x)}$ for $x \in K^*$, where v denotes the discrete valuation on K (by convention, $v(0) = \infty$ and $|0| = 0$).

The absolute value on L extends uniquely to any finite extension of L, and hence to L^{s} and L^{alg}. We denote by $\widehat{L^{\mathrm{alg}}}$ the completion of L^{alg}, and by $\widehat{L^{\mathrm{s}}}$ the closure of L^{s} in $\widehat{L^{\mathrm{alg}}}$; these are again non-archimedean fields. We denote by L^o the valuation ring $\{x \in L \mid |x| \leq 1\}$, by L^{oo} its maximal ideal $\{x \in L \mid |x| < 1\}$, and by $\widetilde{L}$ the residue field L^o/L^{oo}. For $L = K$ we have $L^o = R$, $L^{oo} = (\pi)$ and $\widetilde{L} = k$.

Since L is endowed with an absolute value, one can use this structure to develop a theory of analytic varieties over L by mimicking the construction over $\mathbf{C}$. Naïvely, we can define analytic functions on open subsets of L^n as

L-valued functions which are locally defined by a convergent power series with coefficients in L. However, we are immediately confronted with some pathological phenomena. Consider, for instance, the p-adic unit disc

$$\mathbf{Z}_p = \{x \in \mathbf{Q}_p \mid |x| \leq 1\}.$$

The partition

$$\{p\,\mathbf{Z}_p, 1 + p\,\mathbf{Z}_p, \ldots, (p-1) + p\,\mathbf{Z}_p\}$$

is an open cover of $\mathbf{Z}_p$ with respect to the p-adic topology. Hence the characteristic function of $p\,\mathbf{Z}_p$ is analytic, according to our naïve definition. This contradicts some elementary properties that one expects an analytic function to have. The cause of this and similar pathologies is the fact that the unit disc $\mathbf{Z}_p$ is totally disconnected with respect to the p-adic topology. In this approach, there are "too many" analytic functions, and "too few" analytic varieties (for instance, with this definition, any compact p-adic manifold is isomorphic to a disjoint union of i unit discs, where $i \in \{0, \ldots, p-1\}$ is its *Serre invariant* [41]).

Rigid geometry is a more refined approach to non-archimedean analytic geometry, turning the unit disc into a connected space. Rigid spaces are endowed with a certain Grothendieck topology, allowing only a special type of covers.

We'll indicate two possible approaches to the theory of rigid varieties over L. The first is due to Tate [42], the second to Raynaud [39]. If we return to our example of the p-adic unit disc $\mathbf{Z}_p$, Tate's construction can be understood as follows. In fact, we already know what the "correct" algebra of analytic functions on $\mathbf{Z}_p$ should be: the power series with coefficients in K which converge *globally* on $\mathbf{Z}_p$. Tate's idea is to start from this algebra and then to construct a space on which these functions live naturally. This is similar to the construction of the spectrum of a ring in algebraic geometry. Raynaud observed that a certain class of Tate's rigid varieties can be characterized in terms of formal schemes.

4.2 Tate algebras

The basic objects in Tate's theory are the *algebras of convergent power series over* L:

$$\begin{aligned} T_m &= L\{x_1, \ldots, x_m\} \\ &= \Big\{\alpha = \sum_{i \in \mathbf{N}^m} \big(\alpha_i \prod_{j=1}^{m} x_j^{i_j}\big) \in L[[x_1, \ldots, x_m]] \;\Big|\; |\alpha_i| \to 0 \text{ as } |i| \to \infty\Big\}, \end{aligned}$$

where $|i| = \sum_{j=1}^{m} i_j$. The convergence condition implies in particular that for each α there exists $i_0 \in \mathbf{N}$ such that for $|i| > i_0$ the coefficient α_i belongs to L^o. The algebra T_m is the algebra of power series over L which converge on the closed unit polydisc $(L^o)^m$ in L^m (since an infinite sum converges in a non-archimedean field if and only if its terms tend to zero). Note that, for $L = K$, $T_m \cong R\{x_1, \ldots, x_m\} \otimes_R K$. Analogously, we can define an algebra of convergent power series $B\{x_1, \ldots, x_m\}$ for any Banach algebra B. The algebra T_m is a Banach algebra for the sup-norm $\|f\|_{\text{sup}} = \max_i |\alpha_i|$. It is Noetherian, and every ideal I is closed, so that the quotient T_m/I is again a Banach algebra with respect to the residue norm.

A *Tate algebra*, or *L-affinoid algebra*, is an L-algebra A isomorphic to such a quotient T_m/I. The residue norm on A depends on the presentation $A \cong T_m/I$. However, any morphism of L-algebras $T_m/I \to T_m/J$ is automatically continuous, so in particular the residue norm on A is well-defined up to equivalence, and the induced topology on A is independent of the chosen presentation. For any maximal ideal y of A, the residue field A/y is a finite extension of L. For proofs of all these facts, we refer to [20, 3.2.1].

By Proposition 1 of [10, 7.1.1], the maximal ideals y of T_m correspond bijectively to $G(L^{\text{alg}}/L)$-orbits of tuples $(z_1, \ldots, z_m)$, with $z_i \in (L^{\text{alg}})^o$, via the map

$$y \mapsto \{(\varphi(x_1), \ldots, \varphi(x_m)) \mid \varphi \colon T_m/y \hookrightarrow L^{\text{alg}}\},$$

where φ runs through the L-embeddings of T_m/y in L^{alg}. In particular, for any morphism of L-algebras $\psi \colon T_m \to L^{\text{alg}}$ and any index i, the element $\psi(x_i)$ belongs to $(L^{\text{alg}})^o$. It follows that ψ is *contractive*, in the sense that $|\psi(a)| \leq \|a\|_{\text{sup}}$ for any a in T_m.

The fact that we obtain tuples of elements in $(L^{\text{alg}})^o$, rather than L^{alg}, might look strange at first; it is one of the most characteristic properties of Tate's rigid varieties. Let us consider an elementary example. If z is an element of L, then $x - z$ is invertible in $L\{x\}$ if and only if z does not belong to L^o. Indeed, for $z \neq 0$ the coefficients of the formal power series $1/(x - z) = -(1/z)\sum_{i \geq 0}(x/z)^i$ tend to zero if and only if $|z| > 1$, i.e. iff $z \notin L^o$. So $(x - z)$ defines a maximal ideal in $L\{x\}$ only if $z \in L^o$.

4.3 Affinoid spaces

The category of *L-affinoid spaces* is by definition the opposite category of the category of Tate algebras over L. For any L-affinoid space X, we shall denote the corresponding Tate algebra by $A(X)$ and call it the *algebra of analytic functions* on X. Conversely, for any Tate algebra A, we denote the corresponding affinoid space by $\mathrm{Sp}A$ (some authors use the notation Spm instead). For any $m \geq 0$, the affinoid space $\mathrm{Sp}\,T_m$ is called the *closed unit disc* of dimension m over L.

To any L-affinoid space $X = \mathrm{Sp}A$ we associate the set $X^\flat$ of maximal ideals of the Tate algebra $A = A(X)$. If we present A as a quotient $T_m/(f_1, \ldots, f_n)$, then the elements of $(\mathrm{Sp}A)^\flat$ correspond bijectively to the $G(L^{\mathrm{alg}}/L)$-orbits of tuples $z = (z_1, \ldots, z_m)$, with $z_i \in (L^{\mathrm{alg}})^o$ and $f_j(z) = 0$ for each j. In particular, if L is algebraically closed and X is the closed unit disc $\mathrm{Sp}\,T_1$, then

$$X^\flat = L^o = \{x \in L \mid |x| \leq 1\}\,.$$

We have seen above that for any maximal ideal x of A, the quotient A/x is a finite extension of L, so it carries a unique prolongation of the absolute value $|.|$ on L. Hence, for any $f \in A$ and any $x \in (\mathrm{Sp}A)^\flat$, we can speak of the value $f(x)$ of f at x (the image of f in A/x) and of its absolute value $|f(x)|$. In this way, the elements of A are viewed as functions on $(\mathrm{Sp}A)^\flat$. Note that if x is a prime ideal of A, there is in general no canonical way to extend the absolute value on L to the extension A/x. This is one of the reasons for working with the maximal spectrum $(\mathrm{Sp}A)^\flat$, rather than the prime spectrum $\mathrm{Spec}A$. In Berkovich's theory (Section 5) the notion of a point is generalized by admitting any prime ideal x *and* specifying an extension of the absolute value on L to A/x.

The *spectral semi-norm* on A is defined by

$$\|f\|_{\sup} := \sup_{x \in X^\flat} |f(x)|\,.$$

It is a norm if and only if A is reduced. By the *maximum modulus principle* [10, 6.2.1.4], this supremum is in fact a maximum, i.e. there is a point x in $X^\flat$ with $|f(x)| = \|f\|_{\sup}$. Moreover, for $A = T_m$, this definition coincides with the one in the previous section, by [10, 5.1.4.6].

We could try to endow $X^\flat$ with the initial topology with respect to the functions $x \mapsto |f(x)|$, where f varies in A. If L is algebraically closed and if we identify $(\mathrm{Sp}\,L\{x\})^\flat$ with L^o, then this topology is simply the topology on L^o defined by the absolute value. It is totally disconnected, so it does not have the nice properties we are looking for.

