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RELATIVE COMPLETIONS OF LINEAR GROUPS

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Here is a question that I've thought about a lot, but I can't seem to solve. The classical Malcev completion of a group is well known. It has a universal mapping property that allows one to generalize the definition as follows. Let k be a field and let G be a group. The *unipotent k-completion* of G is a prounipotent k-group $\mathcal{U}$ that is universal among such groups admitting a map from G. The Malcev completion is the case $k = \mathbf{Q}$.

One possible problem with this construction is that it might be trivial; that is, the group $\mathcal{U}$ may consist of a single element. This happens, for example, when $H_1(G,k) = 0$. To get around this, there is a generalization (due to Deligne) called the *relative completion*. The set-up is the following. Suppose G is a discrete group and that $\rho \colon G \to S$ is a representation of G in a semisimple algebraic k-group S. Assume that the image of ρ is Zariski dense. The *completion of G relative to ρ* is a proalgebraic k-group $\mathcal{G}$ that is an extension of S by a prounipotent k-group $\mathcal{U}$:

$$1 \longrightarrow \mathcal{U} \longrightarrow \mathcal{G} \longrightarrow S \longrightarrow 1\,,$$

along with a lift $\tilde{\rho} \colon G \to \mathcal{G}$ of ρ. The group $\mathcal{G}$ should satisfy the obvious universal mapping property. If S is the trivial group, then this reduces to the unipotent completion. Full details about this construction may be found in [1], [2].

Consider the group $G = \mathrm{SL}_n(k[t])$ with the map $\rho \colon \mathrm{SL}_n(k[t]) \to \mathrm{SL}_n(k)$ induced by setting $t = 0$.

QUESTION 43.1. *What is the completion of G relative to ρ?*

There is an obvious guess, namely the group $\mathrm{SL}_n(k[[T]])$, and this turns out to be correct sometimes.

I proved this when k is a number field or a finite field, and $n \geq 3$ [2]. The proof goes like this. Let K be the kernel of ρ; this is the *congruence subgroup of the ideal* (t). Filter K by powers of (t): $K^i = \{A \in K : A \equiv I \mod t^i\}$. Then it is easy to see that for each i, $K^i/K^{i+1} \cong \mathfrak{sl}_n(k)$. Moreover, the filtration $K^\bullet$ turns out to be the lower central series in this case, and so it follows that the unipotent k-completion of K is $\varprojlim K/K^i = \ker\{\mathrm{SL}_n(k[[T]]) \xrightarrow{T=0} \mathrm{SL}_n(k)\}$. General properties of the relative completion (e.g., it is always a *split* extension) then imply that the correct answer is $\mathrm{SL}_n(k[[T]])$.

This approach fails for other fields though. Here's why. Denote the lower central series of K by $\Gamma^\bullet$. For any field, there is a short exact sequence

$$1 \longrightarrow K^2/\Gamma^2 \longrightarrow H_1(K, \mathbf{Z}) \longrightarrow K/K^2 \longrightarrow 1 .$$

The last group is $\mathfrak{sl}_n(k)$, and most of the time, the kernel K^2/Γ^2 surjects onto the module $\Omega^1_{k/\mathbf{Z}}$ [4]. Recall that this is the k-module generated by symbols df, where the f range over k, subject to the relations $d(fg) = f\,dg + g\,df$ for $f, g \in k$, and $dr = 0$ for $r \in \mathbf{Z}$ (here, we mean the image of r under the map $\mathbf{Z} \to k$). For finite fields and number fields, this is no obstruction since it is easily seen that $\Omega^1_{k/\mathbf{Z}} = 0$, but for $k = \mathbf{C}$, for example, we see that K^2/Γ^2 is very large. So $K^\bullet$ differs wildly from $\Gamma^\bullet$ and it is therefore not easy to compute the unipotent completion of K.

Still, I conjecture that $\mathrm{SL}_n(k[[T]])$ is the correct answer all the time. In fact, I make the following, more ambitious, conjecture.

CONJECTURE 43.2. *Let k be a field and let C be a smooth affine curve over k. Denote the coordinate ring of C by A and assume that C has a k-rational point with associated maximal ideal $\mathfrak{m} \subset A$. Let $\rho\colon \mathrm{SL}_n(A) \to \mathrm{SL}_n(k)$ be induced by the isomorphism $A/\mathfrak{m} \to k$. Finally, let $\widehat{A}$ be the $\mathfrak{m}$-adic completion of A. Then the completion of $\mathrm{SL}_n(A)$ relative to ρ is the group $\mathrm{SL}_n(\widehat{A})$.*

I proved [2] that this is true if we replace A by the localization of A at $\mathfrak{m}$. And, not surprisingly, it is true when k is a number field [3].

REFERENCES

[1] HAIN, R. M. Completions of mapping class groups and the cycle $C - C^-$. In: *Mapping Class Groups and Moduli Spaces of Riemann Surfaces (Göttingen, 1991 and Seattle, WA, 1991)*, 75–105. Contemp. Math. *150.* Amer. Math. Soc., 1993.

[2] KNUDSON, K. P. Relative completions and the cohomology of linear groups over local rings. *J. London Math. Soc. (2) 65* (2002), 183–203.

[3] —— Relative completions and K_2 of curves. Preprint arXiv: math.KT/0502553 (2005).

[4] KRUSEMEYER, M. I. Fundamental groups, algebraic K-theory, and a problem of Abhyankar. *Invent. Math. 19* (1973), 15–47.

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