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Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **54 (2008)**

Heft 1-2

PDF erstellt am: **26.04.2024**

Persistenter Link: https://doi.org/10.5169/seals-109916

L'Enseignement Mathématique (2) 54 (2008), 135–137

47

ISOPERIMETRIC INEQUALITIES AND THE ASYMPTOTIC GEOMETRY OF HADAMARD SPACES

by Urs LANG and Stefan WENGER

The conjecture we describe here deals with isoperimetric fillings of k-cycles in a proper cocompact CAT(0)-space X, where it is assumed that k is greater than or equal to the *Euclidean rank* of X, i.e. the maximal $n \in \mathbf{N}$ for which $\mathbf{R}^n$ isometrically embeds into X. In order to state the conjecture let us fix the following notation. Given a complete metric space X and $k \in \mathbf{N}$ we define the *filling volume function* FV_{k+1} of X by

$$FV_{k+1}(s) := \sup\{\mathrm{FillVol}(T) : T \text{ is a } k\text{-cycle in } X \text{ with } \mathrm{Vol}(T) \leq s\},$$

where FillVol(T) is the least volume of a $(k+1)$-chain with boundary T. In this generality, a suitable chain complex is provided by the metric integral currents introduced by Ambrosio and Kirchheim in [1]. Alternatively, one may work with a simplicial approximation (e.g. a Rips complex) of X and then use Lipschitz chains or simplicial chains.

In his seminal paper [2] Gromov proved that every *Hadamard manifold*, i.e. complete simply-connected Riemannian manifold of non-positive sectional curvature, admits a Euclidean isoperimetric inequality for k-cycles for every $k \geq 1$, thus

$$FV_{k+1}(s) \leq Cs^{\frac{k+1}{k}}$$

for all $s \geq 0$ and for some constant C. More generally, this holds true for CAT(0)-spaces, and even for metric spaces admitting cone type inequalities for l-cycles, $l = 1, \dots, k$, as was shown by Wenger in [7]. The latter property is shared for example by all geodesic metric spaces with convex distance function and all Banach spaces.

If X is a CAT(κ)-space with $\kappa < 0$, i.e. has a strictly negative upper curvature bound, then it is not difficult to show, see [8], that X admits a linear isoperimetric inequality for k-cycles for every $k \geq 1$, i.e.

$$FV_{k+1}(s) \leq Cs$$

for all s and for some constant C.

Now, one of the rough guiding principles in the theory of non-positively curved spaces is that their asymptotic geometry should exhibit hyperbolic behavior in the dimensions above the rank. The following conjecture appears, though somewhat implicitly, in Gromov's book [4].

CONJECTURE 47.1. *Every proper cocompact* CAT(0)*-space X of Euclidean rank r admits a linear isoperimetric inequality for k-cycles for every $k \geq r$.*

Instead of assuming X to be proper, cocompact and of Euclidean rank $\leq k$, one may also look at the larger class of CAT(0)-spaces all of whose asymptotic cones have geometric dimension at most k. For a proper cocompact CAT(0)-space X, the Euclidean rank r equals 1 if and only if X is hyperbolic in the sense of Gromov. Then, for $k = 1$, a linear isoperimetric inequality holds, as is well known, see [3]. More generally, in geodesic Gromov hyperbolic spaces satisfying suitable conditions on the geometry on small scales (not necessarily CAT(0)), linear isoperimetric inequalities for k-cycles hold for all $k \geq 1$. This was shown, in a simplicial setup, by Lang in [5]. In particular, the conjecture holds in the case $r = 1$, as follows from [5] and the Lipschitz extension results of [6].

As for the case $r > 1$, the conjecture is known to hold for symmetric spaces of non-compact type. In fact, if X is a symmetric space of non-compact type and $F \subset X$ is a maximal flat of dimension r, the orthogonal projection onto F decreases r-dimensional volume exponentially with the distance from the flat. This can be used to produce fillings with a linear volume bound.

A consequence of the above conjecture would be that isoperimetric inequalities detect the Euclidean rank. This also follows from the following result, which has recently been proved by Wenger in [9]: Let $k \in \mathbf{N}$ and let X be a quasiconvex metric space admitting cone type inequalities for l-cycles for $l = 1, \ldots, k$. Then X admits a 'sub-Euclidean' isoperimetric inequality for k-cycles, i.e.

$$\limsup_{s \to \infty} \frac{FV_{k+1}(s)}{s^{\frac{k+1}{k}}} = 0,$$

if and only if every asymptotic cone of X has dimension at most k. As it stands the conjecture remains open for most cases even in the context of Hadamard manifolds.

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