If $\varphi\colon A \to B$ is a morphism of L-affinoid algebras then, for any maximal ideal x in B, $\varphi^{-1}(x)$ is a maximal ideal in A, since B/x is a finite extension of L. Hence, any morphism of L-affinoid spaces $h\colon X \to Y$ induces a map $h^\flat\colon X^\flat \to Y^\flat$ on the associated sets. A morphism of L-affinoid spaces $h\colon X \to Y$ is called a *closed immersion* if the corresponding morphism of L-affinoid algebras $A(Y) \to A(X)$ is surjective.

4.4 Open covers

A morphism $h\colon Y \to X$ of L-affinoid spaces is called an *open immersion* if it satisfies the following universal property: for any morphism $g\colon Z \to X$ of L-affinoid spaces such that the image of $g^\flat$ is contained in the image of $h^\flat$ in $X^\flat$, there is a unique morphism $g'\colon Z \to Y$ such that $g = h \circ g'$. If h is an open immersion, the image D of $h^\flat$ in $X^\flat$ is called an *affinoid domain*. One can show that the map $h^\flat$ is always injective [10, 7.2.2.1], so it identifies the set D with $Y^\flat$. The L-affinoid space Y and the open immersion $h\colon Y \to X$ are uniquely determined by the affinoid domain D, up to canonical isomorphism. With a slight abuse of notation, we will identify the affinoid domain D with the L-affinoid space Y, so that we can think of an affinoid domain as an affinoid space sitting inside X, and we can speak of the Tate algebra $A(D)$ of analytic functions on D. If E is a subset of D, then E is an affinoid domain in D if and only if it is an affinoid domain in X. In this case, the universal property yields a restriction map $A(D) \to A(E)$. The intersection of two affinoid domains is again an affinoid domain, but this does not always hold for their union. If $h\colon Z \to X$ is a morphism of L-affinoid spaces, then the inverse image of an affinoid domain in X is an affinoid domain in Z.

Example 4. Consider the closed unit disc $X = \mathrm{Sp}\, L\{x\}$. For $a \in L^o$ and r in the value group $|L^*|$, we denote by $D(a,r)$ the "closed disc" $\{z \in X^\flat \mid |x(z) - a| \leq r\}$, and by $D^-(a,r)$ the "open disc" $\{z \in X^\flat \mid |x(z) - a| < r\}$. We will see below that the disc $D(a,r)$ is an affinoid domain in X, with $A(D(a,r)) = L\{x,T\}/(x - a - \rho T)$, where ρ is any element of L with $|\rho| = r$. On the other hand, the disc $D^-(a,r)$ cannot be an affinoid domain in X, since the function $|x(.) - a|$ does not reach its maximum on $D^-(a,r)$.

Assume now that L is algebraically closed. By Theorem 2 in [10, 9.7.2] the affinoid domains in X are the finite disjoint unions of subsets of the form

$$D(a_0, r_0) \setminus \bigcup_{i=1}^{q} D^-(a_i, r_i)$$

with a_i in L^o and r_i in $|L^*| \cap]0,1]$ for $i = 0, \ldots, q$.

An *affinoid cover* of X is a *finite* set of open immersions $u_i\colon U_i \to X$ such that the images of the maps $(u_i)^\flat$ cover $X^\flat$. A special kind of affinoid cover is constructed as follows: take analytic functions $f_1, \ldots, f_n$ in $A(X)$, and suppose that these elements generate the unit ideal $A(X)$. Consider, for each $i = 1, \ldots, n$, the L-affinoid space U_i given by

$$A(U_i) = A(X)\{T_1, \ldots, T_n\}/(f_j - T_j f_i)_{j=1,\ldots,n}\,.$$

The obvious morphism $u_i\colon U_i \to X$ is an open immersion, and U_i is called a *rational subspace* of X. The image of $(u_i)^\flat$ is the set of points x in $X^\flat$ such that $|f_i(x)| \geq |f_j(x)|$ for $j = 1,\dots,n$. Indeed, using the fact that a morphism of L-algebras $T_m \to L^{\mathrm{alg}}$ is contractive (Section 4.2) and the assumption that $f_1,\dots,f_n$ generate $A(X)$, one shows that a morphism of L-algebras $\psi\colon A(X) \to L^{\mathrm{alg}}$ factors through a morphism of L-algebras $\psi_i\colon A(U_i) \to L^{\mathrm{alg}}$ if and only if $\psi(f_i) \neq 0$ and $\psi(T_j) = \psi(f_j)/\psi(f_i)$ belongs to $(L^{\mathrm{alg}})^o$, i.e. $|\psi(f_j)| \leq |\psi(f_i)|$. In this case, ψ_i is unique.

The set of morphisms $\{u_1,\dots,u_n\}$ is an affinoid cover and is called a *standard cover*. It is a deep result that any affinoid domain of X is a finite union of rational subsets of X, and any affinoid cover of X can be refined by a standard cover [10, 7.3.5.3+8.2.2.2].

One of the cornerstones in the theory of rigid varieties is Tate's Acyclicity Theorem [10, 8.2.1.1]. It states that analytic functions on any affinoid cover $\{u_i\colon U_i \to X\}_{i\in I}$ satisfy the *glueing property*: the sequence

$$A(X) \to \prod_{i\in I} A(U_i) \rightrightarrows \prod_{(i,j)\in I^2} A(U_i \cap U_j)$$

is exact.

Now we can define, for each L-affinoid space X, a topology on the associated set $X^\flat$. It will not be a topology in the classical sense, but a Grothendieck topology, a generalization of the topological concept in the framework of categories. A *Grothendieck topology* specifies a class of opens (*admissible opens*) and, for each admissible open, a class of covers (*admissible covers*). These have to satisfy certain axioms which allow one to develop a theory of sheaves and cohomology in this setting. A space with a Grothendieck topology is called a *site*. Any topological space (in the classical sense) can be viewed as a site in a canonical way: the admissible opens and admissible covers are the open subsets and the open covers. For our purposes we do not need the notion of Grothendieck topology in its most abstract and general form: a sufficient treatment is given in [10, 9.1.1].

The *weak* G-topology on an L-affinoid space X is defined as follows: the admissible open sets of $X^\flat$ are the affinoid domains, and the admissible covers are the affinoid covers [10, 9.1.4]. Any morphism h of L-affinoid spaces is continuous with respect to the weak G-topology (meaning that the inverse image under $h^\flat$ of an admissible open is again an admissible open, and the inverse image of an admissible cover is again an admissible cover). We can define a presheaf of L-algebras $\mathcal{O}_X$ on $X^\flat$ with respect to this topology by putting $\mathcal{O}_X(D) = A(D)$ for any affinoid domain D of X (with the natural restriction maps). By Tate's Acyclicity Theorem, $\mathcal{O}_X$ is a sheaf. Note that the exact definition of the weak G-topology varies in the literature: sometimes the admissible opens are taken to be the finite unions of rational subsets in X, and the admissible covers are the covers by admissible opens with a finite subcover (e.g. in [20, § 4.2]).

In the theory of Grothendieck topologies, there is a canonical way to refine the topology without changing the associated category of sheaves [10, 9.1.2]. This refinement is important to get good glueing properties for affinoid spaces, and to obtain continuity of the analytification map (Section 4.6). This leads to the following definition of the *strong* G-topology on an L-affinoid space X.

- The admissible open sets are (possibly infinite) unions $\bigcup_{i\in I} D_i$ of affinoid domains D_i in X such that, for any morphism of L-affinoid spaces $h\colon Y \to X$, the image of $h^\flat$ in $X^\flat$ is covered by a finite number of D_i.
- An admissible cover of an admissible open subset $V \subset X^\flat$ is a (possibly infinite) set of admissible opens $\{V_j \mid j \in J\}$ in $X^\flat$ such that $V = \bigcup_j V_j$, and such that, for any morphism of L-affinoid spaces $\varphi : Y \to X$ with $\mathrm{Im}(\varphi^\flat) \subset V$, the cover $\{(\varphi^\flat)^{-1}(V_j) \mid j \in J\}$ of Y can be refined by an affinoid cover.

Any morphism of L-affinoid spaces is continuous with respect to the strong G-topology. The strong G-topology on $X = \mathrm{Sp} A$ is finer than the Zariski topology on the maximal spectrum of A (this does not hold for the weak G-topology). From now on, we will endow all L-affinoid spaces X with the strong G-topology. The structure sheaf $\mathcal{O}_X$ of X extends uniquely to a sheaf of L-algebras with respect to the strong G-topology, which is called the *sheaf of analytic functions* on X. One can show that its stalks are local rings. In this way we associate to any L-affinoid space X a locally ringed site in L-algebras $(X^\flat, \mathcal{O}_X)$.

