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Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **52 (2006)**

Heft 1-2: **L'enseignement mathématique**

PDF erstellt am: **23.09.2024**

Persistenter Link: https://doi.org/10.5169/seals-2228

L'Enseignement Mathématique (2) 52 (2006), 143–158

A SHORT SOLUTION TO HIRONAKA'S POLYHEDRA GAME

by Dominik ZEILLINGER *)

INTRODUCTION

In section 1 the polyhedra game is defined. After an example in section 2 a winning strategy for the first player is presented in section 3.

The formulation of the polyhedra game in this paper is motivated by the connection to the resolution of singularities as shown in section 4 (and also given for instance in the article of Herwig Hauser, see [1]). There are small (but not substantial) differences to the original version of the game which was proposed by Heisuke Hironaka in 1970 and solved by Mark Spivakovsky in 1983 (see [3]). Section 5 gives the original version of the game and a sketch of Spivakovsky's solution, which is then compared with this paper's solution.

1. THE POLYHEDRA GAME

We start with a finite set of points $A = \{a_1, \ldots, a_d\} \subseteq \mathbf{N}^n$. The positive convex hull $N(A)$ of A in $\mathbf{R}^n$, that is

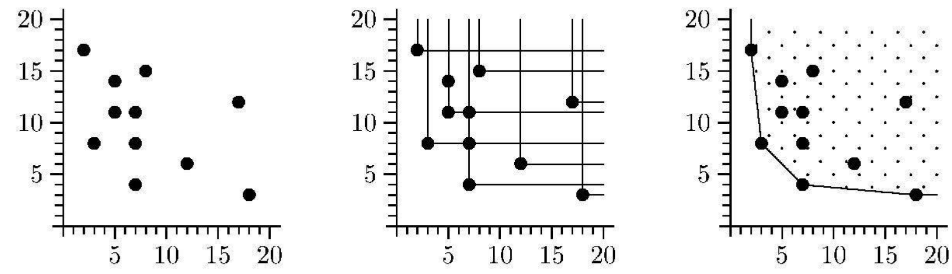
$$N(A) := \operatorname{conv}(A + \mathbf{R}^n_{\geq 0}),$$

is called the Newton polyhedron generated by A (see Figure 1).

Moreover, we define the linear function $\tau_{J,j}\colon \mathbf{Z}^n \to \mathbf{Z}^n\colon w \to w'$ for a non-empty subset $J \subseteq \{1, \ldots, n\}$ and $j \in J$ as follows: Substitute the j-th component of $w \in \mathbf{Z}^n$ by the sum of the components w_i with index i in J and leave the other components unchanged (see Figure 2):

$$\begin{aligned} w'_j &:= \sum_{i \in J} w_i\,, \\ w'_k &:= w_k \quad \text{for } k \neq j\,. \end{aligned}$$

*) Supported in part by FWF – Project P-15551 of the Austrian Ministry of Science.



The set A The polyhedron $N(A)$ The sum $A + \mathbf{R}_1^2$

FIGURE 1
Newton polyhedron N generated by A

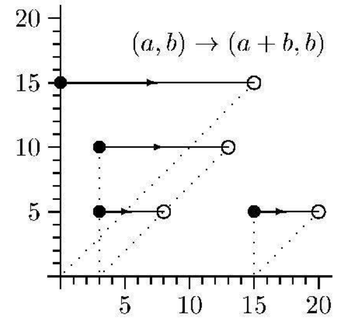


FIGURE 2
Movements of points under $\tau_{\{1,2\},1}$

Two players, $\mathcal{P}_1$ and $\mathcal{P}_2$, make the following moves:

1. Player $\mathcal{P}_1$ chooses, in consideration of A respectively $N(A)$, a non-empty set $J \subseteq \{1, \ldots, n\}$.
2. Player $\mathcal{P}_2$ chooses, in consideration of A and J, a $j \in J$ and replaces the set $A = \{a_1, \ldots, a_d\}$ by the set

$$A' := \tau_{J,j}(A) = \{\tau_{J,j}(a_1), \ldots, \tau_{J,j}(a_d)\} = \{a'_1, \ldots, a'_d\}\,.$$

(So the Newton polyhedron $N(A)$ is replaced by the Newton polyhedron $N' = N(A')$.)

The next round starts over again, with A resp. N replaced by A' resp. N': Player $\mathcal{P}_1$ chooses a subset J' of $\{1, \ldots, n\}$, and $\mathcal{P}_2$ picks j' in J' as before. The set A' resp. the polyhedron N' is replaced by $A^2 := A''$ resp. $N^2 := N''$. The game proceeds in this way. Player $\mathcal{P}_1$ wins if, after finitely many moves, the polyhedron N has been transformed into an orthant. That is, there are $m \in \mathbf{N}$ and $\alpha \in \mathbf{R}^n_{\geq 0}$ such that

$$N^m = \alpha + \mathbf{R}^n_{\geq 0}\,.$$

If this never occurs, player $\mathcal{P}_2$ will have won.

