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A HOMOTOPIC INTERSECTION THEORY ON SURFACES: APPLICATIONS TO MAPPING CLASS GROUP AND BRAIDS

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§0 INTRODUCTION

(0.1) Let $S_{g,b}$ be a compact, connected, oriented surface of genus g, with b boundary components ($b \geq 1$). Denote by $S_{g,b,n}$ ($n \geq 0$) the surface $S_{g,b}$ with n points $\{P_1, P_2, \ldots, P_n\}$ removed in the interior of $S_{g,b}$. Fix a base point s_0 in one of the boundary components and set $\Gamma = \pi_1(S_{g,b,n}\,;\, s_0)$. Thus Γ is a free group with $m = 2g + n + b - 1$ generators. Denote by H the abelianization of Γ. Denote by $\mathcal{M}_{g,b,n}$ the mapping class group of the surface $S_{g,b,n}$, i.e. the group of isotopy classes of homeomorphisms of $S_{g,b,n}$, equal to identity on the boundary and keeping invariant the set of points $\{P_1, P_2, \ldots, P_n\}$.

(0.2) There is a well-known integral, skew-symmetric bilinear form on H, denoted by $\langle\ ,\ \rangle$, defined as follows: let $\alpha, \beta \in H$ be represented by immersed, oriented loops with only transversal self-intersection points. Put α, β in general position. Then define $\langle \alpha, \beta \rangle$ by

$$\langle \alpha, \beta \rangle = \sum_{P \in \alpha \pitchfork \beta} \varepsilon_P\,,$$

where the sum is taken over all intersection points P and ε_P is equal to $+1$ (resp. -1) if the framing $(\overrightarrow{T_P\alpha}, \overrightarrow{T_P\beta})$ gives the right (resp. opposite) orientation of $S_{g,b,n}$, where $\overrightarrow{T_P\alpha}$ denotes a tangent vector to α at P, with the orientation given by α.

(0.3) Let $\mathbf{Z}[\Gamma]$ be the group ring of Γ with integer coefficients: this is the free abelian group generated by Γ, with the multiplication inherited from that of Γ. Let $e\colon \mathbf{Z}[\Gamma] \longrightarrow \mathbf{Z}$ be the canonical homomorphism defined by

$$e\Big(\sum_i n_i g_i\Big) = \sum_i n_i .$$

Then the main result of this paper is

THEOREM 0.1. *There exists a map* $\omega\colon \Gamma \times \Gamma \longrightarrow \mathbf{Z}[\Gamma]$ *(the homotopic intersection form) making the following diagram commutative:*

$$\begin{tikzcd} \Gamma \times \Gamma \arrow[r, "\omega"] \arrow[d] & \mathbf{Z}[\Gamma] \arrow[d, "e"] \\ H \times H \arrow[r, "\langle\,,\rangle"] & \mathbf{Z} \end{tikzcd}$$

and such that:

1) $\omega(y,x) = -\overline{\omega(x,y)} + (y-1)(x^{-1} - 1)$,

2) $\omega(xy,z) = \omega(x,z) + x\omega(y,z)$,

3) $\omega(x,yz) = \omega(x,y) + \omega(x,z)y^{-1}$,

where $\overline{(\ )}\colon \mathbf{Z}[\Gamma] \longrightarrow \mathbf{Z}[\Gamma]$ *is the anti-isomorphism given by* $\overline{\sum n_i\, g_i} = \sum n_i\, g_i^{-1}$.

REMARK. ω can be extended by bilinearity to $\mathbf{Z}[\Gamma] \times \mathbf{Z}[\Gamma]$.

(0.4) In the terminology of ([Pap], §4), ω is a biderivation on $\mathbf{Z}[\Gamma]$, using points 2) and 3) of Theorem 0.1.

Given a free basis $(z_1, \ldots, z_m)$ of Γ, we associate to ω a $m \times m$ matrix Λ with coefficients in $\mathbf{Z}[\Gamma]$, whose (i,j) entry is $\omega(z_i, z_j)$. Standard arguments of Fox free differential calculus show that

$$\omega(x,y) = \partial x^t \times \Lambda \times \overline{\partial y}\,,$$

where ∂x is the column $\Big(\dfrac{\partial x}{\partial z_1}, \ldots, \dfrac{\partial x}{\partial z_m}\Big)^t$. (Here $\dfrac{\partial}{\partial z_i}$ denotes the Fox partial derivative; see [F].)

(0.5) In a suitable free basis, the above matrix Λ takes a particularly simple form in the case of $S_{g,1,0}$ (see Lemma 2.4) and in the case of $S_{0,1,n}\,(= D^2 - \{P_1, \ldots, P_n\})$ (see Lemma 2.5).

(0.6) The main interest of Theorem 0.1 is perhaps in its applications. It allows us to reprove very quickly and simply some classical results and to prove some new ones.

Here are some of these applications.

(0.7) APPLICATION A). The intersection form ω is closely related to the Reidemeister pairing (see [Pap] or [H]).

Let S be a compact, connected, oriented surface, s_0 a base point and N a normal subgroup of $\Gamma = \pi_1(S\,;\,s_0)$. Denote by T the quotient Γ/N and let $\chi\colon \Gamma \longrightarrow T$ be the canonical map. Let $\widetilde{S}$ be the regular covering of S corresponding to N.

(0.8) The Reidemeister pairing is the bilinear map

$$\phi\colon H_1(\widetilde{S}\,;\,\mathbf{Z}) \times H_1(\widetilde{S}\,;\,\mathbf{Z}) \longrightarrow \mathbf{Z}[T]$$

defined by

$$\phi(\alpha,\beta) = \sum_{t\in T} \langle \alpha, t.\beta\rangle t\,,$$

where $t.\beta$ denotes the action of $t \in T$ on the 1-chain β and $\langle\ ,\ \rangle$ is the usual algebraic intersection form on $\widetilde{S}$.

Denote by Φ the composition

$$N \times N \longrightarrow H_1(\widetilde{S}\,;\,\mathbf{Z}) \times H_1(\widetilde{S}\,;\,\mathbf{Z}) \xrightarrow{\phi} \mathbf{Z}[T]\,.$$

(0.9) Then elementary properties of coverings show that (Lemma 2.6)

$$\Phi = \chi \circ \omega \mid N \times N\,.$$

Using (0.4) and (0.5) we immediately get the fundamental formula of ([Pap], Theorem 10.13; see also [H], Theorem 3.3),

$$(*) \qquad \Phi(u,v) = \chi(\partial u^t \times \Lambda \times \partial \overline{v}) \text{ for } u,v \in N.$$

From this formula $(*)$, Papakyriakopoulos obtains the main result of ([Pap], Theorem 11.1), which is a necessary and sufficient condition for the covering $\widetilde{S}$ to be planar.

(0.10) APPLICATION B). From the very definition of the form ω on $S_{g,b,n}$, we have for any (isotopy class of) homeomorphism f of $S_{g,b,n}$, equal to identity on ∂S and permuting the points $\{P_1,\dots,P_n\}$:

$$(**) \qquad \omega(f(x),f(y)) = f\omega(x,y) \text{ for any } x,y \in \Gamma = \pi_1(S_{g,b,n}\,;\,s_0)$$

(f is identified with the isomorphism induced on Γ).

(0.11) Given a free basis $\{z_i\,;\, i = 1, \dots, m\}$ of Γ, define the Fox matrix $B(f)$ of f by

$$\begin{array}{cc} & j \\ i & \left(\begin{array}{ccc} & \vdots & \\ \cdots & \overline{\dfrac{\partial f(z_j)}{\partial z_i}} & \cdots \\ & \vdots & \end{array} \right) \end{array} =: B(f) \in GL_m(\mathbf{Z}[\Gamma])\,.$$

(0.12) In the case of $S_{g,1,0}$, for the free basis given in Lemma 2.4, the relation $(**)$ above translates into matrix language using (0.4) as

$$^t\overline{B(f)} \times \Lambda \times B(f) = {}^f\Lambda\,.$$

We thus recover almost tautologically Theorem 5.3 of [Mo$_1$]. This shows the "symplectic" character of the Fox matrix of the elements of the mapping class group $\mathcal{M}_{g,1}$ of $S_{g,1}$ ($\mathcal{M}_{g,1}$ stands for $\mathcal{M}_{g,1,0}$).