For any morphism of L-affinoid spaces $h\colon Y \to X$, there is a morphism of sheaves of L-algebras $\mathcal{O}_X \to (h^\flat)_*\mathcal{O}_Y$ which defines a morphism of locally ringed spaces $(Y^\flat, \mathcal{O}_Y) \to (X^\flat, \mathcal{O}_X)$ (if $D = \mathrm{Sp} A$ is an affinoid domain in X, then $(h^\flat)^{-1}(D)$ is an affinoid domain $\mathrm{Sp} B$ in Y and there is a natural morphism of L-algebras $A \to B$ by the universal property defining affinoid domains). This construction defines a functor from the category of L-affinoid spaces to the category of locally ringed sites in L-algebras, and this functor is fully faithful [10, 9.3.1.1], i.e. every morphism of locally ringed sites in L-algebras $((\mathrm{Sp} B)^\flat, \mathcal{O}_{\mathrm{Sp} B}) \to ((\mathrm{Sp} A)^\flat, \mathcal{O}_{\mathrm{Sp} A})$ is induced by a morphism of L-algebras $A \to B$. With a slight abuse of notation, we will also call the objects in its essential image L-affinoid spaces, and we will identify an L-affinoid space X with its associated locally ringed site in L-algebras $(X^{\flat}, \mathcal{O}_X)$.

If D is an affinoid domain in X, then the strong G-topology on X restricts to the strong G-topology on D, and the restriction of $\mathcal{O}_X$ to D is the sheaf of analytic functions $\mathcal{O}_D$.

One can check that the affinoid space $\mathrm{Sp}\, T_m$ is connected with respect to the strong and the weak G-topology, for any $m \geq 0$. More generally, connectedness of an L-affinoid space $X = \mathrm{Sp} A$ is equivalent for the weak G-topology, the strong G-topology, and the Zariski topology [10, 9.1.4, Prop. 8], and it is also equivalent to the property that the ring A has no non-trivial idempotents; so the G-topologies nicely reflect the algebraic structure of A.

EXAMPLE 5. Let X be the closed unit disc $\mathrm{Sp} L\{x\}$. The set

$$U_1 = \{z \in X \mid |x(z)| = 1\} = \{z \in X \mid |x(z)| \geq 1\}$$

is a rational domain in X, so it is an admissible open already for the weak G-topology. The algebra of analytic functions on U_1 is given by

$$\mathcal{O}_X(U_1) = L\{x, T\}/(xT - 1).$$

The set

$$U_2 = \{z \in X \mid |x(z)| < 1\}$$

is not an admissible open for the weak G-topology (it cannot be affinoid since the function $|x(.)|$ does not reach a maximum on U_2), but it is an admissible open for the strong G-topology: we can write it as an infinite union of rational domains

$$U_2^{(n)} = \{z \in X \mid |x(z)|^n \leq |a|\},$$

where n runs through $\mathbf{N}^*$ and a is any non-zero element of L^{oo}.

This family satisfies the finiteness condition in the definition of the strong G-topology: if $Y \to X$ is any morphism of L-affinoid spaces whose image is contained in U_2, then by the maximum principle (Section 4.3) the pull-back of the function $|x(.)|$ to Y reaches its maximum on Y, so the image of Y is contained in $U_2^{(n)}$ for n sufficiently large.

The algebra $\mathcal{O}_X(U_2)$ of analytic functions on U_2 consists of the elements $\sum_{i \geq 0} a_i x^i$ of $L[[x]]$ such that $|a_i| r^i$ tends to zero as $i \to \infty$, for any $r \in]0, 1[$.

Hence, we can write X as a disjoint union $U_1 \sqcup U_2$ of admissible opens. This does not contradict the fact that X is connected, because $\{U_1, U_2\}$ is *not* an admissible cover, since it cannot be refined by a (finite!) affinoid cover.

4.5 Rigid varieties

Now we can give the definition of a general *rigid variety* over L. It is a set X, endowed with a Grothendieck topology[2]) and a sheaf of L-algebras $\mathcal{O}_X$, such that X has an admissible cover $\{U_i\}_{i \in I}$ with the property that each locally ringed space $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic to an L-affinoid space. An admissible open U in X is called an *affinoid domain* in X if $(U, \mathcal{O}_X|_U)$ is isomorphic to an L-affinoid space. If X is affinoid, this definition is compatible with the previous one. A morphism $Y \to X$ of rigid varieties over L is a morphism of locally ringed spaces in L-algebras.

A rigid variety over L is called *quasi-compact* if it is a finite union of affinoid domains. It is called *quasi-separated* if the intersection of any pair of affinoid domains is quasi-compact, and *separated* if the diagonal morphism is a closed immersion.

4.6 Analytification of an L-variety

For any L-scheme X of finite type, we can endow the set X^o of closed points of X with the structure of a rigid L-variety.

More precisely, by [8, 0.3.3] and [30, 5.3] there exists a functor

$$(\,.\,)^{\mathrm{an}} \colon (\mathit{ft}\text{–}\mathit{Sch}/L) \to (\mathit{Rig}/L)$$

from the category of L-schemes of finite type to the category of rigid L-varieties, such that

1. for any L-scheme of finite type X, there exists a natural morphism of locally ringed sites
$$i \colon X^{\mathrm{an}} \to X$$
which induces a bijection between the underlying set of X^{an} and the set X^o of closed points of X. The couple (X^{an}, i) satisfies the following universal property: for any rigid variety Z over L and any morphism of locally ringed sites $j \colon Z \to X$, there exists a unique morphism of rigid varieties $j' \colon Z \to X^{\mathrm{an}}$ such that $j = i \circ j'$;
2. if $f \colon X' \to X$ is a morphism of L-schemes of finite type, the square
$$\begin{array}{ccc} (X')^{\mathrm{an}} & \xrightarrow{f^{\mathrm{an}}} & X^{\mathrm{an}} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ X' & \xrightarrow{f} & X \end{array}$$
commutes;

[2]) To be precise, this Grothendieck topology should satisfy certain additional axioms; see [10, 9.3.1.4].

3. the functor $(\,.\,)^{\mathrm{an}}$ commutes with fibred products and takes open (resp. closed) immersions of L-schemes to open (resp. closed) immersions of rigid L-varieties. In particular, X^{an} is separated if X is separated.

We call X^{an} the *analytification* of X. It is quasi-compact if X is proper over L, but not in general. The analytification functor has the classical GAGA properties: if X is proper over L, then analytification induces an equivalence between coherent $\mathcal{O}_X$-modules and coherent $\mathcal{O}_{X^{\mathrm{an}}}$-modules, and the cohomology groups agree; a closed rigid subvariety of X^{an} is the analytification of an algebraic subvariety of X; and for any L-variety Y, all morphisms $X^{\mathrm{an}} \to Y^{\mathrm{an}}$ are algebraic. These results can be deduced from Grothendieck's Existence Theorem; see [31, 2.8].

EXAMPLE 6. Let D be the closed unit disc $\mathrm{Sp}\,L\{x\}$, and consider the endomorphism σ of D mapping x to $a{\cdot}x$, for some non-zero $a \in L^{oo}$. Then σ is an isomorphism from D onto the affinoid domain $D(0,|a|)$ in D (notation as in Example 4). The *rigid affine line* $(\mathbf{A}^1_L)^{\mathrm{an}}$ is the limit of the direct system

$$D \xrightarrow{\ \sigma\ } D \xrightarrow{\ \sigma\ } \dots$$

in the category of locally ringed sites in L-algebras. Intuitively, it is obtained as the union of an infinite number of concentric closed discs whose radii tend to ∞.

4.7 RIGID SPACES AND FORMAL SCHEMES

Finally, we come to a second approach to the theory of rigid spaces, due to Raynaud [39]. We will deal only with the case where $L = K$ is a complete discretely valued field, but the theory is valid in greater generality (see [13]).

We have seen before that the underlying topological space of an *stft* formal R-scheme X_∞ coincides with the underlying space of its special fibre X_0. Nevertheless, the structure sheaf of X_∞ contains information on an infinitesimal neighbourhood of X_0, so one might try to construct the generic fibre X_η of X_∞. As it turns out, this is indeed possible, but we have to leave the category of (formal) schemes: this generic fibre X_η is a rigid variety over K.

4.8 THE AFFINE CASE

Let A be an algebra topologically of finite type over R, and consider the affine formal scheme $X_\infty = \mathrm{Spf}\,A$. The tensor product $A \otimes_R K$ is a K-affinoid algebra, and the generic fibre X_η of X_∞ is simply the K-affinoid space $\mathrm{Sp} A \otimes_R K$.

Let K' be any finite extension of K, and denote by R' the normalization of R in K'. There exists a canonical bijection between the set of morphisms of formal R-schemes $\mathrm{Spf} R' \to X_\infty$ and the set of morphisms of rigid K-varieties $\mathrm{Sp} K' \to X_\eta$. Consider a morphism of formal R-schemes $\mathrm{Spf} R' \to X_\infty$ or, equivalently, a morphism of R-algebras $A \to R'$. Tensoring with K yields a morphism of K-algebras $A \otimes_R K \to R' \otimes_R K \cong K'$, and hence a K'-point of X_η. Conversely, for any morphism of K-algebras $A \otimes_R K \to K'$, the image of A will be contained in R', since we have already seen in Section 4.2 that the image of $R\{x_1, \ldots, x_m\}$ under any morphism of K-algebras $T_m \to K^{\mathrm{alg}}$ is contained in the normalization R^{alg} of R in K^{alg}.