PROBLEM. Show that player $\mathcal{P}_1$ can always win. More precisely: Show that player $\mathcal{P}_1$ has a winning strategy for every set A resp. for every polyhedron N. That is, player $\mathcal{P}_1$ has a procedure for choosing $J \subseteq \{1, \dots, n\}$ at each stage of the game starting with A resp. N such that following it guarantees the victory of $\mathcal{P}_1$ in a finite number of moves, regardless of the responses of $\mathcal{P}_2$.

2. AN EXAMPLE AND FIRST OBSERVATIONS

Let there be given

$$A = \{(0,6), (3,2), (4,1), (8,0)\}\,.$$

Figure 3 shows that A is identical with the set of vertices of the Newton polyhedron $N(A)$. Note that a point $\alpha \in \mathbf{R}^n$ is a vertex of a polyhedron if and only if there is an affine hyperplane whose section with the polyhedron is exactly α.

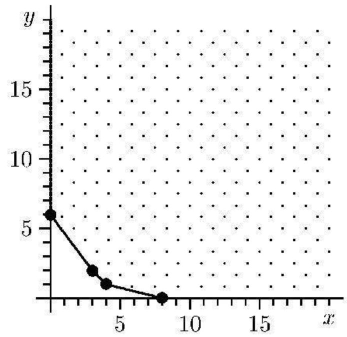


FIGURE 3
The polyhedron $N(A)$

Moreover we have the following lemma.

LEMMA. *Let A be a finite subset of $\mathbf{N}^n$, let $N = N(A)$ be the Newton polyhedron generated by A and let E be the set of vertices of N.*

1. *We have $E \subseteq A$ and $N(E) = N(A)$.*
2. *If α is a vertex of N, then $N(E) \neq N(E \setminus \{\alpha\})$.*

Proof. These statements are well known and can be found, for instance, in [5].

Let us assume that player $\mathcal{P}_1$ chooses $J = \{1,2\}$ (in fact, this is his only reasonable choice in dimension 2, because with $J = \{1\}$ or $J = \{2\}$ nothing will change after one move in the game). If player $\mathcal{P}_2$ chooses $j = 2$, the set A is replaced by

$$\begin{aligned} A^1 &= \tau_{J,j}(A) = \tau_{\{1,2\},2}(A) \\ &= \big\{(0,6),(3,5),(4,5),(8,8)\big\}\,. \end{aligned}$$

Player $\mathcal{P}_1$ has not won yet, because the Newton polyhedron $N' = N^1$ generated by A^1 is not an orthant. It has the two vertices $(0,6)$ and $(3,5)$ (see Figure 4) and so

$$N^1 = N(A^1) = N\big(\{(0,6),(3,5)\}\big)\,.$$

Note that the two vertices $(0,6)$ and $(3,2)$ of N are the "ancestors" of the two vertices of N^1 and that $(4,1)$ and $(8,0)$ have moved to the interior of N^1.

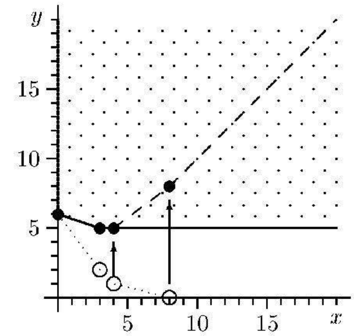


FIGURE 4

The polyhedron $N(A^1)$; the empty circles and the dotted lines give the position of the polyhedron $N(A)$ and the dashed lines show the position of the edges of $N(A)$ after the transformation $\tau_{\{1,2\},2}$ of the game

Now the game can proceed as follows: Choosing $J = \{1,2\}$ and $j = 1$ results in (see Figure 5)

$$A^2 = \{(6,6),(8,5),(9,5),(16,8)\}$$

and if the players continue with $J = \{1,2\}$ and $j = 1$ we have (see Figure 6)

$$A^3 = \{(12,6),(13,5),(14,5),(24,8)\}.$$

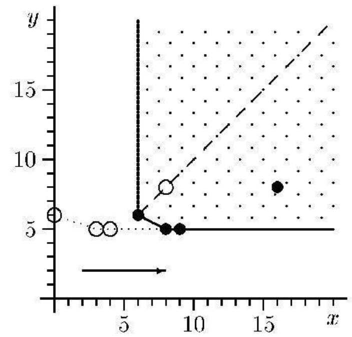


FIGURE 5
The polyhedron $N(A^2)$

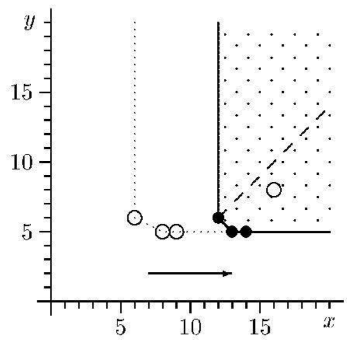


FIGURE 6
The polyhedron $N(A^3)$

Finally the game ends, if the players choose $J = \{1,2\}$ and $j = 2$, which leads to the result