(0.13) Application C). Applying the abelianization homomorphism

$$\Gamma = \pi_1(S_{g,1}\,;\, s_0) \longrightarrow H = H_1(S_{g,1}\,;\, \mathbf{Z})\,,$$

the Fox matrix $B(f)$ of $f \in \mathcal{M}_{g,1}$ becomes a matrix $B^{ab}(f) \in GL_{2g}(\mathbf{Z}[H])$. It is easy to see that the map

$$B^{ab} \colon \mathcal{M}_{g,1} \longrightarrow GL_{2g}(\mathbf{Z}[H])\,,$$

when restricted to the Torelli subgroup $\mathcal{I}_{g,1}$ of $\mathcal{M}_{g,1}$, is a homomorphism (recall that $f \in \mathcal{I}_{g,1}$ if f induces identity at the homological level).

(0.14) S. Morita ([Mo$_1$], problem 6.23) asks whether

$$B^{ab} \colon \mathcal{I}_{g,1} \longrightarrow GL_{2g}(\mathbf{Z}[H])$$

is injective or not.

In [Su], Suzuki exhibits an element of $\mathcal{I}_{g,1}$ in the kernel of B^{ab}, using lengthy computations.

We produce here, using the form ω, a geometric way of obtaining elements in the kernel of B^{ab}, explaining geometrically, without computation why Suzuki's example works.

Moreover we obtain many more explicit elements in the kernel of B^{ab}.

(0.15) APPLICATION D). Observe that the mapping class group $\mathcal{M}_{0,1,n}$ is the usual braid group B_n with n strings (see [Bi]).

For $\sigma \in B_n$, we define as above the matrix $B^{ab}(\sigma) \in GL_n(\mathbf{Z}[H])$, where $H = H_1(D^2 - \{P_1, \dots, P_n\}\,; \mathbf{Z})$. This matrix is known as the Gassner matrix of σ. The map $B^{ab}\colon B_n \longrightarrow GL_n(\mathbf{Z}[H])$ becomes a (true) homomorphism, when restricted to the pure braid group $P_n \subset B_n$.

(0.16) Translated into matrix language, the relation (**) of (0.10) above becomes

$$\overline{B^{ab}(\sigma)}^{\,t} \times \Omega_n \times B^{ab}(\sigma) = \Omega_n$$

for any $\sigma \in P_n \subset B_n$, where Ω_n is the following matrix of $GL_n(\mathbf{Z}[H])$, mentioned in (0.5):

$$\Omega_n = \begin{pmatrix} 1-u_1^{-1} & 0 & \dots & 0 & \dots & 0 \\ (1-u_2)(1-u_1^{-1}) & 1-u_2^{-1} & & 0 & & 0 \\ \vdots & (1-u_3)(1-u_2^{-1}) & & & \dots & 0 \\ \vdots & \vdots & & & \ddots & \vdots \\ (1-u_n)(1-u_1^{-1}) & (1-u_n)(1-u_2^{-1}) & \dots & \dots & & 1-u_n^{-1} \end{pmatrix}$$

(0.17) Under the mapping $H \longrightarrow \mathbf{Z} = \langle t \rangle$ sending each generator $u_i (i = 1, \dots, n)$ of H onto t, the Gassner matrix of $\sigma \in B_n$ becomes the (unreduced) Burau matrix $Bu(\sigma)$ (see [Bi], Chapter 3). The matrix Ω_n of (0.16) becomes, after simplification by $(1-t^{-1})$,

$$\widetilde{\Omega}_n = \begin{pmatrix} 1 & 0 & & 0 & \dots & 0 \\ (1-t) & 1 & & 0 & \dots & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & (1-t) & & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \vdots \\ (1-t) & (1-t) & \dots & (1-t) & \dots & 1 \end{pmatrix}$$

(0.18) REMARK. Our Gassner and Burau matrices are those of [Bi], up to transposition.

(0.19) By considering the "hermitian" matrix $\Omega_n + \overline{\Omega}_n^t$ (resp. $\widetilde{\Omega}_n + \overline{\widetilde{\Omega}}_n^t$), and sending $u_i (i = 1, \dots, n)$ (resp. t) to appropriate complex numbers of the unit circle, we can see that the group of Gassner (resp. Burau) matrices is conjugated to a subgroup of the unitary group $U_n(\mathbf{C})$.

In the case of Burau, this was first proved by Squier [Sq]. But our matrix $\widetilde{\Omega}_n$ is much simpler than that of Squier: $\widetilde{\Omega}_n$ is triangular and belongs to $GL_n(\mathbf{Z}[t, t^{-1}])$, instead of $GL_n(\mathbf{Z}[t^{\pm 1/2}])$.

(0.20) The fact that Ω_n (resp. $\widetilde{\Omega}_n$) is triangular imposes strong constraints on a matrix to be a Gassner (resp. Burau) matrix. In the case of Burau, these contraints are stronger that the one imposed by Squier. For example, we have a generalization of Theorem 1.1 of [LP].

PROPOSITION. *Let B be the Gassner (resp. Burau) matrix of some $\sigma \in B_n$ such that*

$$B = \begin{pmatrix} a_1 & & ? & & ? & ? & ? & ? & ? \\ 0 & \ddots & & & & & & & \\ \vdots & & \ddots & & & & & & \\ \vdots & & & \ddots & & & & & \\ 0 & & \vdots & & a_p & ? & ? & ? & ? \\ \vdots & & \vdots & & \vdots & & & & \\ \vdots & & \vdots & & \vdots & & A_{n-p} & & \\ \vdots & & \vdots & & \vdots & & & & \\ 0 & & 0 & & 0 & & & & \end{pmatrix}$$

Then B must be equal to

$$\begin{pmatrix} 1 & & & 0 & 0 & 0 \\ \vdots & \ddots & & & & \\ 0 & & 1 & 0 & 0 & 0 \\ \vdots & & 0 & & & \\ \vdots & & \vdots & & A_{n-p} & \\ 0 & & 0 & & & \end{pmatrix}$$

§ 1 DEFINITION OF THE INTERSECTION FORM ω

We are going to prove Theorem 0.1.

(1.1) Let $x, y \in \Gamma = \pi_1(S_{g,b,n}\,;\, s_0)$ be represented by oriented immersed closed curves with normal crossings, based at $s_0 \in \partial S_{g,b,n}$.

Push a little bit the base point s_0 into a point $s_0' \in \partial S_{g,b,n}$, in the positive direction of $\partial S_{g,b,n}$ (oriented like the boundary of $S_{g,b,n}$). Push also a little bit the curve y onto a curve y' based at s_0' (see Figure 1). Put x and y' in general position and define $\omega(x, y)$ by

$$\omega(x, y) = \sum_{P \in x \pitchfork y'} \varepsilon_P \; g_P \in \mathbf{Z}[\Gamma]\,,$$

where ε_P was defined in (0.2) and $g_P \in \Gamma$ is defined as follows: go from s_0 to P along x in the positive direction of x, then go from P to s_0' along y' in the negative direction, then go from s_0' to s_0 along $\partial S_{g,n,b}$ in the negative direction of $\partial S_{g,n,b}$ (Figure 2).

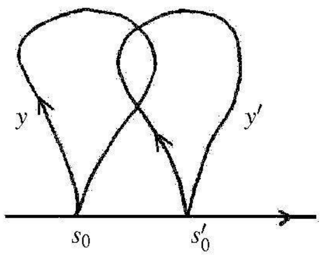


FIGURE 1

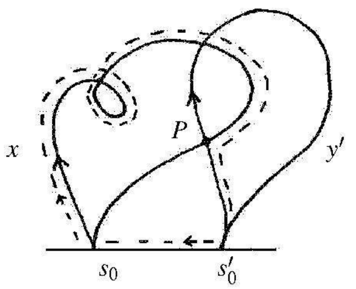


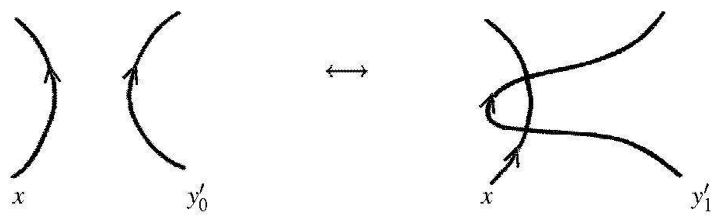
FIGURE 2

(g_P is represented in Figure 2 by the oriented dotted line)

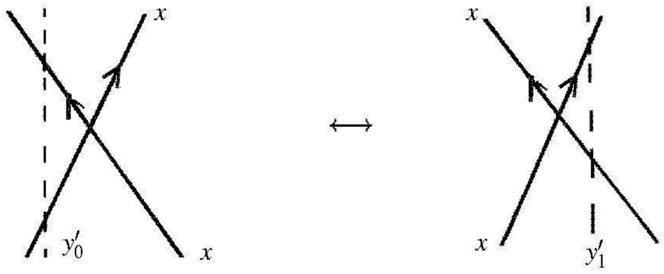
(1.2) We first have to prove that ω is well-defined, that is depends only on the homotopy classes of x and y. For this, fix x and let y_0, y_1 be two oriented immersed curves with normal crossings which are homotopic in $S_{g,b,n}$. Deform y_0 and y_1 onto y_0' and y_1' as above.