To any R'-section on X_∞ we can associate a point of X_∞, namely the image of the singleton $|\mathrm{Spf} R'|$. In this way, we obtain a *specialization map* of sets

$$sp\colon |X_\eta| \to |X_\infty| = |X_0| .$$

4.9 The general case

The construction of the generic fibre for general *stft* formal R-schemes X_∞ is obtained by glueing the constructions on affine charts. The important point here is that the specialization map sp is *continuous*: if $X_\infty = \mathrm{Spf} A$ is affine then, for any open formal subscheme U_∞ of X_∞, the inverse image $sp^{-1}(U_\infty)$ is an admissible open in X_η; in fact, if $U_\infty = \mathrm{Spf} B$ is affine then $sp^{-1}(U_\infty)$ is an affinoid domain in X_η, canonically isomorphic to $U_\eta = \mathrm{Sp} B \otimes_R K$. Hence, the generic fibres of the members $U_\infty^{(i)}$ of an affine open cover of an *stft* formal R-scheme X_∞ can be glued along the generic fibres of the intersections $U_\infty^{(i)} \cap U_\infty^{(j)}$ to obtain a rigid K-variety X_∞, and the specialization maps glue to a continuous map

$$sp\colon |X_\eta| \to |X_\infty| = |X_0| .$$

This map can be enhanced to a morphism of ringed sites by considering the unique morphism of sheaves

$$sp^\sharp\colon \mathcal{O}_{X_\infty} \to sp_* \mathcal{O}_{X_\eta}$$

which is given by the natural map

$$\mathcal{O}_{X_\infty}(U_\infty) = A \to A \otimes_R K = sp_* \mathcal{O}_{X_\eta}(U_\infty)$$

on any affine open formal subscheme $U_\infty = \mathrm{Spf} A$ of X_∞.

The generic fibre of an *stft* formal R-scheme is a separated, quasi-compact rigid K-variety. The formal scheme X_∞ is called a *formal R-model* for the rigid K-variety X_η. Since the generic fibre is obtained by inverting π, it is clear that the generic fibre does not change if we replace X_∞ by its maximal flat closed formal subscheme (by killing π-torsion). If K' is a finite extension of K and R' the normalization of R in K', then we still have a canonical bijection $X_\infty(R') = X_\eta(K')$.

The construction of the generic fibre is functorial: a morphism of *stft* formal R-schemes $h\colon Y_\infty \to X_\infty$ induces a morphism of rigid K-varieties $h_\eta\colon Y_\eta \to X_\eta$, and the square

$$\begin{array}{ccc} Y_\eta & \xrightarrow{h_\eta} & X_\eta \\ {\scriptstyle sp}\downarrow & & \downarrow{\scriptstyle sp} \\ Y_\infty & \xrightarrow{h} & X_\infty \end{array}$$

commutes. We get a functor

$$(\,.\,)_\eta\colon (\mathit{stft}\text{–}\mathit{For}/R) \to (\mathit{sqc}\text{–}\mathit{Rig}/K) : X_\infty \mapsto X_\eta$$

from the category of *stft* formal R-schemes to the category of separated, quasi-compact rigid K-varieties.

For any locally closed subset Z of X_0, the inverse image $sp^{-1}(Z)$ is an admissible open in X_∞, called the *tube* of Z in X_∞ and denoted by $]Z[$. If Z is open in X_0 then $]Z[$ is canonically isomorphic to the generic fibre of the open formal subscheme $Z_\infty = (|Z|, \mathcal{O}_{X_\infty}|_Z)$ of X_∞. The tube $]Z[$ is quasi-compact if Z is open, but not in general.

Berthelot showed in [8, 0.2.6] how to construct the generic fibre of a broader class of formal R-schemes, not necessarily *tft*. If Z is closed in X_0, then $]Z[$ is canonically isomorphic to the generic fibre of the formal completion of X_∞ along Z (this formal completion is the locally topologically ringed space with underlying topological space $|Z|$ and structure sheaf

$$\varprojlim_n \mathcal{O}_{X_\infty}/\mathcal{I}_Z^n,$$

where $\mathcal{I}_Z$ is the defining ideal sheaf of Z in X_∞). In particular, if Z is a closed point x of X_0 then $]x[$ is the generic fibre of the *formal spectrum* of the completed local ring $\widehat{\mathcal{O}}_{X_\infty,x}$ with its *adic topology* (we did not define this notion; see [23, 10.1]).

EXAMPLE 7. Let X_∞ be an affine *stft* formal R-scheme, say $X_\infty = \operatorname{Spf} A$. Consider a tuple of elements $f_1, \dots, f_r$ in A, and denote by Z the closed subscheme of X_0 defined by the residue classes $\overline{f}_1, \dots, \overline{f}_r$ in A_0. The tube $]Z[$ of Z in X_∞ consists of the points x of X_η with $|f_i(x)| < 1$ for $i = 1, \dots, r$ (since this condition is equivalent to $f_i(x) \equiv 0 \mod (K^{\text{alg}})^{oo}$).

If $X_\infty = \operatorname{Spf} R\{x\}$ then X_η is the closed unit disc $\operatorname{Sp} K\{x\}$ and the special fibre X_0 is the affine line $\mathbf{A}^1_k$. If we denote by O the origin in X_0 and by V its complement, then $]V[$ is the affinoid domain

$$U_1 = \operatorname{Sp} K\{x, T\}/(xT - 1)$$

from Example 5 (the "boundary" of the closed[3]) unit disc), and $]O[$ is the open unit disc U_2 from the same example. The first one is quasi-compact, the second one is not.

4.10 LOCALIZATION BY FORMAL BLOW-UPS

The functor $(\,.\,)_\eta$ is not an equivalence. One can show that formal blow-ups are turned into isomorphisms [13, 4.1]. Intuitively, this is clear: the centre $\mathcal{I}$ of a formal blow-up contains a power of π, so it becomes the unit ideal after inverting π.

In some sense, this is the only obstruction. Denote by $\mathcal{C}$ the category of flat *stft* formal R-schemes, *localized with respect to the formal blow-ups*. This means that we artificially add inverse morphisms for formal blow-ups, thus turning them into isomorphisms. The objects of $\mathcal{C}$ are simply the flat *stft* formal R-schemes, but a morphism in $\mathcal{C}$ from Y_∞ to X_∞ is given by a triple $(Y'_\infty, \varphi_1, \varphi_2)$, where $\varphi_1 \colon Y'_\infty \to Y_\infty$ is a formal blow-up and $\varphi_2 \colon Y'_\infty \to X_\infty$ a morphism of *stft* formal R-schemes. We identify this triple with another triple $(Y''_\infty, \psi_1, \psi_2)$ if there exist a third triple $(Z_\infty, \chi_1, \chi_2)$ and morphisms of *stft* formal R-schemes $Z_\infty \to Y'_\infty$ and $Z_\infty \to Y''_\infty$ such that the obvious triangles commute.

Since admissible blow-ups are turned into isomorphisms by the functor $(\,.\,)_\eta$, it factors through a functor $\mathcal{C} \to (sqc\text{–}Rig/K)$. Raynaud [39] showed that this is an equivalence of categories (a detailed proof is given in [13]). This means that the category of separated, quasi-compact rigid K-varieties can be described entirely in terms of formal schemes. To give an idea of this dictionary between formal schemes and rigid varieties, we list some results. Let X be a separated, quasi-compact rigid variety over K.

[3]) The notion of *boundary* is well defined only if you specify a centre of the disc, since any point of a closed disc can serve as a centre, due to the ultrametric property of the absolute value.

- [13, 4.1(e), 4.7] There exists a flat *stft* formal R-scheme X_∞ such that X is isomorphic to X_η.
- [13, 4.1(c+d)] If X_∞ and Y_∞ are *stft* formal R-schemes and $\varphi\colon Y_\eta \to X_\eta$ is a morphism of rigid K-varieties, then in general φ will not extend to a morphism $Y_\infty \to X_\infty$ on the R-models. However, by Raynaud's result, there exist a formal blow-up $f\colon Y'_\infty \to Y_\infty$ and a morphism of *stft* formal R-schemes $g\colon Y'_\infty \to X_\infty$ such that $\varphi = g_\eta \circ (f_\eta)^{-1}$. If φ is an isomorphism, we can find (Y'_∞, f, g) with both f and g formal blow-ups.
- [13, 4.4] For any affinoid cover $\mathfrak{U}$ of X, there exist a formal model X_∞ of X and a Zariski cover $\{U_1, \dots, U_s\}$ of X_0 such that $\mathfrak{U} = \{\,]U_1[, \dots,]U_s[\,\}$.

See [13, 14, 15, 16] for many other results.

EXAMPLE 8. Consider the *stft* formal R-schemes

$$X_\infty = \operatorname{Spf} R\{x\}/(x^2 - 1),$$
$$Y_\infty = \operatorname{Spf} R\{x\}/(x^2 - \pi^2).$$

The generic fibres Y_η and X_η are isomorphic (both consist of two points $\operatorname{Sp} K$), but it is clear that there is no morphism of *stft* formal R-schemes $Y_\infty \to X_\infty$ which induces an isomorphism between the generic fibres. The problem is that the section x/π is not defined on Y_∞; however, blowing up the ideal (x, π) adds this section to the ring of regular functions, and the formal blow-up scheme is isomorphic to X_∞.