$$A^4 = \{(12,18),(13,18),(14,19),(24,32)\}.$$

Now Player $\mathcal{P}_1$ has won because the Newton polyhedron generated by A^4 has only one vertex, namely $(12,18)$ (see Figure 7). All other elements of A^4 are contained in the orthant $(12,18) + \mathbf{R}^n_{\geq 0}$.

A close study of this example motivates the following lemma:

LEMMA 2.1. *Let A be a finite subset in $\mathbf{N}^n$ and $N = N(A)$ be the Newton polyhedron generated by A. Moreover let E be the set of vertices of N and $E(N')$ the set of vertices of N'. Then we have*

$$E(N') \subseteq (E(N))' = E'$$

and therefore

$$\#E(N') \leq \#E(N).$$

Proof. The statement follows from the linearity of the transformation $\tau_{J,j}\colon \alpha \to \alpha'$.

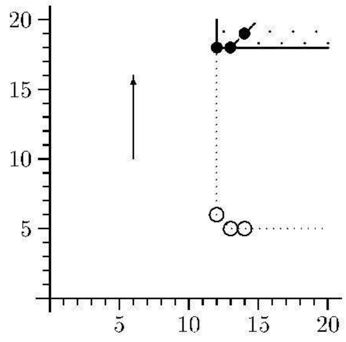


FIGURE 7
The polyhedron $N(A^4)$ is an orthant

3. A WINNING STRATEGY

To describe how player $\mathcal{P}_1$ should choose $J \subseteq \{1, \ldots, n\}$ when given a Newton polyhedron N, we use the concept of characteristic vectors $v_N \in \mathbf{Z}^n$ of N. These vectors are defined by using the notions of length and surface area of a vector.

DEFINITION 3.1. Let w be a vector in $\mathbf{Z}^n$ and N be a Newton polyhedron in $\mathbf{R}^n_{\geq 0}$.

1. The length $\mathsf{L}(w)$ of w is the difference of a maximal and a minimal component of w, thus

$$\mathsf{L}(w) := \max_{1 \leq i \leq n} w_i - \min_{1 \leq i \leq n} w_i .$$

2. Maximal and minimal components of w are called extremal components of w.

3. Let $\mathsf{S}(w)$, the surface area of w, denote the number of extremal components of w, thus

$$\mathsf{S}(w) := \#\{k : w_k = \min_{1 \leq i \leq n} w_i\} + \#\{k : w_k = \max_{1 \leq i \leq n} w_i\} .$$

4. Let B be the set of all vectors connecting two vertices of N, i.e.,

$$B := \{\alpha - \beta : \alpha, \beta \text{ vertices of } N\} .$$

A characteristic vector v_N of N is a minimal vector of B with respect to the inequality

$$\big(\mathsf{L}(v_N), \mathsf{S}(v_N)\big) \leq_{\text{lex}} \big(\mathsf{L}(w), \mathsf{S}(w)\big)$$

for all vectors $w \in B$. Here $<_{\text{lex}}$ denotes the lexicographical order on $\mathbf{N}^2$.

If v is a characteristic vector of a Newton polyhedron N, $-v$ is also a characteristic vector of N. There may be several characteristic vectors, as the following example shows.

EXAMPLE 3.2. Let N be the Newton polyhedron generated by

$$A = \{(0,0,4), (5,0,1), (1,5,1), (0,25,0)\}.$$

The set of vertices of N is equal to A. The following table helps us to find the characteristic vectors of N:

Vector v between two vertices of N	$\mathsf{L}(v)$	$\mathsf{S}(v)$	$n - \mathsf{S}(v)$
$\pm(5,0,-3)$	8	1	2
$\pm(1,5,-3)$	8	1	2
$\pm(4,-5,0)$	9	1	2
$\pm(1,-20,1)$	21	2	1
$\pm(0,25,-4)$	29	1	2
$\pm(5,-25,1)$	30	1	2

So the characteristic vectors of N are $\pm(5,0,-3)$ and $\pm(1,5,-3)$.

These notations suffice to define a winning strategy for player $\mathcal{P}_1$.

THEOREM 3.3 (Winning strategy). *Let N be a Newton polyhedron in $\mathbf{R}^n_{\geq 0}$. Player $\mathcal{P}_1$ wins the polyhedra game if he uses the following strategy:*

If N is an orthant, the game is already won. If N is not an orthant, choose

$$J := \{k, l\} \subseteq \{1, \ldots, n\}$$

with $1 \leq k \neq l \leq n$ such that w_k is a minimal and w_l a maximal component of a characteristic vector $w = (w_1, \ldots, w_n)$ of N.

