

# Two CAT(0) group questions

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### TWO CAT(0) GROUP QUESTIONS

by Kim RUANE

We say a group  $G$  acts *geometrically* on a complete, proper, geodesic metric space  $X$  if  $G$  acts properly discontinuously and cocompactly by isometries on  $X$ . If  $G$  acts geometrically on a CAT(0) space, then  $G$  is called a CAT(0) *group*. Recall that if  $G$  acts geometrically on a  $\delta$ -hyperbolic metric space, then  $G$  is word hyperbolic.

For  $G$  word hyperbolic, the following facts are well known and can be found in [5]. Of course, many careful proofs have been written down in other places, most notably [1] and [4].

1. Any Cayley graph of  $G$  is  $\delta$ -hyperbolic using the corresponding word metric.
2. The *boundary* of  $G$ , denoted  $\partial G$ , is well-defined up to homeomorphism — i.e., if  $G$  acts geometrically on spaces  $X$  and  $Y$ , then  $X$  and  $Y$  are quasi-isometric and this quasi-isometry extends to an (equivariant) homeomorphism of boundaries  $\partial X \rightarrow \partial Y$ .
3.  $G$  acts as a convergence group on  $\partial G$ .
4.  $G$  satisfies the Tits Alternative.
5. Any finite index subgroup and any finite extension of  $G$  are again word hyperbolic.

The two questions here involve the last two facts listed above, but for CAT(0) groups as opposed to word hyperbolic groups. If  $G$  is word hyperbolic, it is easy to see that any finite index subgroup or any finite extension of  $G$  is again word hyperbolic. Indeed, any such group is quasi-isometric to  $G$  and thus inherits word hyperbolicity via the quasi-isometry.

If  $G$  is a CAT(0) group acting on a CAT(0) space  $X$  and  $H$  is a finite index subgroup of  $G$ , then  $H$  is again a CAT(0) group. Indeed, any subgroup of  $G$  will again act properly discontinuously and by isometries. Since  $H$  is

of finite index,  $H$  will also act cocompactly. But if  $K$  is a finite extension of  $G$ , then the question remains:

QUESTION 56.1. *Suppose  $K$  is a finite extension of a CAT(0) group  $G$ . Is  $K$  also a CAT(0) group?*

The main problem here is that there is no geometric construction that models the group theoretic finite extension. It is still the case that  $K$  and  $G$  are quasi-isometric groups, but there is no natural candidate for a CAT(0) space for  $K$  to act on. Well, that isn't quite true... there is a candidate. Suppose  $G$  is a finite index normal subgroup of  $K$  of index  $D$  and suppose  $G$  acts on a topological space  $X$ . A construction of Serre gives an action of  $K$  on the direct product of  $D$  copies of  $X$  (this construction can be found in [3]). In our setting, if  $G$  acts geometrically on  $X$ , then Serre's construction will produce a properly discontinuous and isometric action of  $K$  on the product of  $D$  copies of  $X$  (which is still CAT(0) using the product metric). The problem is finding a convex subspace on which  $K$  acts cocompactly.

The second question concerns the Tits Alternative for CAT(0) groups. Recall that a group  $G$  satisfies the *Tits Alternative* if for every subgroup  $H$  of  $G$ , either  $H$  is virtually solvable or  $H$  contains a free subgroup of rank 2.

If  $G$  is word hyperbolic, then  $G$  satisfies the Tits Alternative. This fact was first proved in [5]. The beauty of this result is that the proof is quite simple using the action of the group  $G$  on its boundary  $\partial G$ . The proof goes like this: suppose  $H$  is an infinite subgroup of  $G$  and consider the closure  $\bar{H}$  of  $H$  inside  $G \cup \partial G$ . The *limit set* of  $H$ , denoted  $\mathcal{L}(H)$ , is  $\bar{H} \cap \partial G$ . One first shows that  $|\mathcal{L}(H)| \geq 2$  — this follows from the fact that any infinite subgroup of  $G$  must contain an element of infinite order [7]. If  $|\mathcal{L}(H)| = 2$ , then  $H$  is virtually  $\mathbf{Z}$ . Otherwise, there must be two infinite order elements  $a, b \in H$  with  $\mathcal{L}(\langle a \rangle) \cap \mathcal{L}(\langle b \rangle) = \emptyset$ . Using the dynamics of the action on  $\partial G$ , one can do a ping-pong argument using carefully chosen open sets around the limit points of these two cyclic subgroups to show that powers of  $a$  and  $b$  generate an  $F_2$  in  $H$ .

For a CAT(0) group  $G$  acting on  $X$ , one could try to use the action of  $G$  on  $\partial X$ . However, this is not a convergence group action. For example, every element of  $\mathbf{Z} \oplus \mathbf{Z}$  acts trivially on  $\partial \mathbf{E}^2 \cong S^1$  which cannot happen in a convergence group action. The most recent result of interest here is from M. Sageev and D. Wise for groups acting properly on CAT(0) cube complexes, see [6]. If such a group  $G$  has a bound on the order of finite subgroups then any subgroup either contains  $F_2$  or is virtually a finitely generated abelian

subgroup. If  $G$  does not have a bound on the order of finite subgroups, then the conclusion does not hold. Thus the following general question remains open :

QUESTION 56.2. *Does the Tits Alternative hold for  $G$  if  $G$  is a CAT(0) group ?*

This question is still open even if  $G$  admits a geometric action on a CAT(0) manifold  $X$ .

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