Then it is easy to see that we can pass from y_0' to y_1' by finite compositions of three types of elementary moves, with respect to x:

Move I



Move II



Move III: Regular homotopy from y'_0 to y'_1 far from x.

It is then easy to see that each of these three operations does not affect the form $\omega(x, y)$.

(1.3) To prove point 1) of Theorem 0.1, we proceed as follows.

To compute $\omega(x, y)$, we use x and y' of Figure 3. To compute $\omega(y, x)$, we can use y'' and x of Figure 3.

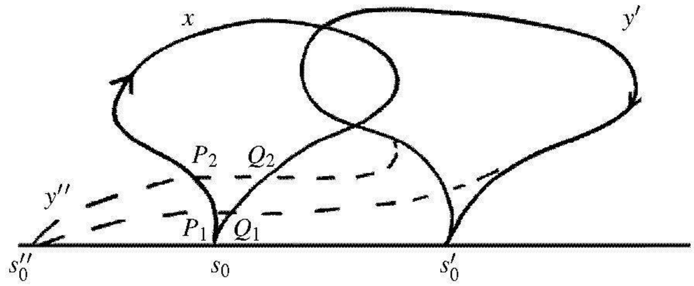


FIGURE 3

This amounts to introducing four extra intersection points between x and y'', P_1, P_2, Q_1, Q_2. For the intersection points P of $x \cap y'$ the contribution in $\omega(x, y)$ is $\varepsilon_P \;\; g_P$, while in $\omega(y, x)$ it is $-\varepsilon_P \;\; g_P^{-1}$. In the configuration of Figure 3 we have for the four extra points of $y'' \pitchfork x$:

$$\varepsilon_{P_1} = -1 = -\varepsilon_{P_2}$$
$$\varepsilon_{Q_1} = 1 = -\varepsilon_{Q_2}$$
$$g_{P_1} = y; \;\; g_{P_2} = 1$$
$$g_{Q_1} = yx^{-1}; \;\; g_{Q_2} = x^{-1}.$$

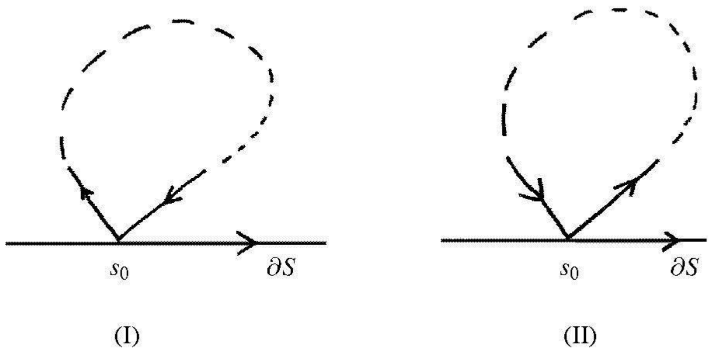
So the contribution of the four extra points in the configuration of Figure 3 is:

$$(y - 1)(x^{-1} - 1).$$

The result is the same in any other configuration (by changing the orientation of x and/or y).

The proof of points 2) and 3) of Theorem 0.1 is straightforward.

DEFINITION 1.1. An oriented embedded loop based at s_0 is said to be *of type I or II*, if in a neighbourhood of s_0 it looks like



The next lemma can be seen as an obstruction for an element of $\pi_1(S_{g,b,n}\,;\,s_0)$ to be represented by an embedded loop.

LEMMA 1.2. *If* $x \in \Gamma = \pi_1(S_{g,b,n}\,;\,s_0)$ *is represented by an embedded based loop then* $\omega(x,x) = 1 - x$ *(resp.* $1 - x^{-1}$*) if* x *is of type I (resp. II).*

Proof. Suppose x is of type I. Then, by pushing x along a normal vector field $\vec{\gamma}$, such that $\vec{\gamma}$ followed by the orientation of x gives the positive orientation of $S_{g,b,n}$, we may suppose that x and x' meet only at two points P and Q shown in Figure 4.

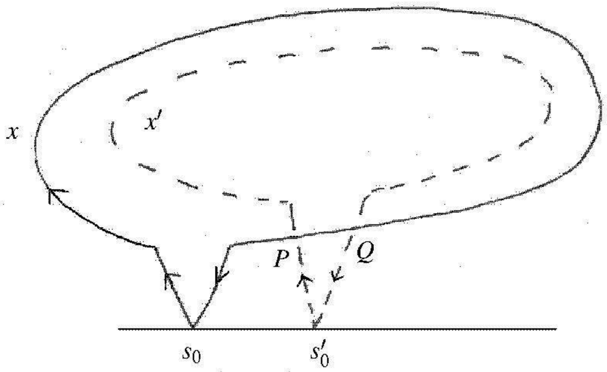


FIGURE 4

Then $\varepsilon_P = -1 = -\varepsilon_Q$, $g_P = x$, $g_Q = 1$. The lemma is proved.

§2 The intersection form ω and Fox free differential calculus

We review some basic facts about Fox free differential calculus [F]. Let Γ be a free group with free basis $z_1, \ldots, z_m$.

(2.1) Definition. A *derivation* (resp. an *antiderivation*) is a map $D\colon \mathbf{Z}[\Gamma] \longrightarrow \mathbf{Z}[\Gamma]$ such that

1) D is additive,

2) $D(uv) = e(v)D(u) + uD(v)$, where $e\colon \mathbf{Z}[\Gamma] \longrightarrow \mathbf{Z}$ is defined in (0.3) (resp. $D(uv) = e(v)D(u) + D(v)\overline{u}$).

The fundamental example of a derivation is the partial derivative $\dfrac{\partial}{\partial z_i}\colon \mathbf{Z}[\Gamma] \longrightarrow \mathbf{Z}[\Gamma]$ defined by:

(i) $\dfrac{\partial z_j}{\partial z_i} = \delta_{ij}$

(ii) $\dfrac{\partial}{\partial z_i}$ is additive

(iii) $\dfrac{\partial uv}{\partial z_i} = e(v)\dfrac{\partial u}{\partial z_i} + \dfrac{\partial v}{\partial z_i}$.

(2.2) Definition. A map $\theta\colon \mathbf{Z}[\Gamma] \times \mathbf{Z}[\Gamma] \longrightarrow \mathbf{Z}[\Gamma]$ is called a *biderivation* if:

1) θ is bilinear for $+$

2) θ is a derivation (resp. antiderivation) with respect to the first (resp. second) variable.

(2.3) Given a biderivation θ and a free basis $z_1, \ldots, z_m$ for Γ, we can associate a $m \times m$ matrix Λ, with coefficients in $\mathbf{Z}[\Gamma]$, in the following way: the (i,j) entry θ_{ij} of Λ is given by $\theta_{ij} = \theta(z_i, z_j)$.

For $u \in \mathbf{Z}[\Gamma]$, let ∂u denote the column $\left(\dfrac{\partial u}{\partial z_1}, \ldots, \dfrac{\partial u}{\partial z_m}\right)^t$. Then we have the following easy lemma.

Lemma 2.3. *Let θ be a biderivation on the free group Γ equipped with a free basis $(z_1, \ldots, z_m)$. Then for $(u, v) \in \mathbf{Z}[\Gamma] \times \mathbf{Z}[\Gamma]$ we have*

$$(*) \qquad \theta(u, v) = \partial u^t \times \Lambda \times \overline{\partial v},$$

where $\overline{\partial v}$ is the column conjugate of ∂v and Λ is the matrix of θ with respect to the free basis $(z_1, \ldots, z_m)$.

Proof. Formula $(*)$ is true, by definition of θ_{ij}, for $u = z_i^{\pm 1}$, $v = z_j^{\pm 1}$. Then proceed by induction on the length of u and v with respect to the free basis $(z_1, \ldots, z_m)$.