The above choice of J is in general not unique. In example 3.2, player $\mathcal{P}_1$ can choose between $J_1 = \{1,3\}$ and $J_2 = \{2,3\}$.

Proof. We assign to each Newton polyhedron N a triple $\delta(N)$ in $\mathbf{N}^3$ and then show that

$$\delta(N') <_{\text{lex}} \delta(N)$$

if N is not an orthant and N' is the polyhedron obtained from N after one move in the game with the given strategy of player $\mathcal{P}_1$ and an arbitrary move by player $\mathcal{P}_2$. Note that $<_{\text{lex}}$ is the lexicographical order on $\mathbf{N}^3$. This proves the theorem because otherwise $\left(\delta(N^m)\right)_{m\in\mathrm{N}}$ would be a strictly decreasing sequence in $(\mathbf{N}^3, <_{\text{lex}})$, which is a contradiction, as $\mathbf{N}^3$ is well ordered with $<_{\text{lex}}$.

Let N be a Newton polyhedron, $w := v_N$ a characteristic vector of N and $E(N)$ the set of vertices of N. We define

$$\delta(N) := \left(\#E(N), \mathsf{L}(v_N), \mathsf{S}(v_N)\right).$$

Note that $\delta(N)$ is independent of the choice of the characteristic vector.

Now let N' be the polyhedron after one move of the game. (Player $\mathcal{P}_1$ chooses J as described above, player $\mathcal{P}_2$ chooses randomly.) Let $E(N')$ be the set of vertices of N'.

If $\#E(N') < \#E(N)$ we are done, so let us assume that $\#E(N') = \#E(N)$. (The number of vertices cannot increase because of Lemma 2.1.) But this implies that for every vertex γ of N', there is a vertex ϵ of N such that $\gamma = \epsilon'$ (i.e., ϵ is the "ancestor" of γ).

Let α, β be two vertices of N such that $w := \alpha - \beta$ is a characteristic vector of N. We will prove that for $w' = \alpha' - \beta'$, the inequality

$$\left(\mathsf{L}(w'), \mathsf{S}(w')\right) <_{\text{lex}} \left(\mathsf{L}(w), \mathsf{S}(w)\right)$$

holds. Then

$$\left(\mathsf{L}(v_{N'}), \mathsf{S}(v_{N'})\right) \leq_{\text{lex}} \left(\mathsf{L}(w'), \mathsf{S}(w')\right),$$

for any characteristic vector $v_{N'}$ of N' and the claim follows.

We first show that the inequality $\mathsf{L}(w') \leq \mathsf{L}(w)$ always holds. Without loss of generality we may suppose that w_1 is a minimal and w_2 a maximal component of w, hence

$$w_1 < 0 < w_2$$

and $\mathsf{L}(w) = w_2 - w_1$.

Following the strategy described in the theorem, player $\mathcal{P}_1$ chooses

$$J = \{1, 2\}.$$

Player $\mathcal{P}_2$ now has two choices. As the argument is the same in both cases, we just look at the choice $j = 1$. We then have

$$w' = (w_1 + w_2, w_2, \ldots, w_n).$$

(Note that w' and w differ only in the first component. So $w'_2 = w_2$ is again a maximal component of w'.) Since w_1 is a minimal and w_2 a maximal component of w by assumption, the inequality

$$w_1 < w_1 + w_2 < w_2 \tag{3.1}$$

holds. Therefore $\mathsf{L}(w') \leq \mathsf{L}(w)$.

If $\mathsf{L}(w) < \mathsf{L}(w')$ we are done. (If we have dimension $n = 2$ this is always the case.) So let us assume $\mathsf{L}(w') = \mathsf{L}(w)$, which implies that there is an index $3 \leq k$ with $w'_k = w_k = w_1$ minimal and $\mathsf{L}(w') = \mathsf{L}(w) = w_2 - w_k$. So w_k is an extremal component of w and w'. Now let $EC(w)$ be the set of all extremal components of w. As inequality (3.1) always holds the component $w'_1 = w_1 + w_2$ is not an extremal component of w'. Thus we have for $EC(w')$, the set of extremal components of w',

$$EC(w') = EC(w) \setminus \{w_1\}$$

and hence

$$\mathsf{S}(w') = \mathsf{S}(w) - 1\,.$$

Therefore we have

$$\big(\mathsf{L}(w'), \mathsf{S}(w')\big) = \big(\mathsf{L}(w), \mathsf{S}(w) - 1\big) <_{\text{lex}} \big(\mathsf{L}(w), \mathsf{S}(w)\big)\,.$$

In conclusion we get the inequality

$$\delta(N') <_{\text{lex}} \delta(N)\,,$$

which proves the theorem.

