Property (T) versus property FW

5

PROPERTY (T) VERSUS PROPERTY FW

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Recall (e.g. from de la Harpe and Valette's book [6] on property (T)) that a countable[1]) group G *has property* (T) if and only if every continuous affine action on a real Hilbert space has a global fixed point. Niblo and Reeves in [7] showed that, for a group satisfying Kazhdan's property (T), every cellular action on a finite dimensional CAT(0) cube complex has a global fixed point. We will look at the following

DEFINITION 5.1. A group G *has property* FW_n if every cellular action of G on every n-dimensional CAT(0) cube complex has a global fixed point. The group G *has property* FW if G has FW_n for all n.

So, according to Niblo and Reeves, if G has Kazhdan's property (T) then G has property FW. Note that the abbreviation FW stands for "fix" and "walls". Recall the following

DEFINITION 5.2 (Haglund and Paulin [5]). A *wall space* is a set Y together with a nonempty collection $\mathcal{H} \subseteq \mathcal{P}(Y)$ of *half-spaces* such that $h \in \mathcal{H} \implies h^c \in \mathcal{H}$ and $\#\{h \in \mathcal{H} : x \in h,\ y \in h^c\} < \infty$ for every $(x,y) \in Y \times Y$. A wall structure endows Y with a pseudo-metric (by counting how many walls separate two points) and yields a metric on a quotient of Y. An *action of a group* G *on the wall space* Y is an action of G on Y that preserves the wall structure, i.e. an action such that $g(h) \in \mathcal{H}$ for every $g \in G$ and $h \in \mathcal{H}$.

[1]) By [3], the assumption of countability cannot be omitted.

It turns out that acting on a wall space is very similar to acting isometrically on a CAT(0) cube complex. It is well known[2]) that an isometric action on a CAT(0) cube complex gives an isometric action on a wall space, and the converse holds as well, as shown in [8] and [1]. Moreover, the distance between a point x and gx is the same in the CAT(0) complex as in the corresponding wall space.

Recently Cherix, Martin, and Valette in [2] showed that a finitely generated group has property (T) if and only if every action on a space with *measured* walls has a global fixed point. A natural question, then, is the following

QUESTION 5.3. *Is* FW *equivalent to* (T), *or (as hinted by Cherix, Martin, and Valette on the second page of their paper) does there exist a group G such that G does not have property* (T), *but G and all its finite index subgroups have property* FW *?*

REMARK 5.4. According to Watatani in [10], groups with Kazhdan's property (T) are also known to have Serre's property FA, but many groups with FA do not have (T). The following generalization of property FA was introduced by Farb: A group is said to have *property* FA_n if every cellular action of the group on every n-dimensional CAT(0) (piecewise-Euclidean or piecewise-hyperbolic) complex has a global fixed point. In particular, FA_n implies FW_n. However, $\mathrm{SL}_m(\mathbf{Z}[\frac{1}{p}])$ has FA_{m-2} (see [4]) but $\mathrm{SL}_m(\mathbf{Z}[\frac{1}{p}])$ acts without a global fixed point on the Bruhat–Tits building for $\mathrm{SL}_m(\mathbf{Q}_p)$. Since this building is an $(m-1)$-dimensional CAT(0) complex, $\mathrm{SL}_m(\mathbf{Z}[\frac{1}{p}])$ does not have FA_{m-1}. Hence FA_n distinguishes between these property (T) groups whereas every property (T) group has FW_n for all n.

REFERENCES

[1] CHATTERJI, I. and G. NIBLO. From wall spaces to CAT(0) cube complexes. *Internat. J. Algebra Comput. 15* (2005), 875–885.

[2] CHERIX, P.-A., F. MARTIN and A. VALETTE. Spaces with measured walls, Haagerup property and property (T). *Ergodic Theory Dynam. Systems 24* (2004), 1895–1908.

[3] DE CORNULIER, Y. Strongly bounded groups and infinite powers of finite groups. *Comm. Algebra 34* (2006), 2337–2345.

[4] FARB, B. Group actions and Helly's theorem. (In preparation.)

[2]) It is unclear who first stated this result in the form given. However, it is implicit in [9].

[5] HAGLUND, F. et F. PAULIN. Simplicité de groupes d'automorphismes d'espaces à courbure négative. *Geom. Topol. Monogr. 1* (1998), Geom. Topol. Publ., Coventry, 181–248.

[6] DE LA HARPE, P. et A. VALETTE. *La propriété* (T) *de Kazhdan pour les groupes localement compacts. Astérisque 175*, Soc. Math. France, 1989.

[7] NIBLO, G. and L. REEVES. Groups acting on CAT(0) cube complexes. *Geom. Topol. 1* (1997), 1–7.

[8] NICA, B. Cubulating spaces with walls. *Algebr. Geom. Topol. 4* (2004), 297–309.

[9] SAGEEV, M. Ends of group pairs and non-positively curved cube complexes. *Proc. London Math. Soc. (3) 71* (1995), 585–617.

[10] WATATANI, Y. Property (T) of Kazhdan implies property FA of Serre. *Math. Japonica 27* (1982), 97–103.

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