(2.4) FUNDAMENTAL REMARK. The intersection form $\omega\colon \Gamma \times \Gamma \longrightarrow \mathbf{Z}[\Gamma]$, extended by bilinearity to $\mathbf{Z}[\Gamma] \times \mathbf{Z}[\Gamma]$ is a biderivation, by points 2) and 3) of Theorem 0.1.

(2.5) FIRST FUNDAMENTAL EXAMPLE. Consider $S_{g,1,0}$ (also denoted by $S_{g,1}$), the compact, connected, oriented surface of genus $g > 0$, with one boundary component. Then $\Gamma = \pi_1(S_{g,1}\,;\, s_0)$ is a free group of rank $2g$. Choose the following "symplectic" free basis $(x_i, y_i\,;\, i = 1, 2, \ldots, g)$ given by Figure 5.

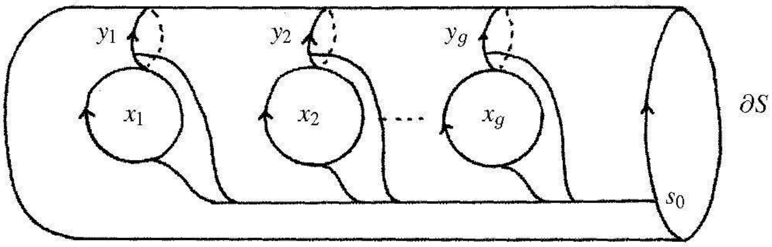


FIGURE 5

LEMMA 2.4. *The matrix $\Lambda_{g,1}$ of the intersection form ω on $\mathbf{Z}[\Gamma]$ with respect to the free basis $(x_i, y_i\,;\, i = 1, 2, \ldots, g)$ given by Figure 5 is*

$$\Lambda_{g,1} = \left(\begin{array}{c|c} J_1 & J_2 \\ \hline J_3 & J_4 \end{array}\right)$$

where J_i, $i = 1, 2, 3, 4$ are the following $g \times g$ matrices with coefficients in $\mathbf{Z}[\Gamma]$:

$$J_1 = \begin{pmatrix} 1 - x_1 & 0 & & 0 & & \\ & & & \vdots & & \\ (1-x_2)(1-x_1^{-1}) & 1 - x_2 & & 0 & & \\ \vdots & & \ddots & & & \\ \vdots & (1-x_3)(1-x_2^{-1}) & & (1-x_i) & & \\ \vdots & \vdots & & \vdots & \ddots & \\ (1-x_g)(1-x_1^{-1}) & (1-x_g)(1-x_2^{-1}) & \ldots & (1-x_g)(1-x_i^{-1}) & \ldots & 1 - x_g \end{pmatrix}$$

$$J_2 = \begin{pmatrix} x_1\ y_1^{-1} & 0 & 0 & 0 \\ & & \vdots & \vdots \\ (1-x_2)(1-y_1^{-1}) & x_2 y_2^{-1} \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \vdots & (1-x_3)(1-y_2^{-1}) & x_i\ y_i^{-1} & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (1-x_g)(1-y_1^{-1}) & (1-x_g)(1-y_2^{-1}) \ldots & (1-x_g)(1-y_i^{-1}) & \ldots & x_g\ y_g^{-1} \end{pmatrix}$$

$$J_3 = \begin{pmatrix} 1-x_1^{-1}-y_1 & 0 & \ldots 0 \ldots & 0 \\ & & \vdots & \vdots \\ (1-y_2)(1-x_1^{-1}) & 1-x_2^{-1}-y_2 & 0 & \vdots \\ \vdots & & \vdots & \vdots \\ \vdots & (1-y_3)(1-x_2^{-1}) & \ddots & 0 \\ \vdots & \vdots & & \vdots \\ (1-y_g)(1-x_1^{-1}) & (1-y_g)(1-x_2^{-1}) \ldots & \ldots & 1-x_g^{-1}-y_g \end{pmatrix}$$

$$J_4 = \begin{pmatrix} 1-y_1^{-1} & 0 & \ldots 0 \ldots & 0 \\ & & \vdots & \vdots \\ (1-y_2)(1-y_1^{-1}) & 1-y_2^{-1} & 0 & \vdots \\ \vdots & & \vdots & \vdots \\ \vdots & (1-y_3)(1-y_2^{-1}) & \ddots & 0 \\ \vdots & \vdots & & \vdots \\ (1-y_g)(1-y_1^{-1}) & (1-y_g)(1-y_2^{-1}) \ldots & \ldots & 1-y_g^{-1} \end{pmatrix}$$

Proof. This follows immediately from the definition and properties of ω, together with Lemma 1.2 (observe that x_i is of type I and y_i of type II).

(2.6) REMARK. The matrix $\Lambda_{g,1}$ appeared first in the work of Papakyriakopoulos ([Pap], §9); see also Hempel ([H], Theorem 3.3), but has no interpretation as the matrix of a geometric biderivation (see §3 below).

(2.7) REMARK. The integer matrix $e(\Lambda_{g,1})$ is of course the standard antisymmetric matrix

$$\left(\begin{array}{c|c} 0 & I_g \\ \hline -I_g & 0 \end{array}\right)$$

where I_g is the $g \times g$ identity matrix.

(2.8) SECOND FUNDAMENTAL EXAMPLE. $S_{0,1,n}$ is the 2-disk with n points $P_1, P_2, \ldots, P_n$ removed. Let $(u_1, \ldots, u_n)$ be the free basis of $\pi_1(S_{0,1,n}\,;\, s_0)$ given by Figure 6.

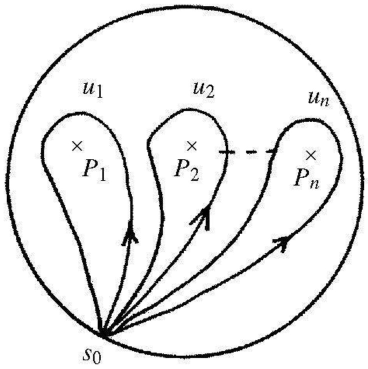


FIGURE 6

LEMMA 2.5. *The matrix, denoted by Ω_n, of the intersection form ω on $S_{0,1,n} = D^2 - \{P_1, \ldots, P_n\}$, with respect to the free basis $(u_1, u_2, \ldots, u_n)$ given by Figure 6 is:*

$$\Omega_n = \begin{pmatrix} 1-u_1^{-1} & 0 & \ldots 0 \ldots & 0 \\ & & & \vdots \\ (1-u_2)(1-u_1^{-1}) & 1-u_2^{-1} & & \vdots \\ \vdots & & & 0 \\ \vdots & (1-u_3)(1-u_2^{-1}) & \ddots & \vdots \\ \vdots & \vdots & & \vdots \\ (1-u_n)(1-u_1^{-1}) & (1-u_n)(1-u_2^{-1}) \ldots & \ldots & 1-u_n^{-1} \end{pmatrix}$$

§3 Relation between the homotopic intersection form and Reidemeister pairing ([Pap], [H])

(3.1) Let S be a compact, connected, oriented surface, s_0 a base point and N a normal subgroup of $\pi_1(S\,;\,s_0)$. Consider the canonical exact sequence

$$0 \longrightarrow N \longrightarrow \pi_1(S\,;\,s_0) \xrightarrow{\chi} T \longrightarrow 0\,.$$

(3.2) Let $\widetilde{S}$ be the covering surface associated to N. Choose a lifting $\widetilde{s_0} \in \widetilde{S}$ of s_0. Of course $N \simeq \pi_1(\widetilde{S}\,;\,\widetilde{s_0})$ and T acts on $\widetilde{S}$ as the group of covering transformations of $\widetilde{S} \longrightarrow S$. Thus $H_1(\widetilde{S}\,;\,\mathbf{Z})$ inherits a structure of $\mathbf{Z}T$-module.

(3.3) Define the Reidemeister pairing $\phi\colon H_1(\widetilde{S},\mathbf{Z}) \times H_1(\widetilde{S},\mathbf{Z}) \longrightarrow \mathbf{Z}[T]$ as

$$\phi(\alpha,\beta) = \sum_{t\in T} \langle \alpha, t.\beta\rangle t\,,$$

where $t.\beta$ denotes the action of t on β and $\langle\ ,\ \rangle$ is the usual algebraic intersection number in $\widetilde{S}$.

Denote by $\Phi\colon N\times N \longrightarrow \mathbf{Z}T$ the composition

$$N\times N \xrightarrow{h\times h} H_1(\widetilde{S}\,;\,\mathbf{Z}) \times H_1(\widetilde{S}\,;\,\mathbf{Z}) \xrightarrow{\phi} \mathbf{Z}T\,,$$

where h is the Hurewicz map (abelianization).