4. The connection to resolution of singularities

The goal of this section is to show the connection between the polyhedra game and blowing up singularities. The expert reader will excuse us if simplicity has been preferred to generality.

Henceforth K denotes an algebraically closed field of characteristic zero, $\mathbf{A}^n$ the n-dimensional affine space and $\mathbf{P}^n$ the n-dimensional projective space. Let $X \subseteq \mathbf{A}^n$ be an affine algebraic set of dimension r defined by $f_1, \ldots, f_k \in K[x_1, \ldots, x_n]$. (By the dimension of X we mean the Krull dimension of its coordinate ring $K[X] = K[x_1, \ldots, x_n]/I(X)$ with $I(X)$ the unique radical vanishing ideal of X in $K[x_1, \ldots, x_n]$.) The set X is singular at a point $a \in \mathbf{A}^n$ if the Jacobian matrix of $f_1, \ldots, f_k$ at a is of rank smaller than $n - r$. The set of singular points of X is called the singular locus of X.

The problem of resolution of singularities is to prove that: For any algebraic variety X over K there exists a proper birational map $f\colon Y \to X$, with Y

non-singular (smooth), such that f is an isomorphism over some open dense subset U of X (i.e., f maps $f^{-1}(U)$ isomorphically onto U).

The most famous and general result concerning this problem was given by Heisuke Hironaka in 1964 (see [2]). He proved that the resolution of singularities can be achieved by a sequence of blowups (see below) if K is of characteristic zero.

Let X, Y be algebraic sets and $\varphi\colon X \to Y$ a continuous map. If for all open (relative to the Zariski topology) $V \subseteq Y$ and for all regular $f\colon V \to K$ the map $f \circ \varphi\colon \varphi^{-1}(V) \to K$ is also regular on $\varphi^{-1}(V)$, the map φ is called a morphism. Moreover a morphism $\varphi\colon X \to Y$ of varieties is birational if there exists an open dense subset $V \subseteq Y$, with $U = \varphi^{-1}(V)$, and a map $\pi\colon V \to U$ which can be given by means of rational functions, such that $\pi \circ \varphi|_U = \mathrm{Id}_U$ and $\varphi|_U \circ \pi = \mathrm{Id}_V$.

To define the blowup of the affine space $\mathbf{A}^n$ with center an ideal $P = \langle g_1, \dots, g_k \rangle \subseteq K[x_1, \dots, x_n]$, resp. with center Z the subvariety $V(P) \subseteq \mathbf{A}^n$, consider the map

$$\begin{aligned} \mathbf{A}^n \setminus Z &\longrightarrow \mathbf{A}^n \times \mathbf{P}^{k-1} \\ \mathbf{x} &\mapsto \big(\mathbf{x}, g_1(\mathbf{x}) : \dots : g_k(\mathbf{x})\big) . \end{aligned}$$

This is a well defined injective morphism. The image is the graph $\Gamma(g)$ of

$$\begin{aligned} g\colon \mathbf{A}^n \setminus Z &\longrightarrow \mathbf{P}^{k-1} \\ \mathbf{x} &\mapsto \big(g_1(\mathbf{x}) : \dots : g_k(\mathbf{x})\big) . \end{aligned}$$

Now the blowup of $\mathbf{A}^n$ with center Z is the Zariski-closure $\widetilde{\mathbf{A}^n}$ of $\Gamma(g)$ in $\mathbf{A}^n \times \mathbf{P}^{k-1}$. The projection

$$\begin{aligned} \pi\colon \widetilde{\mathbf{A}^n} &\longrightarrow \mathbf{A}^n \\ (\mathbf{x}, z_1 : \dots : z_n) &\mapsto \mathbf{x} \end{aligned}$$

is the associated blowup map. It is a birational morphism. Note that $\pi\colon \widetilde{\mathbf{A}^n} \setminus \pi^{-1}(Z) \to \mathbf{A}^n \setminus Z$ is an isomorphism. The subvariety $\pi^{-1}(Z)$ of $\widetilde{\mathbf{A}^n}$ which is contracted under π to Z is a hypersurface and is called the exceptional divisor E of the blowup.

For X a subvariety of $\mathbf{A}^n$ we obtain in a similar manner the Zariski-closure $\widetilde{X}$ of the image of

$$\begin{aligned} X \setminus Z &\longrightarrow X \times \mathbf{P}^{k-1} \\ \mathbf{x} &\mapsto \big(\mathbf{x}, g_1(\mathbf{x}) : \dots : g_k(\mathbf{x})\big) , \end{aligned}$$

which is called the blowup of X with center Z. Again $\pi\colon \widetilde{X} \to X$ is the associated blowup map.

