

The Kropholler conjecture

Autor(en): **Niblo, Graham A. / Sageev, Michah**

Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **54 (2008)**

Heft 1-2

PDF erstellt am: **26.04.2024**

Persistenter Link: <https://doi.org/10.5169/seals-109922>

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

53

THE KROPHOLLER CONJECTURE

by Graham A. NIBLO and Michah SAGEEV

A finitely generated group G is said to *split over a subgroup* H if and only if G may be decomposed as an amalgamated free product $G = A *_H B$ (with $A \neq H \neq B$) or as an HNN extension $G = A *_H$. The Kropholler conjecture is concerned with the existence of such splittings.

Given a subgroup H of a finitely generated group G the *invariant* $e(G, H)$ is defined to be the number of *Freudenthal (topological) ends* of the quotient of the Cayley graph of G under the action of the subgroup H . This number does not depend on the (finite) generating set chosen for G (see [3]) so it is an invariant of the pair (G, H) .

For example, if G is a free abelian group and H is an infinite cyclic subgroup then $e(G, H) = 0$ if G has rank 1, $e(G, H) = 2$ if G has rank 2 and $e(G, H) = 1$ if G has rank greater than or equal to 3.

This invariant generalises Stallings' definition of the number of ends of the group G since if $H = \{1\}$ then $e(G, H) = e(G)$.

In [4] Stallings showed that the group G splits over some finite subgroup C if and only if $e(G) \geq 2$. There are several important generalisations of this fact, the most wide ranging being the algebraic torus theorem, established by Dunwoody and Swenson [1]. This states that, under suitable additional hypotheses, if G contains a polycyclic-by-finite subgroup H of Hirsch length n with $e(G, H) \geq 2$ then either

- (1) G is virtually polycyclic of Hirsch length $n + 1$,
- (2) G splits over a virtually polycyclic subgroup of Hirsch length n , or
- (3) G is an extension of a virtually polycyclic group of Hirsch length $n - 1$ by a Fuchsian group.

This theorem generalises the classical torus theorem from low-dimensional topology which asserts that a closed 3-manifold which admits an immersed incompressible torus either admits an embedded incompressible torus or has a Seifert fibration. These topological conclusions imply the algebraic conclusions for the fundamental group of the manifold.

An important ingredient of the proof of the algebraic torus theorem is a special case of the so-called Kropholler conjecture. Its original formulation relies on the following observation of Scott:

A subgroup H of a finitely generated group G satisfies $e(G, H) \geq 2$ if and only if G admits a subset A satisfying the following:

- (1) $A = HA$,
- (2) A is H -almost invariant, and
- (3) A is H -proper, i.e., neither A nor $G - A$ is H -finite.

We will refer to the subset A as a *proper H -almost invariant subset*.

In his proof of the algebraic torus theorem for Poincaré duality groups Kropholler observed that, under certain additional hypotheses, if G admits a proper H -almost invariant subset A such that $A = AH$, then G admits a splitting over some subgroup $C < G$ related to H (see [2] for an outline of the proof). He conjectured that the additional hypotheses were inessential. Specifically:

CONJECTURE 53.1 (The Kropholler conjecture). *Let G be a finitely generated group and $H < G$. If G contains a proper H -almost invariant subset A such that $A = AH$ then G admits a non-trivial splitting over a subgroup C which is commensurable with a subgroup of H .*

The conjecture is known to hold when G is a Poincaré duality group or when G is word hyperbolic and H is a quasi-convex subgroup. In general it is known (for an arbitrary finitely generated group G) whenever H is a subgroup which satisfies the following descending chain condition:

Every descending chain of subgroups $H = H_0 \geq H_1 \geq H_2 \geq \dots$ such that H_{i+1} has infinite index in H_i eventually terminates.

This condition holds for example for the class of finitely generated polycyclic groups, in which class the Hirsch length is the factor controlling the length of such a chain. This is a key ingredient in the proof of the full algebraic torus theorem.

An alternative, more geometric, point of view on the conjecture is provided by the following characterisation:

THEOREM 53.2. *Given a finitely generated group G and a subgroup $H < G$ the invariant $e(G, H)$ is greater than or equal to 2 if and only if G acts with no global fixed point on a $CAT(0)$ cubical complex with one orbit of hyperplanes, and so that H is a hyperplane stabiliser. H admits a right invariant, proper H -almost invariant subset if and only if the action can be chosen so that H has a fixed point in the complex.*

REFERENCES

- [1] DUNWOODY, M.J. and E.L. SWENSON. The algebraic torus theorem, *Invent. Math.* 140 (2000), 605–637.
- [2] KROPHOLLER, P.H. A group theoretic proof of the torus theorem. In: *Geometric Group Theory, Sussex 1991, Vol. 1*, (eds. G.A. Niblo and M.A. Roller), 138–158. London Math. Soc. Lecture Note Ser. 181. Cambridge Univ. Press, Cambridge, 1993.
- [3] SCOTT, G.P. Ends of pairs of groups. *J. Pure Appl. Algebra* 11 (1977), 179–198.
- [4] STALLINGS, J.R. On torsion-free groups with infinitely many ends. *Ann. of Math.* 88 (1968), 312–334.

G.A. Niblo

School of Mathematics
University of Southampton
Southampton SO17 1BJ
United Kingdom
e-mail: g.a.niblo@soton.ac.uk

M. Sageev

Technion
Israel Institute of Technology
Haifa 32000
Israel
e-mail: sageevm@techunix.technion.ac.il