(3.4) The relation between the intersection form ω and Reidemeister pairing is given by

Lemma 3.1. *If S is a surface with boundary, then $\Phi(u,v) = \chi(\omega(u,v))$ for any $u,v \in N$ (where χ is the canonical map $\pi_1(S\,;\,s_0) \longrightarrow T$).*

Proof. Let u,v be oriented based loops, v' be the loop based at s_0', by pushing v slightly as in §1. Suppose u and v' are in general position and let P be a point of $u \pitchfork v'$.

Call $\widetilde{u},\widetilde{v}$ (resp. $\widetilde{v'}$) the lifts of u,v (resp. v') starting from $\widetilde{s_0}$ (resp. $\widetilde{s_0'}$).

(3.5) Let $\widetilde{s_0'}$ be the lift of s_0' close to $\widetilde{s_0}$. Let u_1 (resp. v_1') be the arc on u (resp. v') going from s_0 (resp. s_0') to P along the positive direction of u (resp. v') (see Figure 7).

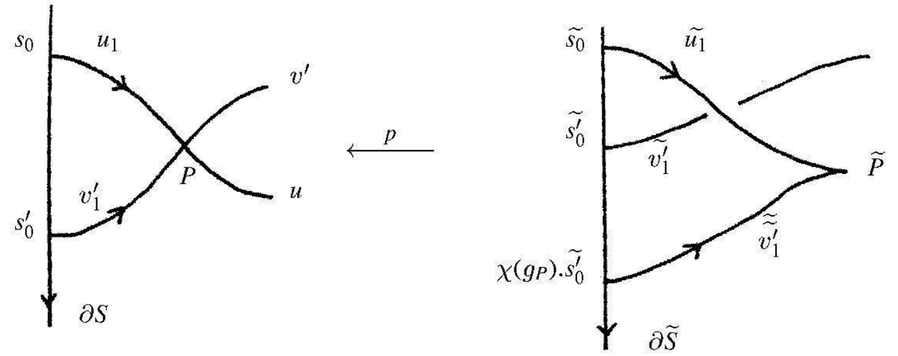


FIGURE 7

Let $\widetilde{u_1}$ (resp. $\widetilde{v'_1}$) be the lift of u_1, starting from $\widetilde{s_0}$ (resp. $\widetilde{s'_0}$) and let $\widetilde{P}$ be the end of $\widetilde{u_1}$. Denote by $\widetilde{\widetilde{v'_1}}$ the lift of v'_1 ending at $\widetilde{P}$. Then the starting point of $\widetilde{\widetilde{v'_1}}$ is $\chi(g_P).\widetilde{s'_0}$, where g_P is the loop $u_1 \circ v_1'^{-1} \circ \gamma$ (here γ denotes the small arc $[s_0, s'_0]$ on ∂S and composition of paths are written from left to right). So $\widetilde{\widetilde{v'_1}} = \chi(g_P).\widetilde{v'_1}$ and

$$\varepsilon_P \chi(g_P) = \langle \widetilde{u}, \chi(g_P).\widetilde{v} \rangle_{\widetilde{P}} \; \chi(g_P)$$

(where $\langle \; , \; \rangle_{\widetilde{P}}$ denotes the algebraic intersection number at $\widetilde{P}$). This proves Lemma 2.6.

(3.6) REMARK. From the Reidemeister pairing Φ defined on $N \times N$, with values in $T = \Gamma/N$, one cannot recover our intersection form ω, since $T = \Gamma/N = \{1\}$ when $N = \Gamma$.

(3.7) In the case of $S_{g,1,0}$, using Lemmas 2.3, 2.4 and 3.1, we recover the fundamental formula of [Pap], Theorem 10.13 (see also [H], Theorem 3.3), given by

COROLLARY 3.2. *With the notations of Lemma 3.1 we have, for* $(u, v) \in N \times N$,

$$(*) \qquad \Phi(u, v) = \chi(\partial u^t \times \Lambda_{g,1} \times \overline{\partial v}).$$

From formula $(*)$ of Corollary 3.2, Papakyriakopoulos deduces the main result of [Pap]:

PROPOSITION 3.3. (Theorem 11.1 of [Pap].) *The covering $\widetilde{S}$, corresponding to the normal subgroup N, is planar (e.g. homeomorphic to a subset of the plane) if and only if $\chi(\partial u^t \times \Lambda \times \overline{\partial v}) = 0$ for any $u, v \in N$.*

Proof. A surface $\widetilde{S}$ is planar if and only $\langle \alpha, \beta \rangle = 0$, for any $\alpha, \beta \in H_1(\widetilde{S}; \mathbf{Z})$, where $\langle \ , \ \rangle$ is the usual algebraic intersection number. Then Proposition 2.8 follows immediately from Corollary 3.2.

§4 APPLICATION TO MAPPING CLASS GROUPS

(4.1) Denote by $\mathcal{M}_{g,b,n}$ the mapping class group of the surface $S_{g,b,n}$, that is the group of isotopy classes of homeomorphisms of the surface $S_{g,b}$, equal to identity on $\partial S_{g,b}$ and preserving (globally) a set of n points in the interior of $S_{g,b}$.

(4.2) Given $f \in \mathcal{M}_{g,b,n}$, we denote by the same letter the isomorphism induced on $\Gamma = \pi_1(S_{g,b,n}\,; s_0)$, since it is well-known that the mapping

$$\mathcal{M}_{g,b,n} \longrightarrow \mathrm{Aut}(\pi_1(S_{g,b,n}\,; s_0))$$

is injective.

LEMMA 4.1. *Let ω be the homotopic intersection form on $S_{g,b,n}$ and $f \in \mathcal{M}_{g,b,n}$. Then $\omega(f(x), f(y)) = f(\omega(x, y))$ for any $x, y \in \Gamma = \pi_1(S_{g,b,n}\,; s_0)$.*

Proof. This is by definition of ω.

(4.3) On the other hand, for f representing an element of $\mathcal{M}_{g,b,n}$, we associate its Fox matrix (see [Mo_1], §5 or [Pe], Chapter 3) as follows.

Choose a free basis $\{x_i, y_i, u_j\,;\ i = 1, 2, \ldots, g,\ j = 1, 2, \ldots, n\}$ for the free group Γ given by Figure 8.

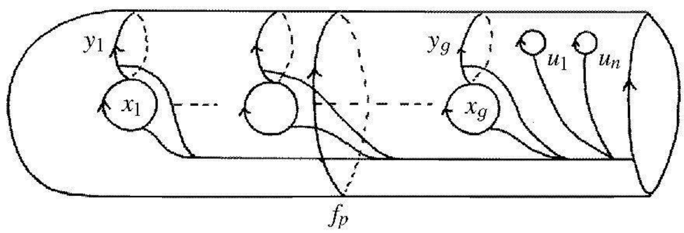


FIGURE 8

Set

$$z_i = \begin{cases} x_i, & i \leq i \leq g \\ y_{i-g}, & g+1 \leq i \leq 2g \\ u_{i-2g}, & 2g+1 \leq i \leq 2g+n\,. \end{cases}$$

Then $f(z_i) \in \Gamma$ is a word in the variables z_j.

The Fox matrix $B(f)$ of f is the $(2g+n)\times(2g+n)$ matrix with coefficients in $\mathbf{Z}[\Gamma]$ given by

$$\begin{array}{cc} & j \\ i & \begin{pmatrix} & \vdots & \\ \cdots & \overline{\dfrac{\partial f(z_j)}{\partial z_i}} & \cdots \\ & \vdots & \end{pmatrix} \end{array} = B(f)\,,$$

where $\overline{(\ \ )}$ is the anti-isomorphism of $\mathbf{Z}[\Gamma]$ defined in Theorem 0.1 and $\frac{\partial}{\partial z_i}$ is the Fox partial derivative defined in §2.

LEMMA 4.2. (Proposition 5.2 of [Mo$_1$], or Lemma 3.2 of [Pe].)
For $f, g \in \mathcal{M}_{g,b,n}$ we have

$$B(f \circ g) = B(f) \times {}^f B(g)\,,$$

where $\times$ denotes the usual matrix multiplication and ${}^f B(g)$ is defined by

$${}^f B(g) = {}^f(a_{ij}) = (f(a_{ij})) \quad (a_{ij} \in \mathbf{Z}[\Gamma])\,.$$

Consequently, $B(f)$ belongs to $GL_{2g+n}(\mathbf{Z}[\Gamma])$, the group of invertible matrices with coefficients in $\mathbf{Z}[\Gamma]$.