Next, consider the *stft* formal R-scheme $Z_\infty = \operatorname{Spf} R\{x\}$ and the standard cover of Z_η defined by the couple (x, π). The cover consists of the closed disc $D(0, |\pi|)$ and the closed annulus $Z_\eta \setminus D^-(0, |\pi|)$ (notation as in Example 4). These sets are not tubes in Z_∞, since by Example 7 both sets have non-empty intersection with the tube $]O[$ but do not coincide with it. But if we take the formal blow-up $Z'_\infty \to Z_\infty$ at the ideal (x, π), then the rational subsets in our standard cover are precisely the generic fibres of the blow-up charts $\operatorname{Spf} R\{x, T\}/(xT - \pi)$ and $\operatorname{Spf} R\{x, T\}/(x - \pi T)$.

4.11 PROPER R-VARIETIES

Now let X be a separated scheme of finite type over R, and denote by X_K its generic fibre. We denote by $(X_K)^o$ the set of closed points of X_K. By [8, 0.3.5], there exists a canonical open immersion $\alpha\colon (\widehat{X})_\eta \to (X_K)^{\mathrm{an}}$. If X is proper over R, then α is an isomorphism.

For a proper R-scheme of finite type, we can describe the specialization map

$$sp\colon (X_K)^o = |(X_K)^{\mathrm{an}}| = |\widehat{X}_\eta| \to |\widehat{X}| = |X_0|$$

as follows: let x be a closed point of X_K, denote by K' its residue field and by R' the normalization of R in K'. The point x defines a morphism $x\colon \operatorname{Spec} K' \to X$. The valuative criterion for properness guarantees that the morphism $\operatorname{Spec} R' \to \operatorname{Spec} R$ lifts to a unique morphism $h\colon \operatorname{Spec} R' \to X$ with $h|_{\operatorname{Spec} K'} = x$. If we denote by 0 the closed point of $\operatorname{Spec} R'$, then $sp(x) = h(0) \in |X_0|$.

In general, the open immersion $\alpha\colon \widehat{X}_\eta \to (X_K)^{\mathrm{an}}$ is strict. Consider, for instance, a proper R-variety X, and let X' be the variety obtained by removing a closed point x from the special fibre X_0. Then $X'_K = X_K$; however, by taking the formal completion $\widehat{X'}$, we lose all the points in $\widehat{X}_\eta$ that map to x under sp, i.e. $\widehat{X'}_\eta = \widehat{X}_\eta \setminus]x[$. We'll see an explicit example in the following section. This is another instance of the fact that the rigid generic fibre $\widehat{X'}_\eta$ is "closer" to the special fibre than the scheme-wise generic fibre X'_K.

4.12 Example: the projective line

Let X be the affine line $\operatorname{Spec} R[x]$ over R; then $X_K = \operatorname{Spec} K[x]$, and $(X_K)^{\mathrm{an}}$ is the rigid affine line $(\mathbf{A}^1_K)^{\mathrm{an}}$ from Example 6. On the other hand, $\widehat{X} = \operatorname{Spf} R\{x\}$ and $\widehat{X}_\eta$ is the closed unit disc $\operatorname{Sp} K\{x\}$. The canonical open immersion $\widehat{X}_\eta \to (X_K)^{\mathrm{an}}$ is an isomorphism onto the affinoid domain in $(X_K)^{\mathrm{an}}$ consisting of the points z with $|x(z)| \leq 1$.

If we remove the origin O from X, we get a scheme X' with $X'_K = X_K$. However, the formal completion of X' is

$$\widehat{X'} = \operatorname{Spf} R\{x, T\}/(xT - 1)$$

and its generic fibre is the complement of $]O[$ in $\widehat{X}_\eta$ (see Example 7).

Now let us turn to the projective line $\mathbf{P}^1_R = \operatorname{Proj} R[x, y]$. The analytic projective line $(\mathbf{P}^1_K)^{\mathrm{an}}$ can be realized in different ways. First, consider the usual affine cover of $\mathbf{P}^1_K$ by the charts $U_1 = \operatorname{Spec} K[x/y]$ and $U_2 = \operatorname{Spec} K[y/x]$. Their analytifications $(U_1)^{\mathrm{an}}$ and $(U_2)^{\mathrm{an}}$ are infinite unions of closed discs (see Example 6) centred at 0, resp. ∞. Glueing along the admissible opens $(U_1)^{\mathrm{an}} - \{0\}$ and $(U_2)^{\mathrm{an}} - \{\infty\}$ in the obvious way, we obtain $(\mathbf{P}^1_K)^{\mathrm{an}}$.

On the other hand, we can look at the formal completion $\widehat{\mathbf{P}^1_R}$. By the results in Section 4.11, we know that its generic fibre is canonically isomorphic to $(\mathbf{P}^1_K)^{\mathrm{an}}$. The *stft* formal R-scheme $\widehat{\mathbf{P}^1_R}$ is covered by the affine charts $V_1 = \operatorname{Spf} R\{x/y\}$ and $V_2 = \operatorname{Spf} R\{y/x\}$ whose intersection is given by

$$V_0 = \operatorname{Spf} R\{x/y, y/x\}/\big((x/y)(y/x) - 1\big)\,.$$

We have seen in Example 7 that the generic fibres of V_1 and V_2 are closed unit discs around $x/y = 0$, resp. $y/x = 0$, and that $(V_0)_\eta$ coincides with their boundaries. So in this way, $(\mathbf{P}^1_K)^{\mathrm{an}}$ is realized as the Riemann sphere obtained by glueing two closed unit discs along their boundaries.

5. Berkovich spaces

We recall some definitions from Berkovich's theory of analytic spaces over non-archimedean fields. We refer to [2], or to [6] for a short introduction. A very nice survey of the theory and some of its applications are given in [19].

For a commutative Banach ring with unity $(\mathcal{A}, \|.\|)$, the *spectrum* $\mathcal{M}(\mathcal{A})$ is the set of all bounded multiplicative semi-norms $x\colon \mathcal{A} \to \mathbf{R}_+$ (where "bounded" means that there exists a number $C > 0$ such that $x(a) \leq C\|a\|$ for all a in $\mathcal{A}$). If x is a point of $\mathcal{M}(\mathcal{A})$, then $x^{-1}(0)$ is a prime ideal of $\mathcal{A}$, and x descends to an absolute value $|.|$ on the quotient field of $\mathcal{A}/x^{-1}(0)$. The completion of this field is called the *residue field of* x and is denoted by $\mathcal{H}(x)$. Hence any point x of $\mathcal{M}(\mathcal{A})$ gives rise to a bounded ring morphism χ_x from $\mathcal{A}$ to the complete valued field $\mathcal{H}(x)$, and x is completely determined by χ_x. In this way, one can characterize the points of $\mathcal{M}(\mathcal{A})$ as equivalence classes of bounded ring morphisms from $\mathcal{A}$ to a complete valued field [2, 1.2.2(ii)], just as one can view elements of the spectrum $\operatorname{Spec} A$ of a commutative ring A either as prime ideals in A or as equivalence classes of ring morphisms from A to a field.

If we denote the image of $f \in \mathcal{A}$ under χ_x by $f(x)$, then $x(f) = |f(x)|$. We endow $\mathcal{M}(\mathcal{A})$ with the weakest topology such that $\mathcal{M}(\mathcal{A}) \to \mathbf{R} : x \mapsto |f(x)|$ is continuous for each f in $\mathcal{A}$. This topology is called the *spectral topology* on $\mathcal{M}(\mathcal{A})$. If $\mathcal{A}$ is not the zero ring, it turns $\mathcal{M}(\mathcal{A})$ into a non-empty compact Hausdorff topological space [2, 1.2.1]. A bounded morphism of Banach algebras $\mathcal{A} \to \mathcal{B}$ induces a continuous map $\mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{A})$ between their spectra. In particular, the spectrum of $\mathcal{A}$ depends only on the equivalence class of $\|.\|$.

If L is a non-archimedean field with non-trivial absolute value and A is an L-affinoid algebra (these are called *strictly* L-affinoid in Berkovich's theory), then A carries a Banach norm, well-defined up to equivalence (Section 4.2). The spectrum $\mathcal{M}(A)$ of A is called a *strictly L-affinoid analytic space*; Berkovich endows these topological spaces with a structure sheaf of analytic functions. General *strictly L-analytic spaces* are obtained by glueing strictly L-affinoid analytic spaces.

Any maximal ideal x of A defines a point of $\mathcal{M}(A)$: the bounded multiplicative semi-norm sending $f \in A$ to $|f(x)|$. This defines a natural injection $\operatorname{Sp} A \to \mathcal{M}(A)$, whose image consists of the points y of $\mathcal{M}(A)$ with $[\mathcal{H}(y) : L] < \infty$. So $\mathcal{M}(A)$ contains the "classical" rigid points of $\operatorname{Sp} A$, but in general also some additional points z with $z^{-1}(0)$ not a maximal ideal. Beware that the natural map

$$\mathcal{M}(A) \to \operatorname{Spec} A : z \mapsto z^{-1}(0)$$

is not injective, in general: if $P \in \operatorname{Spec} A$ is not a maximal ideal, there may be several bounded absolute values on A/P extending the absolute value on L.