Let us assume that the subvariety X is the zero set of the polynomial $f(x_1,\ldots,x_n) \in K[x_1,\ldots,x_n]$ and that Z is defined by the ideal $P = \langle x_1,\ldots,x_k\rangle$. The standard charts of $\mathbf{P}^{k-1}$ induce charts on $\mathbf{A}^n \times \mathbf{P}^{k-1}$. The l-th affine chart expression of the blowup of X with center Z, with $1 \le l \le k$, is now given by

$$f(x_1x_l + t_1x_l,\ldots,x_{l-1}x_l + t_{l-1}x_l, x_l, x_{l+1}x_l + t_{l+1}x_l,\ldots,x_kx_l + t_kx_l, x_{k+1},\ldots,x_n),$$

with $(t_1,\ldots,t_k) \in K^k$. If we just look at the origin of the l-th affine chart expression, we have

$$f_l(\mathbf{x}) := (x_1x_l,\ldots,x_{l-1}x_l, x_l, x_{l+1}x_l,\ldots,x_kx_l, x_{k+1},\ldots,x_n).$$

To connect the technique of blowing up with the polyhedra game, we associate to the polynomial $f = \sum_{\alpha\in\mathbf{N}^n} c_\alpha \mathbf{x}^\alpha$ the Newton polyhedron $N(f)$ generated by the support $\mathrm{supp}(f) := \{\alpha \in \mathbf{N}^n : c_\alpha \neq 0\}$ of f. Now let us look at the Newton polyhedron $N(f_l)$. As we have

$$f = \sum_{\alpha\in\mathbf{N}^n} c_\alpha x_1^{\alpha_1}\cdots x_{l-1}^{\alpha_{l-1}} x_l^{\sum_{m=1}^k \alpha_k} x_{l+1}^{\alpha_{l+1}}\cdots x_n^{\alpha_n},$$

the support of f_l is given by

$$\mathrm{supp}(f_l) = \left\{\left(\alpha_1,\ldots,\alpha_{l-1}, \sum_{m=1}^k \alpha_m, \alpha_{l+1},\ldots,\alpha_n\right) : \alpha \in \mathrm{supp}(f)\right\}.$$

Therefore we have

$$N(f_l) = N(f)'$$

with $N(f)'$ the Newton polyhedron obtained from $N(f)$ after one move of the polyhedra game with $J = \{1,\ldots,k\}$ and $j = l$.

Conclusion: One move in the polyhedra game is equivalent to taking a look at the origin of one chart expression of a blowup with a chosen center Z.

5. Hironaka's game and Spivakovsky's solution

We first give the original version of the polyhedra game as in [3]: Let N be the Newton polyhedron generated by a finite set $A \subseteq \mathbf{Q}^n$. We assume that $\sum_{i=1}^n x_i > 1$ for all $(x_1,\ldots,x_n) \in N$. Two players, $\mathcal{P}_1$ and $\mathcal{P}_2$, make the following moves:

1. Player $\mathcal{P}_1$ chooses a non-empty set $J \subseteq \{1, \dots, n\}$ such that $\sum_{i \in J} x_i \geq 1$ for all $(x_1, \dots, x_n) \in N$ (such a J is called permissible).
2. Player $\mathcal{P}_2$ chooses an element $j \in J$ and replaces N by $N' := \sigma_{J,j}(N)$, where $\sigma_{J,j} \colon \mathbf{R}^n \to \mathbf{R}^n$ is defined by

$$x'_k := x_k \text{ if } k \neq j,$$
$$x'_j := \sum_{i \in J} x_i - 1.$$

The game proceeds in this way. Player $\mathcal{P}_1$ wins if, after finitely many moves, the polyhedron N has been transformed into a polyhedron M which contains a point $(x_1, \dots, x_n)$ such that $\sum_{i=1}^n x_i \leq 1$. If this never occurs, player $\mathcal{P}_2$ will have won.

Mark Spivakovsky presented a solution in 1983 (see [3]). To decide which J to choose, Spivakovsky uses the following definitions (see also Figure 8).

Let $N \subseteq \mathbf{R}_+^n$ be a Newton polyhedron with $\sum_{i=1}^n x_i > 1$ for all $(x_1, \dots, x_n) \in N$. By E we denote the set of vertices of N.

1. The vector $\omega(N) = (\omega_1(N), \dots, \omega_n(N)) \in \mathbf{R}_+^n$ is defined by

$$\omega_i := \min\{a_i \,:\, (a_1, \dots, a_n) \in E\}, \qquad \text{for } i = 1, \dots, n.$$

2. Let $\widetilde{N}$ be the translation of N by $\omega(N)$, say

$$\widetilde{N} := N - \omega(N).$$

(Therefore we have $\omega(\widetilde{N}) = (0, \dots, 0)$.) Likewise, we define for a vertex a of N

$$\widetilde{a} := a - \omega(N).$$

3. Let $d(N)$ denote the smallest "distance" of a vertex of N to the origin, more precisely:

$$d(N) := \min\left\{\sum_{i=1}^n a_i \,:\, a \in E\right\}.$$

(The number $d(\widetilde{N})$ is the first important numerical character of N.)