Combining Lemmas 2.3, 2.4 and 4.1, we get a tautological proof of Theorem 5.3 of [Mo$_1$]:

PROPOSITION 4.3. *Let $f \in \mathcal{M}_{g,1}$. Then*

$${}^t\overline{B(f)} \times \Lambda_{g,1} \times B(f) = {}^f\Lambda_{g,1}\,,$$

where $\Lambda_{g,1}$ is the matrix of the form ω with respect to the free basis $\{x_i, y_i\,; i = 1, 2, \ldots, g\}$ given in Lemma 2.4 (${}^f\Lambda_{g,1}$ is defined in Lemma 4.2).

(4.4) Note that $\mathcal{M}_{0,1,n}$ is the usual braid group B_n (see [Bi]). Using Lemmas 2.3, 2.4, 4.1 again we get

COROLLARY 4.4. *Let $f \in \mathcal{M}_{0,1,n} = B_n$. Then*

$$ {}^t\overline{B(f)} \times \Omega_n \times B(f) = {}^f\Omega_n \,, $$

where Ω_n is the matrix defined in Lemma 2.5.

(4.5) Denote by H the abelianization of Γ. For $f \in \mathcal{M}_{g,b,n}$, denote by f_* the isomorphism of H induced by f, and by $B^{ab}(f)$ the image of the Fox matrix of f under the canonical homomorphism $GL(\mathbf{Z}[\Gamma]) \longrightarrow GL(\mathbf{Z}[H])$.

The advantage in considering $\mathbf{Z}[H]$ is that it is a commutative ring, but of course we lose information.

For $u \in \mathbf{Z}[\Gamma]$, denote by $\partial_H u$ the image of the column ∂u (defined in § 2) under the map $\mathbf{Z}[\Gamma] \longrightarrow \mathbf{Z}[H]$.

Lemma 4.2, Proposition 4.3 and Corollary 4.4 become under the mapping $\Gamma \longrightarrow H$:

(i) $B^{ab}(f \circ g) = B^{ab}(f) \times {}^{f_*}B^{ab}(g)$.

(ii) ${}^t\overline{B^{ab}(f)} \times \Lambda^{ab}_{g,1} \times B^{ab}(f) = {}^{f_*}\Lambda^{ab}_{g,1}$ $(f \in \mathcal{M}_{g,1})$.

(iii) ${}^t\overline{B^{ab}(f)} \times \Omega_n \times B^{ab}(f) = {}^{f_*}\Omega^{ab}_n$ $(f \in B_n)$.

(4.6) Let $\mathcal{I}_{g,b,n}$ be the normal subgroup of $\mathcal{M}_{g,b,n}$ consisting of homeomorphisms such that $f_* = \mathrm{id}_H$. This subgroup is usually called the Torelli group of $S_{g,b,n}$.

In the case $g = 0, b = 1, n > 0$, $\mathcal{I}_{0,1,n}$ is usually denoted by P_n, the pure braid group of index n (recall that $\mathcal{M}_{0,1,n}$ is the braid group B_n).

By (i) of (4.5), $B^{ab}\colon \mathcal{I}_{g,b,n} \longrightarrow GL_{2g+n}(\mathbf{Z}[H])$ is a true homomorphism.

In the case $g = 0$, $g = 1$, $n > 1$, $B^{ab}\colon P_n \longrightarrow GL_n(\mathbf{Z}[H])$ is the so-called Gassner representation.

In ([Mo$_1$], problem 6.23, § 6.8), Morita asks the following question:

Is the representation $B^{ab}\colon \mathcal{I}_{g,1} = \mathcal{I}_{g,1,0} \longrightarrow GL_{2g}(\mathbf{Z}[H])$ injective ?

In [Su], Suzuki answers this question negatively, by exhibiting, using lengthy computations, a non-zero element in the kernel of B^{ab}, for any $g \geq 2$.

In the remaining part of this paragraph, using our form ω, we produce a geometric way of obtaining elements in the kernel of B^{ab}, explaining geometrically, without computation, why Suzuki's example works.

REMARK. The question of injectivity of $B^{ab}\colon \mathcal{I}_{0,1,n} = P_n \longrightarrow GL_n(\mathbf{Z}[H])$ (the Gassner representation) is still open for $n \geq 4$.

(4.7) The following lemma holds:

LEMMA 4.5. *Let Γ be a free group with basis $(z_1, \dots, z_p)$, H its abelianization. Let $\alpha \in \Gamma$ be homologous to 0 (e.g. the image of α in H is 0). Then for any $g \in \Gamma$,*

$$\partial_H(g\alpha g^{-1}) = g\partial_H(\alpha)\,,$$

where $\partial(\alpha)$ is the p-column $(\dfrac{\partial\alpha}{\partial z_1}, \dots, \dfrac{\partial\alpha}{\partial z_p})^t$ and $\partial_H(\alpha)$ is its image under the map $\Gamma \longrightarrow H$.

Proof. This follows easily from the properties of the partial derivative.

(4.8) Let c be a simple closed curve on a surface $S_{g,1} = S_{g,1,0}$. Denote by $D(c)$ the Dehn twist along c (see [Bi], Chapter 4) defining an element of $\mathcal{M}_{g,1}$. By a bounding curve of genus p we mean a curve bounding a subsurface of genus p. Then we have:

PROPOSITION 4.6. *Let c be a simple closed bounding curve on a surface $S_{g,1}$. Then $B^{ab}(D(c)) = I_{2g} + \overline{\partial_H\alpha} \times (\partial_H\alpha)^t \times \Lambda^{ab}_{g,1}$, where α (defined up to conjugation and orientation) is the element of $\Gamma = \pi_1(S_{g,1}\,;\, s_0)$ represented by c, and $\Lambda^{ab}_{g,1}$ is the image under $\Gamma \longrightarrow H$ of the matrix $\Lambda_{g,1}$ given in Lemma 2.4.*

Proof. Observe that $\overline{\partial_H\alpha} \times (\partial_H\alpha)^t$ depends only on the conjugacy class of α, by Lemma 4.5 and does not depend on the orientation of α.

We first prove Proposition 4.6 for $c = f_p$, where f_p is the closed simple bounding curve of genus p defined by Figure 8.

(4.9) The circle f_p, oriented and equipped with the path as indicated by Figure 8, represents the element

$$[y_p, x_p]\,[y_{p-1}, x_{p-1}] \dots [y_1, x_1]$$

of $\Gamma = \pi_1(S_{g,1}\,;\, s_0)$, where $[a, b]$ denotes the commutator $a\,b\,a^{-1}\,b^{-1}$. The action of the Dehn twist $D(f_p)$ on the free basis $\{x_i, y_i\,;\, 1, 2, \dots, g\}$ is given by

$$D(f_p)(z_k) = f_p\, z_k f_p^{-1} \ \text{ for } z_k = x_k,\ y_k\ (1 \leq k \leq p)\,,$$

$$D(f_p)(z_k) = z_k \ \text{ for } z_k = x_k,\ y_k\ (p < k \leq g)\,.$$

(4.10) For simplicity, we will make computations in case $p = 1$ (so we may suppose $g = 1$).

Easy computations show that

$$B^{ab}(D(f_1)) = \begin{pmatrix} 1-(1-x_1^{-1})(1-y_1^{-1}) & -(1-y_1^{-1})^2 \\ (1-x_1^{-1})^2 & 1+(1-x_1^{-1})(1-y_1^{-1}) \end{pmatrix}$$

$$\overline{\partial f_1} \times \partial f_1^t = \begin{pmatrix} y_1^{-1}-1 \\ 1-x_1^{-1} \end{pmatrix} (y_1-1,\ 1-x_1)$$

$$\Lambda^{ab}_{g,1} = \begin{pmatrix} 1-x_1 & x_1\, y_1^{-1} \\ 1-x_1^{-1}-y_1 & 1-y_1^{-1} \end{pmatrix} \qquad \text{(by Lemma 2.4)}.$$

Proposition 4.6 is easily verified for $c = f_1$.

(4.11) Let c be a simple, closed, bounding curve of genus p. Then it is easy to find a homeomorphism φ of $\mathcal{M}_{g,1}$ such that $c = \varphi(f_p)$.