For a Hausdorff strictly L-analytic space X, the set of rigid points

$$X_{\mathrm{rig}} := \{x \in X \mid [\mathcal{H}(x) : L] < \infty\}$$

can be endowed with the structure of a quasi-separated rigid variety over L in a natural way. Moreover, the functor $X \mapsto X_{\mathrm{rig}}$ induces an equivalence between the category of paracompact strictly L-analytic spaces, and the category of quasi-separated rigid varieties over L which have an admissible affinoid covering of finite type [3, 1.6.1]. The space X_{rig} is quasi-compact if and only if X is compact.

The big advantage of Berkovich spaces is that they carry a "true" topology instead of a Grothendieck topology, with very nice features (Hausdorff, locally arcwise connected, ...). As we have seen, Berkovich obtains his spaces by adding points to the points of a rigid variety (not unlike the generic points in algebraic geometry) which have an interpretation in terms of valuations. We refer to [2, 1.4.4] for a description of the points and the topology of the closed unit disc $D = \mathcal{M}(L\{x\})$.

To give a taste of these Berkovich spaces, let us explain how two points of D_{rig} can be joined by a path in D. We assume, for simplicity, that L is algebraically closed. For each point a of D_{rig} and each $\rho \in [0,1]$ we define $D(a,\rho)$ as the set of points z in D_{rig} with $|x(z) - x(a)| \leq \rho$. This is not an affinoid domain if $\rho \notin |L^*|$. Any such disc $E = D(a,\rho)$ defines a bounded multiplicative semi-norm $|.|_E$ on the Banach algebra $L\{x\}$, by mapping $f = \sum_{n=0}^{\infty} a_n(T-a)^n$ to

$$|f|_E = \sup_{z\in E} |f(z)| = \max_n |a_n|\rho^n ,$$

and hence E defines a Berkovich-*point* of D. Now a path between two points a, b of $D_{\mathrm{rig}} = L^o$ can be constructed as follows: put $\delta = |x(a) - x(b)|$ and consider the path

$$\gamma\colon [0,1] \to D : t \mapsto \begin{cases} D(a, 2t\delta)\,, & \text{if } 0 \leq t \leq 1/2\,, \\ D(b, 2(1-t)\delta)\,, & \text{if } 1/2 \leq t \leq 1\,. \end{cases}$$

Geometrically, this path can be seen as a closed disc around a, growing continuously in time t until it contains b, and then shrinking to b.

A remarkable feature of Berkovich's theory is that it can also be applied to the case where L carries the trivial absolute value. If k is any field, and X is an algebraic variety over k, then we can endow k with the trivial absolute value and consider the Berkovich analytic space X^{an} associated to X over k [2, 3.5]. Surprisingly, the topology of X^{an} contains some non-trivial information on X. For instance, if $k = \mathbf{C}$, then the rational singular cohomology $H_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbf{Q})$ of X^{an} is canonically isomorphic to the weight-zero part of the rational singular cohomology of the complex analytic space $X(\mathbf{C})$ [7, 1.1(c)]. We refer to [36] and [43] for other applications of analytic spaces with respect to trivial absolute values.

Let us mention that there are still alternative approaches to non-archimedean geometry, such as Fujiwara and Kato's *Zariski-Riemann spaces* [21], or Huber's *adic spaces* [27]. See [38] for a (partial) comparison.

6. Some applications

6.1 Relation to arc schemes and the Milnor fibration

6.1.1 Arc spaces. Let k be any field, and let X be a separated scheme of finite type over k. Put $R = k[[t]]$. For each $n \geq 1$, we define a functor

$$F_n\colon (k\text{-}Alg) \to (Sets) : A \mapsto X(A \otimes_k R_n)$$

from the category of k-algebras to the category of sets. It is representable by a separated k-scheme of finite type $\mathcal{L}_n(X)$ (this is nothing but the *Weil restriction* of $X \times_k R_n$ to k). For any pair of integers $m \geq n \geq 0$, the truncation map $R_m \to R_n$ induces, by Yoneda's Lemma, a morphism of k-schemes

$$\pi_n^m\colon \mathcal{L}_m(X) \to \mathcal{L}_n(X)\,.$$

It is easily seen that these morphisms are affine, and hence we can consider the projective limit

$$\mathcal{L}(X) := \varprojlim_{n} \mathcal{L}_n(X)$$

in the category of k-schemes. This scheme is called the *arc scheme* of X. It satisfies $\mathcal{L}(X)(k') = X(k'[[t]])$ for any field k' over k (these points are called *arcs* on X) and comes with natural projections

$$\pi_n \colon \mathcal{L}(X) \to \mathcal{L}_n(X)\,.$$

In particular, we have a morphism $\pi_0 \colon \mathcal{L}(X) \to \mathcal{L}_0(X) = X$. For any subscheme Z of X, we put $\mathcal{L}(X)_Z = \mathcal{L}(X) \times_X Z$. By Yoneda's Lemma, a morphism of separated k-schemes of finite type $h \colon Y \to X$ induces k-morphisms $h \colon \mathcal{L}_n(Y) \to \mathcal{L}_n(X)$ and, by passage to the limit, a k-morphism $h \colon \mathcal{L}(Y) \to \mathcal{L}(X)$.

If X is smooth over k, the schemes $\mathcal{L}_n(X)$ and $\mathcal{L}(X)$ are fairly well understood: if X has pure dimension d then, for each pair of integers $m \geq n \geq 0$, π_n^m is a locally trivial fibration with fibre $\mathbf{A}_k^{d(m-n)}$ (with respect to the Zariski topology). If x is a singular point of X, however, the scheme $\mathcal{L}(X)_x$ is still quite mysterious. It contains a lot of information on the singular germ (X, x); interesting invariants can be extracted by the theory of motivic integration (see [17, 18, 44]).

The schemes $\mathcal{L}(X)_x$ and $\mathcal{L}(X)$ are not Noetherian, in general, which complicates the study of their geometric properties. Already the fact that they have only finitely many irreducible components if k has characteristic zero, is a non-trivial result. We will show how rigid geometry allows one to translate questions concerning the arc space into arithmetic problems on rigid varieties.

6.1.2 THE RELATIVE CASE. Let k be any algebraically closed field of characteristic zero[4]), and put $R = k[[t]]$. For each integer $d > 0$, $K = k((t))$ has a unique extension $K(d)$ of degree d in a fixed algebraic closure K^{alg} of K, obtained by joining a d-th root of t to K. We denote by $R(d)$ the normalization of R in $K(d)$. For each $d > 0$, we choose a d-th root of t in K^{alg}, denoted by $\sqrt[d]{t}$, such that $(\sqrt[de]{t})^e = \sqrt[d]{t}$ for each $d, e > 0$. This choice defines an isomorphism of k-algebras $R(d) \cong k[[\sqrt[d]{t}]]$. It also induces an isomorphism of R-algebras

$$\varphi_d \colon R(d) \to R(d)' : \sum_{i \geq 0} a_i (\sqrt[d]{t})^i \mapsto \sum_{i \geq 0} a_i t^i\,,$$

[4]) This condition is imposed only to simplify the arguments.

where $R(d)'$ is the ring R with R-algebra structure given by

$$R \to R : \sum_{i \geq 0} b_i t^i \mapsto \sum_{i \geq 0} b_i t^{id} .$$

Let X be a smooth irreducible variety over k, endowed with a dominant morphism $f\colon X \to \mathbf{A}^1_k = \operatorname{Spec} k[t]$. We denote by $\widehat{X}$ the formal completion of the R-scheme $X_R = X \times_{k[t]} k[[t]]$; we will also call this the *t-adic completion* of f. Its special fibre X_0 is simply the fibre of f over the origin.

There exists a tight connection between the points on the generic fibre $\widehat{X}_\eta$ of $\widehat{X}$, and the arcs on X. For any integer $d > 0$, we denote by $\mathcal{X}(d)$ the closed subscheme of $\mathcal{L}(X)$ defined by

$$\mathcal{X}(d) = \{\psi \in \mathcal{L}(X) \mid f(\psi) = t^d\} .$$

We will construct a canonical bijection

$$\varphi\colon \widehat{X}_\eta(K(d)) \to \mathcal{X}(d)(k)$$

such that the square

$$\begin{array}{ccc} \widehat{X}_\eta(K(d)) & \xrightarrow{\ \varphi\ } & \mathcal{X}(d)(k) \\ {\scriptstyle sp}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ X_0(k) & \xrightarrow{\ =\ } & X_0(k) \end{array}$$

commutes.