4. We define a partition of $\{1, \dots, n\}$:

$$S(N) := \big\{i \in \{1, \dots, n\} \,:\, \text{there is an } a \in E \text{ with } a_1 + \dots + a_n = d(N) \text{ and } \widetilde{a}_i \neq 0\big\},$$

$$I_1(N) := \{1, \dots, n\} \setminus S(N).$$

5. For $l \in \mathbf{Z}$ and two complementary subsets S and I of $\{1, \dots, l\}$, we define the subset $M_S^l \subseteq \mathbf{R}_{\geq 0}^l$ by
$$M_S^l := \left\{(x_1, \dots, x_l) \in \mathbf{R}_{\geq 0}^l \ : \ \sum_{i \in S} x_i < 1\right\},$$
and the projection $P_S \colon M_S^l \to \mathbf{R}_{\geq 0}^I$ as follows: For $\alpha \in \mathbf{R}_{\geq 0}^I, \beta \in \mathbf{R}_{\geq 0}^S$ with $\sum_{i=1}^n \beta_i < 1$, we set
$$P_S(\alpha, \beta) = \frac{\alpha}{1 - \sum \beta_i}.$$

With these notations for a Newton polyhedron $N \subseteq \mathbf{R}_{\geq 0}^n$, we define a Newton polyhedron $N_1 \subseteq \mathbf{R}_{\geq 0}^{I_1}$ by
$$N_1 := P_{S(N)}\left(M_{S(N)}^n \cap \left[\frac{\widetilde{N}}{d(\widetilde{N})} \cup N\right]\right).$$
Then Spivakovsky repeats the same procedure with N_1, namely, he defines $\widetilde{N_1}$, $I_2 = I_1 \backslash S(N_1)$, etc. He thus obtains a sequence of polyhedra $N_1, N_2, \dots, N_r$ and a decreasing sequence $I_1 \supseteq I_2 \supseteq \dots \supseteq I_r$ of subsets of $\{1, \dots, n\}$ such that

$N_i \subseteq \mathbf{R}_{\geq 0}^{I_i}$ for $i = 1, \dots, r$,

$\omega(N_i) = (\omega_k)_{k \in I_i}, \quad \omega_k = \min\{x_k \ : \ x \in N_i\}, \ k \in I_i$,

$\widetilde{N_i} = N_i - \omega(N_i)$,

$S(\omega_i) = \{k \in I_i \ : \ \exists\, w \in N_i \text{ such that } \sum w_l = d(N_i) \text{ and } \widetilde{w_k} \neq 0\}$,

$I_{i+1} = I_i \setminus S(N_i)$ for $i = 1, \dots, r-1$,

$N_{i+1} = P_{S(N_i)}\left(M_{S(N_i)}^{I_i} \cap \left[\frac{\widetilde{N_i}}{d(\widetilde{N_i})} \cup N\right]\right).$

The procedure stops when either $d(\widetilde{N_r}) = 0$ or $N_r = \varnothing$.

After these preparations the choice J of player $\mathcal{P}_1$ is given by:

- If $N_r = \varnothing$, player $\mathcal{P}_1$ chooses $J = \{1, \dots, n\} \setminus I_r$.
- If $N_r \neq \varnothing$, player $\mathcal{P}_1$ first chooses a subset $J_r \subseteq I_r$ which is minimal among those permissible for N_r (i.e. $\sum_{i \in J_r} x_i \geq 1$ for all $(x_1, \dots, x_{\#I_r}) \in N_r$ and no proper subset of I_r is permissible). Then, he sets
$$J = (\{1, \dots, n\} \setminus I_r) \cup I_r.$$

To prove that the given procedure allows player $\mathcal{P}_1$ to win, Spivakovsky associates to every Newton polyhedron N the $2r + 2$ numbers
$$\delta(N) := \left(d(\widetilde{N}), \#I_1, d(\widetilde{N_1}), \dots, \#I_r, d(\widetilde{N_r}), d(N_r)\right).$$

Spivakovsky shows that after one move in the game the inequality

$$\delta(N) <_{\text{lex}} \delta(N')$$

holds for the resulting polyhedron N'.

Let's compare Spivakovsky's solution to the solution given in this paper.