We claim that:

(i) $D(c) = \varphi \circ D(f_p) \circ \varphi^{-1}$,

(ii) $\partial_H \alpha = \overline{B^{ab}(\varphi)} \times {}^{\varphi_*}\partial_H f_p$,

where $\alpha \in \Gamma$ is represented by c (up to conjugation and orientation), φ_* is the isomorphism induced by φ at the homological level, and ${}^{\varphi_*}\partial_H f_p$ is the image of the column $\partial_H f_p$ under φ_*.

(i) is well known.

(ii) follows from the more general formula

$$\partial\varphi(u) = \overline{B(\varphi)} \times {}^{\varphi}\partial u\,,$$

where $u \in \Gamma$. This last formula can easily be proved by induction on the length of u, in terms of a basis of Γ.

(4.12) We are now ready to prove Proposition 4.6:

$$\begin{aligned} B^{ab}(D(c)) &= B^{ab}(\varphi \circ D(f_p) \circ \varphi^{-1}) \\ &= B^{ab}(\varphi) \times {}^{\varphi_*}B^{ab}(D(f_p)) \times {}^{\varphi_* \circ D(f_p)_*}\, B^{ab}(\varphi^{-1}) \\ &= B^{ab}(\varphi) \times {}^{\varphi_*}\, B^{ab}(D(f_p)) \times {}^{D(c)_*}\, B^{ab}(\varphi)^{-1} \end{aligned}$$

(since $B(\varphi^{-1}) = ({}^{\varphi^{-1}}B(\varphi))^{-1}$ by Lemma 4.2). Hence

$$B^{ab}(D(c)) = B^{ab}(\varphi) \times {}^{\varphi_*}\ B^{ab}(D(f^p)) \times B^{ab}(\varphi)^{-1}$$

(since $D(c)_* = \mathrm{id}_H$)

$$= B^{ab}(\varphi) \times {}^{\varphi_*}[I + \overline{\partial_H\, f_p} \times \partial_H\, f_p^t \times \Lambda_{g,1}] \times B^{ab}(\varphi)^{-1}$$
$$= I + \overline{\partial_H \alpha} \times (\partial_H \alpha)^t \times \overline{(B^{ab}(\varphi)^{-1})^{t\varphi_*}}\ \Lambda_{g,1} \times B^{ab}(\varphi)^{-1}$$

(by (4.11) (ii))

$$= I + \overline{\partial_H \alpha} \times (\partial_H \alpha)^t \times \Lambda_{g,1}$$

(by Proposition 4.3).

COROLLARY 4.7. *Let c, d be two simple, closed, bounding curves. Then*

$$\begin{aligned} B^{ab}(D(c) \circ D(d)) = {} & -I + B^{ab}(D(c)) + B^{ab}(D(d)) \\ & + \omega_H(\alpha, \beta)\ \overline{\partial \alpha} \times \partial \beta^t \times \Lambda^{ab}_{g,1}, \end{aligned}$$

where $\alpha \in \Gamma$ (resp. β) is represented by c (resp. d), up to conjugation and orientation.

Proof. By Lemma 4.2 ($D(c)_* = \mathrm{id}_H$) and Proposition 4.6 we have

$$\begin{aligned} B^{ab}(D(c) \circ D(d)) = {} & -I + B^{ab}(D(c)) + B^{ab}(D(d)) \\ & + \overline{\partial_H \alpha} \times (\partial_H \alpha^t \times \Lambda^{ab}_{g,1} \times \overline{\partial_H \beta}) \times \partial_H\ \beta^t \times \Lambda^{ab}_{g,1}\,. \end{aligned}$$

The parenthesis in the last term above is exactly $\omega_H(\alpha, \beta)$, by Lemmas 2.3 and 2.4.

COROLLARY 4.8. *Let c, d be two simple closed, bounding curves of $S_{g,1}$ such that $\omega_H(\alpha, \beta) = 0$. Then $B^{ab}(D(c))$ and $B^{ab}(D(d))$ commute (α, β are defined as in Corollary 4.7).*

PROPOSITION 4.9. *The homomorphism $B^{ab}\colon \mathcal{I}_{g,1} \longrightarrow GL_{2g}(\mathbf{Z}[H])$ is not injective, for $g \geq 2$.*

Proof. By Corollary 4.8, it is sufficient to find a pair of simple, closed, bounding curves c, d such that $\omega_H(\alpha, \beta) = 0$, and such that $D(c)$ and $D(d)$ do not commute. Here are three examples of such pairs.

EXAMPLE 1. c is the circle given by Figure 9:

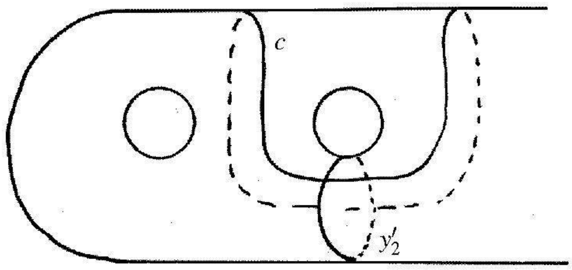


FIGURE 9

d is the image of c by the Dehn twist along the circle y_2'.

EXAMPLE 2. c and d are given by Figure 10:

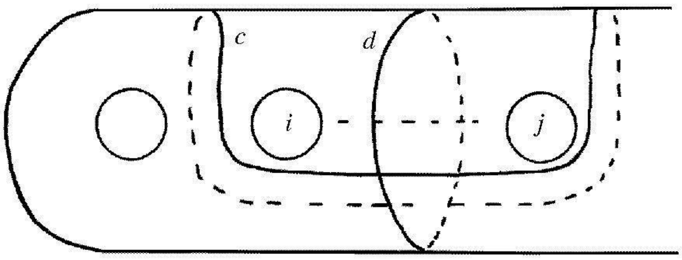


FIGURE 10

EXAMPLE 3. c and d are given by Figure 11:

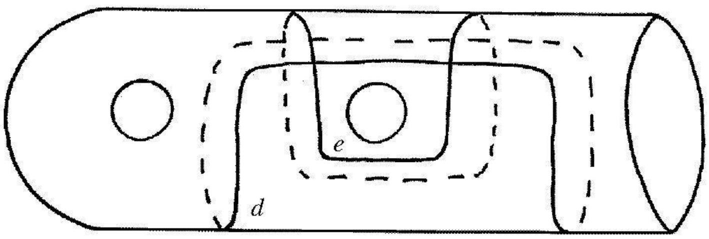


FIGURE 11

In each case, it is easy to see that the corresponding $\omega_H(\alpha, \beta)$ is zero in $\mathbf{Z}[H]$, while $D(c)$ and $D(d)$ do not commute. To prove this last point, we use the well-known result (see, for example [PaR], Prop. 3.7) that two Dehn twists commute if and only if the curves c and d can be disjoined, up to isotopy.

The first example is given in [Su] where it is shown that $B^{ab}(D(c))$ and $B^{ab}(D(d))$ commute, using lengthy explicit computations.

REMARK. Our method for proving the non injectivity of B^{ab} resembles the method used by Moody [Moo], Long-Paton [LP] and Bigelow [Bg] to prove the non injectivity of the Burau representation. Our Corollary 4.8 plays the role of Theorem 1 of [Moo], and Theorem 1.5 of [LP].

§5 APPLICATIONS TO THE BRAID GROUPS

(5.1) Recall that the braid group B_n is the mapping class group $\mathcal{M}_{0,1,n}$, e.g. the group of homeomorphisms of the 2-disk D^2, fixing pointwise the boundary ∂D^2 and leaving invariant a set of n points $P_1, P_2, \ldots, P_n$ in the interior of D^2.

(5.2) A set of generators of B_n is defined as follows. Let $P_1, P_2, \ldots, P_n$ be n points in the horizontal diameter of D^2 (Figure 12).

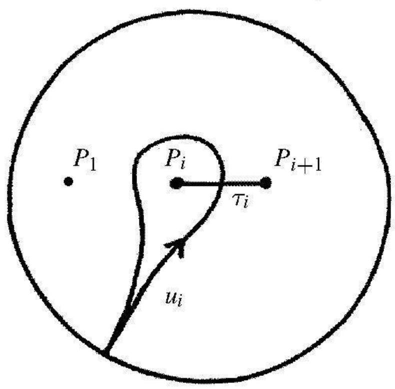


FIGURE 12

Let σ_i be the "half Dehn twist" along the segment $\tau_i = [P_i, P_{i+1}]$, equal to identity outside a regular neighbourhood of τ_i and which sends a vertical segment meeting τ_i as indicated by Figure 13.