As we saw in Section 4.9, the specialization morphism of ringed sites $sp\colon \widehat{X}_\eta \to \widehat{X}$ induces a bijection $\widehat{X}_\eta(K(d)) \to \widehat{X}(R(d))$, and the morphism sp maps a point of $\widehat{X}_\eta(K(d))$ to the reduction modulo $\sqrt[d]{t}$ of the corresponding point of $\widehat{X}(R(d))$. By Grothendieck's Existence Theorem (Section 3.4), the completion functor induces a bijection $(X_R)(R(d)) \to \widehat{X}(R(d))$. Finally, the R-isomorphism $\varphi_d\colon R(d) \to R(d)'$ induces a bijection

$$(X_R)(R(d)) \to (X_R)(R(d)') = \mathcal{X}(d)(k) .$$

In other words, if we take an arc $\psi\colon \operatorname{Spec} R \to X$ with $f(\psi) = t^d$, then the morphism $\widehat{\psi}_\eta$ yields a $K(d)$-point on X_η, and this correspondence defines a bijection between $\mathcal{X}(d)(k)$ and $\widehat{X}_\eta(K(d))$. Moreover, the image of ψ under the projection $\pi_0\colon \mathcal{L}(X) \to X$ is nothing but the image of the corresponding element of $\widehat{X}_\eta(K(d))$ under the specialization map $sp\colon |\widehat{X}_\eta| \to |\widehat{X}| = |X_0|$.

The Galois group $G(K(d)/K) = \mu_d(k)$ acts on $\widehat{X}_\eta(K(d))$, and its action on the level of arcs is easy to describe: if ψ is an arc $\operatorname{Spec} R \to X$ with $f(\psi) = t^d$, and ξ is an element of $\mu_d(k)$, then $\xi.\psi(t) = \psi(\xi.t)$.

The spaces $\mathcal{X}(d)$, with their $\mu_d(k)$-action, are quite close to the arc spaces appearing in the definition of the motivic zeta function associated to f [18, 3.2]. In fact, the motivic zeta function can be realized in terms of the motivic integral of a Gelfand-Leray form on $\widehat{X}_\eta$, and the relationship between arc schemes and rigid varieties can be used in the study of motivic zeta functions and the monodromy conjecture, as is explained in [35, 34].

6.1.3 THE ABSOLUTE CASE. This case is easily reduced to the previous one. Let X be any separated k-scheme of finite type, and consider its base change $X_R = X \times_k R$. We denote by $\widehat{X}$ the formal completion of X_R.

There exists a canonical bijection between the sets $\mathcal{L}(X)(k)$ and $X_R(R)$. Hence, by the results in the previous section, k-rational arcs on X correspond to K-points on the generic fibre $\widehat{X}_\eta$ of $\widehat{X}$, by a canonical bijection

$$\varphi\colon \mathcal{L}(X)(k) \to \widehat{X}_\eta(K),$$

and the square

$$\begin{array}{ccc} \mathcal{L}(X)(k) & \xrightarrow{\varphi} & \widehat{X}_\eta(K) \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle sp} \\ X(k) & \xrightarrow{=} & X(k) \end{array}$$

commutes. So the rigid counterpart of the space $\mathcal{L}(X)_Z := \mathcal{L}(X) \times_X Z$ of arcs with origin in some closed subscheme Z of X, is the tube $]Z[$ of Z in $\widehat{X}_\eta$ (or rather, its set of K-rational points).

Of course, the scheme structure on $\mathcal{L}(X)$ is very different from the analytic structure on $\widehat{X}_\eta$. Nevertheless, the structure on $\widehat{X}_\eta$ seems to be much richer than the one on $\mathcal{L}(X)$, and one might hope that some essential properties of the non-Noetherian scheme $\mathcal{L}(X)$ are captured by the more "geometric" object $\widehat{X}_\eta$. Moreover, there exists a satisfactory theory of étale cohomology for rigid K-varieties (see for instance [3] or [27]), making it possible to apply cohomological techniques to the study of the arc space.

6.1.4 THE ANALYTIC MILNOR FIBRE. Let $g\colon \mathbf{C}^m \to \mathbf{C}$ be an analytic map; we denote by Y_0 the analytic space defined by $g = 0$. Let x be a point of Y_0. Consider an open disc $D := B(0, \eta)$ of radius η around the origin in $\mathbf{C}$, and an open disc $B := B(x, \varepsilon)$ in $\mathbf{C}^m$. We denote by $D^\times$ the punctured disc $D - \{0\}$, and we put

$$X' := B \cap g^{-1}(D^\times).$$

Then, for $0 < \eta \ll \varepsilon \ll 1$, the induced map

$$g' : X' \longrightarrow D^{\times}$$

is a C^{∞} locally trivial fibration, called the *Milnor fibration* of g at x. It is trivial if g is smooth at x. Its fibre at a point t of $D^{\times}$ is denoted by $F_x(t)$, and it is called the (topological) *Milnor fibre* of g at x (with respect to t). To remove the dependency on the base point, one constructs the *canonical Milnor fibre* F_x by considering the fibre product

$$F_x := X' \times_{D^{\times}} \widetilde{D^{\times}} ,$$

where $\widetilde{D^{\times}}$ is the universal covering space

$$\widetilde{D^{\times}} = \{z \in \mathbf{C} \mid \Im(z) > -\log \eta\} \to D^{\times} : z \mapsto \exp(iz) .$$

Since this covering space is contractible, F_x is homotopically equivalent to $F_x(t)$. The group of covering transformations $\pi_1(D^{\times})$ acts on the singular cohomology of F_x; the action of the canonical generator $z \mapsto z + 2\pi$ of $\pi_1(D^{\times})$ is called the *monodromy transformation* of g at x. The Milnor fibration g' was devised in [32] as a tool for gathering information on the topology of Y_0 near x.

We return to the algebraic setting: let k be an algebraically closed field of characteristic zero, let $R = k[[t]]$, let X be a smooth irreducible variety over k, and let $f : X \to \mathbf{A}^1_k = \operatorname{Spec} k[t]$ be a dominant morphism. As before, we denote by $\widehat{X}$ the formal t-adic completion of f, with generic fibre $\widehat{X}_{\eta}$. For any closed point x on X_0, we put $\mathcal{F}_x :=]x[$, and we call this rigid K-variety the *analytic Milnor fibre* of f at x. This object was introduced and studied in [33, 34]. We consider it as a bridge between the topological Milnor fibration and arc spaces; a tight connection between these data is predicted by the *motivic monodromy conjecture*. See [35] for more on this point of view.

The topological intuition behind the construction is the following: the formal neighbourhood $\operatorname{Spf} R$ of the origin in $\mathbf{A}^1_k = \operatorname{Spec} k[t]$ corresponds to an infinitesimally small disc around the origin in $\mathbf{C}$. Its inverse image under f is realized as the t-adic completion of the morphism f; the formal scheme $\widehat{X}$ should be seen as a tubular neighbourhood of the special fibre X_0 defined by f on X. The inverse image of the punctured disc becomes the "complement" of X_0 in $\widehat{X}$; this "complement" makes sense in the category of rigid spaces, and we obtain the generic fibre $\widehat{X}_{\eta}$ of $\widehat{X}$. The specialization map *sp* can be seen as a canonical "contraction" of $\widehat{X}_{\eta}$ on X_0, such that $\mathcal{F}_x$ corresponds to the topological space X' considered above. Note that this is not really the Milnor fibre yet: we had to base-change to a universal cover of $D^{\times}$, which corresponds to considering $\mathcal{F}_x \widehat{\times}_K \widehat{K^{\mathrm{alg}}}$ instead of $\mathcal{F}_x$, by the dictionary between finite covers of $D^\times$ and finite extensions of K. The monodromy action is translated into the Galois action of $G(K^{\mathrm{alg}}/K) \cong \widehat{\mathbf{Z}}(1)(k)$ on $\mathcal{F}_x \widehat{\times}_K \widehat{K^{\mathrm{alg}}}$.

It follows from the results in Section 6.1.2 that, for any integer $d > 0$, the points in $\mathcal{F}_x(K(d))$ correspond canonically to the arcs

$$\psi \colon \operatorname{Spec} k[[t]] \to X$$

satisfying $f(\psi) = t^d$ and $\pi_0(\psi) = x$. Moreover, by Berkovich's comparison result in [5, 3.5] (see also Section 6.3), there are canonical isomorphisms

$$H^i_{\text{ét}}(\mathcal{F}_x \widehat{\times}_K \widehat{K^{\mathrm{alg}}}, \mathbf{Q}_\ell) \cong R^i\psi_\eta(\mathbf{Q}_\ell)_x$$

such that the Galois action of $G(K^{\mathrm{alg}}/K)$ on the left-hand side corresponds to the monodromy action of $G(K^{\mathrm{alg}}/K)$ on the right. Here $H^*_{\text{ét}}$ is *étale ℓ-adic cohomology*, and $R\psi_\eta$ denotes the *ℓ-adic nearby cycle functor* associated to f. In particular, if $k = \mathbf{C}$, this implies that $H^i_{\text{ét}}(\mathcal{F}_x \widehat{\times}_K \widehat{K^{\mathrm{alg}}}, \mathbf{Q}_\ell)$ is canonically isomorphic to the singular cohomology $H^i_{\mathrm{sing}}(F_x, \mathbf{Q}_\ell)$ of the canonical Milnor fibre F_x of f at x, and that the action of the canonical topological generator of $G(K^{\mathrm{alg}}/K) = \widehat{\mathbf{Z}}(1)(\mathbf{C})$ corresponds to the monodromy transformation, by Deligne's classical comparison theorem for étale and analytic nearby cycles [1, XIV]. In view of the motivic monodromy conjecture, it is quite intriguing that $\mathcal{F}_x$ relates certain arc spaces to monodromy action; see [35] for more background on this perspective.