Note that the two versions of the game – we will call them Spivakovsky's game and this paper's game – are equivalent in the following sense: Every winning strategy for one version of the game can be used to win the other version. This can be seen as follows: On the one hand this paper's game is in essence a special case of Spivakovsky's game: If it is not possible to win this paper's game for a given $A \subseteq \mathbf{N}^n$ by using a winning strategy for Spivakovsky's game then it would also be impossible to win Spivakovsky's game with $A \subseteq \mathbf{N}^n \subseteq \mathbf{Q}^n_{\geq 0}$.

Let on the other hand A be a finite subset of $\mathbf{Q}^n_{\geq 0}$. As A is finite, there is a $q \in \mathbf{N}$ such that $q \cdot A$ is a subset of $\mathbf{N}^n$. Therefore if we use a winning strategy for this paper's game in Spivakovsky's game we will get – after finitely many moves in the game – an orthant $\widetilde{N}$ which is generated by just one vertex α. Spivakovsky's game is won, if we have $\sum_{i=1}^n \alpha_i \leq 1$. But if we have $\sum_{i=1}^n \alpha_i > 1$, it is an easy exercise to find a strategy to win the game in the end (note that an orthant remains an orthant after one move in the game).

Moreover we have the following common properties and differences:

COMMON PROPERTIES. • Both strategies are algorithms for determining the right choice for player $\mathcal{P}_1$. The choice depends only on the given polyhedron N and *not* on the history of the preceding moves.

• In general the strategies do not prescribe a unique choice of J but an option of several possible J, any one of which ultimately guarantees victory.

DIFFERENCES. • To determine the right move for player $\mathcal{P}_1$, Spivakovsky uses induction on the dimension n. The solution presented above does not need induction and is thus easier to describe.

• Spivakovsky's solution chooses J as big as possible (for instance $J = \{1, \ldots, n\}$ can occur). This paper's solution always gives a choice J which consists of only two elements.

• The winning strategies for the ployhedra game can be used to decide which center to choose when blowing up a singular affine variety. Spivakovsky's strategy is closer to the choice given by Heisuke Hironaka than this paper's solution.

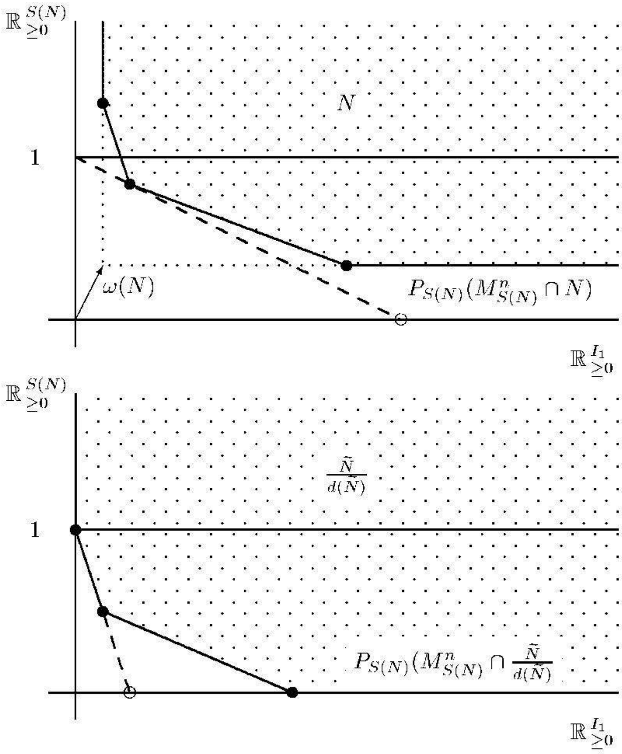


FIGURE 8
Illustration of Spivakovsky's notation

This comparison makes clear that the solution presented in this paper and Spivakovsky's solution are quite different. Which strategy should player $\mathcal{P}_1$ choose? It seems that the solution in this paper is easier to use. But maybe player $\mathcal{P}_1$ wants to finish the game as quickly as possible. It would be interesting to study which solution brings victory faster.

Here is a hint for anyone who wants to study the complexity of this paper's solution: If the given Newton polyhedron has just two vertices α and β, all the information about it is given by the characteristic vector $w = \alpha - \beta$. And it is possible to give an upper bound for the number of moves necessary for player $\mathcal{P}_1$ to win the game by using the above strategy: Let M be the maximum of the absolute values of all components of w, say

$$M := \max_{1 \leq i \leq n} |w_i| .$$

Then the game requires at most $(n-1) \cdot M$ moves (because the length of the characteristic vector drops by at least one after at most $n-1$ moves).

This upper bound is sharp in certain cases; consider for instance the example $w = (-1, M, \dots, M)$.

ACKNOWLEDGEMENTS

I want to thank Herwig Hauser who brought the game to my attention and encouraged me to find a new solution. Moreover he was a great help in making the paper more readable. I also want to thank the anonymous referee whose suggestions polished the whole paper.

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(Reçu le 24 février 2006)

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