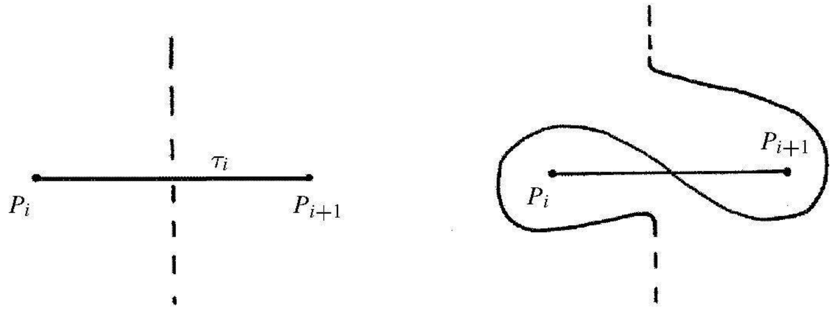


FIGURE 13

(5.3) The action of σ_i on the free basis $(u_1, \ldots, u_n)$ of $\Gamma = \pi_1(D^2 - \{P_i\}; s_0)$ defined by Figure 12 is as follows:

$$\begin{aligned} \sigma_i(u_j) &= u_j \quad j \neq i,\ i+1 \\ \sigma_i(u_i) &= u_i^{-1}\, u_{i+1}\, u_i \\ \sigma_i(u_{i+1}) &= u_i \end{aligned}$$

(composition of paths are from left to right).

So the Fox matrix of σ_i with respect to the free basis $\{u_i \ ; \ 1, 2, \ldots, n\}$ is given by

$$B(\sigma_i) = \begin{array}{c} \begin{array}{cccc} \phantom{I_{i-1}} & i & i+1 & \phantom{I_{n-i-2}} \end{array} \\ \left(\begin{array}{c|cc|c} I_{i-1} & 0 & 0 & 0 \\ \hline 0 & -u_i + u_{i+1}^{-1} u_i & 1 & 0 \\ 0 & u_i & 0 & 0 \\ \hline 0 & 0 & 0 & I_{n-i-2} \end{array}\right) \end{array} \begin{array}{c} \\ \\ i \\ i+1 \\ \\ \end{array}$$

(5.4) For $\sigma \in B_n$, denote by σ_* the isomorphism induced at the homological level; of course σ_* is a permutation matrix, corresponding to the permutation of the points $\{P_1, \ldots, P_n\}$ under σ. By definition the pure braid group P_n is the normal subgroup of B_n, consisting of braids such that $\sigma_* = \mathrm{id}$.

By Lemma 4.2, the map $B^{ab}\colon P_n \longrightarrow GL_n(\mathbf{Z}[H])$ is a homomorphism, called the Gassner representation of P_n.

PROPOSITION 5.1. *For any $\sigma \in P_n$, the Gassner representation B^{ab} satisfies*

$$\overline{B}^{ab}(\sigma)^t \times \Omega_n^{ab} \times B^{ab}(\sigma) = \Omega_n^{ab},$$

where Ω_n is the $n \times n$ matrix defined in Lemma 2.5.

Proof. Follows immediately from Corollary 4.4.

(5.5) REMARK. By sending u_i on appropriate $\tau_i \in S^1 \subset \mathbf{C}$ (so that $\tau_i^{-1} = \overline{\tau}_i$), and considering the "hermitian" matrix $\Omega_n^{ab} + (\overline{\Omega_n}^{ab})^t$, the above proposition shows that the image of the Gassner representation is conjugated to a subgroup of the unitary group $U_n(\mathbf{C})$.

(5.6) REMARK. The fact that Ω_n^{ab} is triangular imposes strong conditions for a matrix of $GL_n(\mathbf{Z}[H])$ to be the Gassner matrix of some pure braid.

For example we have the following easy lemma which generalizes Theorem 1.1 of [LP].

COROLLARY 5.2. *Suppose the Gassner matrix M of $\sigma \in P_n$ has the form*

$$M = \begin{pmatrix} \alpha_1 & & X & X & X \\ \vdots & \ddots & & & \\ 0 & & \alpha_p & X & X \\ \vdots & & \vdots & & \\ 0 & & 0 & & \\ \vdots & & \vdots & (A_{n-p}) & \\ 0 & & 0 & & \end{pmatrix}$$

Then M is equal to

$$M = \left(\begin{array}{c|c} I_p & 0\dots0 \\ \hline 0 \quad 0 & \\ \vdots & A_{n-p} \\ 0 \quad 0 & \end{array}\right)$$

Proof: Using Proposition 5.1 and the fact that Ω_n^{ab} is triangular, it is easy to see that

$$M = \begin{pmatrix} \alpha_1 & & & 0\dots0 \\ \vdots & \ddots & & \\ 0 & & \alpha_p & \dots0\dots \\ \vdots & & \vdots & A_{n-p} \\ 0 & & 0 & \end{pmatrix}$$

To show that $\alpha_1 = 1$, proceed as follows. By definition of M, its first column is $\left(\overline{\dfrac{\partial\sigma(u_1)}{\partial u_1}}^{ab}, \dots, \overline{\dfrac{\partial\sigma(u_1)}{\partial u_n}}^{ab}\right)^t$.

By the fundamental formula of Fox differential calculus,

$$\sigma(u_1) - 1 = \sum_{i=1}^{n} \frac{\partial\sigma(u_1)}{\partial u_i}(u_1 - 1),$$

we have $\sigma(u_1)^{ab} - 1 = \overline{\alpha_1}(u_1 - 1)$. But since $\sigma \in P_n$, $\sigma(u_1)^{ab} = u_1$.

(5.7) Let $\theta\colon H = \langle u_1, \dots, u_n\rangle \longrightarrow \mathbf{Z} =< t >$ be the homomorphism defined by $\theta(u_i) = t$.

By definition of B_n, any element $\sigma \in B_n$ makes the following diagram commutative:

$$\begin{array}{ccc} H & \xrightarrow{\sigma_*} & H \\ \theta\searrow & & \swarrow\theta \\ & \mathbf{Z} & \end{array}$$

The homomorphism θ induces a homomorphism $\theta\colon \mathbf{Z}[H] \longrightarrow \mathbf{Z}[\mathbf{Z}] \simeq \mathbf{Z}[t, t^{-1}]$.

Denote by $Bu\colon B_n \longrightarrow GL_n(\mathbf{Z}[t, t^{-1}])$ the (true) homomorphism defined by the composition

$$Bu\colon B_n \xrightarrow{B^{ab}} GL_n(\mathbf{Z}[H]) \xrightarrow{\theta} GL_n(\mathbf{Z}[t, t^{-1}]).$$

This is the (unreduced) Burau representation of B_n.

From Corollary 4.4 we deduce

COROLLARY 5.3. *For any $\sigma \in B_n$, its Burau matrix satisfies*

$$\overline{Bu(\sigma)}^t \times \widetilde{\Omega}_n \times B_n(\sigma) = \widetilde{\Omega}_n,$$

where

$$\widetilde{\Omega}_n = \begin{pmatrix} 1 & & & 0 & \dots & 0 \\ & \ddots & & & & \\ 1-t & & 1 & & & \vdots \\ \vdots & & & \ddots & & 0 \\ & & & & 1 & \\ 1-t & \dots & 1-t & & & 1 \end{pmatrix}$$

REMARK. Squier [Sq] gives an "hermitian" matrix M_n such that

$$\overline{Bu(\sigma)}^t \times M_n \times Bu(\sigma) = M_n ,$$

but our matrix $\widetilde{\Omega n}$ is much simpler for two reasons:

(a) $\widetilde{\Omega}_n \in GL_n(\mathbf{Z}[t, t^{-1}])$, whereas $M_n \in GL_n(\mathbf{Z}[t^{\pm 1/2}])$;

(b) $\widetilde{\Omega}_n$ is triangular.

The fact that $\widetilde{\Omega}_n$ is triangular imposes more constraints on a matrix to be a Burau matrix, than that of Squier. This will help to understand the group of Burau matrices (recall that we know that the Burau representation is not faithful for $n \geq 5$ by [Moo], [L; P], [Bg]).

COROLLARY 5.4. *Corollary 5.2 is true, if Gassner matrices are replaced by Burau matrices.*

Added in proof. After this paper had been written, the author was informed (in June 2005) that Theorem 0.1 and Lemma 1.2 were obtained previously by V. Turaev in a paper "Intersection loops in two-dimensional manifolds", which appeared in *Mathematics of the USSR Sbornik 35* (1979).

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