6.2 Deformation theory and lifting problems

Suppose that R has mixed characteristic, and let X_0 be a scheme of finite type over the residue field k. In [28, 5.1] Illusie sketches the following problem: is there a flat scheme X of finite type over R such that $X_0 = X \times_R k$? Grothendieck suggested the following approach: first, try to construct an inductive system X_n of flat R_n-schemes of finite type such that $X_n \cong X_m \times_{R_m} R_n$ for $m \geq n \geq 0$. In many situations, the obstructions to lifting X_n to X_{n+1} live in a certain cohomology group of X_0, and when these obstructions vanish, the isomorphism classes of possible X_{n+1} correspond to elements in another appropriate cohomology group of X_0. Once we find such an inductive system, its direct limit is a flat formal R-scheme X_∞, topologically of finite type. Next, we need to know if this formal scheme is *algebraizable*, i.e. if there exists an R-scheme X whose formal completion $\widehat{X}$ is isomorphic to X_∞. This scheme X would be a solution to our lifting problem. A useful criterion for proving the existence of X is the one quoted in Section 3.4: if X_0 is proper and carries an ample line bundle that lifts to a line bundle on X_∞, then X_∞ is algebraizable. Moreover, the algebraic model X is unique up to isomorphism by Grothendieck's Existence Theorem (Section 3.4). For more concrete applications of this approach, we refer to Section 5 of [28].

6.3 Nearby cycles for formal schemes

Berkovich used his étale cohomology theory for non-archimedean analytic spaces, developed in [3], to construct nearby and vanishing cycle functors for formal schemes [4, 5]. His formalism applies, in particular, to *stft* formal R-schemes X_∞ and to formal completions of such formal schemes along closed subschemes of the special fibre X_0. Let us denote by $R\psi_\eta$ the functor of nearby cycles, both in the algebraic and in the formal setting. Suppose that k is algebraically closed. Let X be a variety over R; we denote by $\widehat{X}$ its formal completion, with generic fibre $\widehat{X}_\eta$. Let Y be a closed subscheme of X_0, and let $\mathcal{F}$ be an étale constructible sheaf of abelian groups on $X \times_R K$, with torsion orders prime to the characteristic exponent of k. Then Berkovich associates to $\mathcal{F}$ in a canonical way an étale sheaf $\widehat{\mathcal{F}}$ on $\widehat{X}_\eta$ and an étale sheaf $\widehat{\mathcal{F}/Y}$ on the tube $]Y[$. His comparison theorem [5, 3.1] states that there are canonical quasi-isomorphisms

$$R\psi_\eta(\mathcal{F}) \cong R\psi_\eta(\widehat{\mathcal{F}}) \quad \text{and} \quad R\psi_\eta(\mathcal{F})|_Y \cong R\psi_\eta(\widehat{\mathcal{F}/Y}).$$

Moreover, by [5, 3.5] there is a canonical quasi-isomorphism

$$R\Gamma(Y, R\psi_\eta(\mathcal{F})|_Y) \cong R\Gamma(]Y[\widehat{\times}_K\widehat{K^s}, \widehat{\mathcal{F}/Y}).$$

In particular, if x is a closed point of X_0, then $R^i\psi_\eta(\mathbf{Q}_\ell)_x$ is canonically isomorphic to the i-th ℓ-adic cohomology space of the tube $]x[\widehat{\times}_K\widehat{K^s}$. Similar results hold for tame nearby cycles and vanishing cycles.

This proves a conjecture of Deligne's, stating that $R\psi_\eta(\mathcal{F})|_Y$ depends only on the formal completion of X along Y. In particular, the stalk of $R\psi_\eta(\mathcal{F})$ at a closed point x of X_0 depends only on the completed local ring $\widehat{\mathcal{O}}_{X,x}$.

6.4 Semi-stable reduction for curves

Bosch and Lütkebohmert show in [12, 11] how rigid geometry can be used to construct stable models for smooth projective curves over a non-archimedean field L, and uniformizations for Abelian varieties. Let us briefly sketch their approach to semi-stable reduction of curves.

If A is a reduced Tate algebra over L, then we define

$$A^o = \{f \in A \mid ||f||_{\sup} \leq 1\},$$
$$A^{oo} = \{f \in A \mid ||f||_{\sup} < 1\}.$$

Note that A^o is a subring of A, and that A^{oo} is an ideal in A^o. The quotient $\widetilde{A} := A^o/A^{oo}$ is a reduced algebra of finite type over $\widetilde{L}$, by [10, 1.2.5.7+6.3.4.3], and $\widetilde{X} := \operatorname{Spec}\widetilde{A}$ is called the *canonical reduction* of the affinoid space $X := \operatorname{Sp}A$. There is a natural reduction map $X \to \widetilde{X}$ mapping points of X to closed points of $\widetilde{X}$. The inverse image of a closed point x of $\widetilde{X}$ is called the *formal fibre* of X at x; it is an open rigid subspace of X.

Let C be a projective smooth geometrically connected curve over L of genus $g \geq 2$; we consider its analytification C^{an}. By a technical descent argument, we may assume that L is algebraically closed. The idea is to construct a finite admissible cover $\mathfrak{U}$ of C^{an} by affinoid domains U whose canonical reductions $\widetilde{U}$ are semi-stable. If the cover $\mathfrak{U}$ satisfies a certain compatibility property, the canonical reductions $\widetilde{U}$ can be glued to a semi-stable $\widetilde{L}$-variety. From this cover $\mathfrak{U}$ one constructs a stable model for C. The advantage of passing to the rigid world is that the Grothendieck topology on C^{an} is much finer than the Zariski topology on C, thus allowing finer patching techniques.

To construct the cover $\mathfrak{U}$, it is proved that smooth points and ordinary double points on $\widetilde{U}$ can be recognized by looking at their formal fibre in U. For instance, a closed point x of $\widetilde{U}$ is smooth if and only if its formal fibre is isomorphic to an open disc of radius 1. An alternative proof based on rigid geometry is given in [20, 5.6].

6.5 Constructing Étale covers, and Abhyankar's Conjecture

Formal and rigid patching techniques can also be used in the construction of Galois covers; see [26] for an introduction to this subject. This approach generalizes the classical Riemann Existence Theorem for complex curves to a broader class of base fields. Riemann's Existence Theorem states that, for any smooth connected complex curve X, there is an equivalence between the category of finite étale covers of X, the category of finite analytic covering spaces of the complex analytic space X^{an}, and the category of finite topological covering spaces of $X(\mathbf{C})$ (with respect to the complex topology). So the problem of constructing an étale cover is reduced to the problem of constructing a topological covering space, where we can proceed locally with respect to the complex topology and glue the resulting local covers. In particular, it can be shown in this way that any finite group is the Galois group of a finite Galois extension of $\mathbf{C}(x)$, by studying the ramified Galois covers of the complex projective line.

The strategy in rigid geometry is quite similar: given a smooth curve X over a non-archimedean field L, we consider its analytification X^{an}. We construct an étale cover Y' of X^{an} by constructing covers locally and glueing them to a rigid variety. Then we use a GAGA-theorem to show that Y' is algebraic, i.e. $Y' = Y^{\mathrm{an}}$ for some curve Y over L; Y is an étale cover of X. Of course, several technical complications have to be overcome to carry out this strategy.

We list some results that can be obtained by means of these techniques, and references to their proofs.

- (Harbater) For any finite group G, there exists a ramified Galois cover $f\colon X \to \mathbf{P}^1_L$ with Galois group G, such that X is absolutely irreducible, smooth, and projective, and such that there exists a point x in $X(L)$ at which f is unramified. An accessible proof is given by Q. Liu in [29]; see also [37, § 3].
- (Abhyankar's Conjecture for the projective line) Let k be an algebraically closed field of characteristic $p > 0$. A finite group G is the Galois group of a covering of $\mathbf{P}^1_k$, ramified only over ∞, if and only if G is generated by its elements of order p^n with $n \geq 1$. This conjecture was proved by Raynaud in [40]. This article also contains an introduction to rigid geometry and étale covers.
- (Abhyankar's Conjecture) Let k be an algebraically closed field of characteristic $p > 0$. Let X be a smooth connected projective curve over k of genus g, let $\xi_0, \ldots, \xi_r$ $(r \geq 0)$ be distinct closed points on X, and let $\Gamma_{g,r}$ be the topological fundamental group of a complex Riemann surface of genus g minus $r+1$ points (it is the free group on $2g+r$ generators). Put $U = X \setminus \{\xi_0, \ldots, \xi_r\}$. A finite group G is the Galois group of an unramified Galois cover of U if and only if every prime-to-p quotient of G is a quotient of $\Gamma_{g,r}$. This conjecture was proved by Harbater in [25].

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(Reçu le 27 février 2007)

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