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ON PRODUCTS OF DISJOINT BLOCKS OF CONSECUTIVE INTEGERS

by Maciej ULAS

ABSTRACT. In this note we disprove the conjecture of Erdős and Graham which states that for fixed $k \geq 2$ and $l \geq 4$ the product of k disjoint blocks of l consecutive integers is a square only finitely many times. We give two infinite families of solutions for $k = l = 4$.

Let $l \geq 2$ be an integer, and set $f(x) = (x+1) \cdot \ldots \cdot (x+l)$. For a fixed integer $k \geq 1$ we consider the diophantine equation

$$y^2 = \prod_{i=1}^{k} f(x_i)\,, \tag{1}$$

with the condition

$$0 < x_1 < \cdots < x_k\,, \quad x_j + l \leq x_{j+1}\,, \quad j = 1, \ldots, k-1\,. \tag{2}$$

Condition (2) ensures that the blocks $f(x_1), \ldots, f(x_k)$ of consecutive integers are disjoint.

Erdős and Selfridge [1] have proved a celebrated theorem implying that for $k = 1$ and each $l \geq 2$ the only solutions of (1) are $x_1 = -1, \ldots, -l$, $y = 0$. On the other hand, if there is more than one block we have

THEOREM 1. *For $l \leq 3$ and each $k \geq 2$, equation* (1) *has infinitely many solutions satisfying condition* (2).

Proof. For $l = 2$ and $k = 2$ we take

$$x_1 = n - 1\,, \quad x_2 = 4n^2 + 4n - 1\,, \quad n \geq 2\,.$$

For $l = 2$ and $k = 3$ we take

$$x_1 = n - 1\,, \quad x_2 = 3n\,, \quad x_3 = \frac{9n(n+1)}{2} - 1\,, \quad n \geq 2\,.$$

Since each $k \geq 2$ is of the form $2a+3b$ with nonnegative a, b, we obtain our conclusion for $l = 2$.

For $l = 3$ and $k = 2$, K. R. S. Sastry takes

$$x_1 = n - 1\,, \quad x_2 = 2n - 1\,,$$

where n, $m \in \mathbf{N}$ satisfy the equation

$$(n+2)(2n+1) = m^2\,,$$

which has infinitely many solutions [3].

For $l = 3$ and $k = 3$ we take, as does Skałba in [4],

$$x_1 = F_{2u-1} - 2\,, \quad x_2 = F_{2u+1} - 2\,, \quad x_3 = F_{2u}^2 - 2\,, \quad u \geq 2\,,$$

where F_n is the n-th term of the Fibonacci sequence.

As in the case $l = 2$, we see that for each $k \geq 2$ equation (1) has infinitely many solutions satisfying (2). □

In contrast to this result Erdős and Graham conjecture in ([2], p. 67) that if $l \geq 4$, then for each fixed k, equation (1) has only finitely many solutions satisfying (2). This conjecture is also mentioned in [3] as Problem D17.

In connection with this problem M. Skałba proved in [4] that if we allow k to vary, the above conjecture is not true.

We disprove the Erdős and Graham conjecture by establishing the following

THEOREM 2. *For $k = l = 4$, equation* (1) *has infinitely many solutions satisfying* (2).

Proof. Let n be an integer greater than 1. We put

$$x_1 = 4n - 1\,, \quad x_2 = 4n + 3\,, \quad x_3 = 4n^2 + 7n - 1\,, \quad x_4 = 8n^2 + 14n + 1\,.$$

Then

$$y^2 = f(x_1)f(x_2)f(x_3)f(x_4)\,,$$

where

$$\begin{aligned} y = {} & 16n(n+1)(2n+1)(2n+3)(4n+1)(4n+3) \\ & \times (4n+5)(4n+7)(4n^2+7n+1)(4n^2+7n+2)\,. \end{aligned}$$

We can also take

$$x_1 = 4n\,, \quad x_2 = 4(n+1)\,, \quad x_3 = 4n^2 + 9n + 1\,, \quad x_4 = 8n^2 + 18n + 5\,,$$

and then

$$y = 16(n+1)(n+2)(2n+1)(2n+3)(4n+1)(4n+3) \\ \times (4n+5)(4n+7)(4n^2+9n+3)(4n^2+9n+4)\,. \qquad \square$$

In connection with the above theorem it is natural to ask which is the smallest integer k such that equation (1) with $l = 4$ has infinitely many solutions satisfying (2).

We looked for solutions with $l = 4$ and $k = 2$ or $k = 3$ using a computer.

For $k = 2$ we computed all solutions of (1) which satisfy $x_1+3 < x_2 < 10^5$. In this range we found only one solution, $x_1 = 32$, $x_2 = 1679$.

For $k = 3$ we computed all solutions of (1) which satisfy condition (2) and $x_3 < 5 \cdot 10^3$. The results are shown in the following table.

x_1	x_2	x_3
3	12	23
3	15	167
4	13	47
7	24	62
7	38	285
9	244	1022
10	66	2207
11	30	152
17	30	339
17	47	167

x_1	x_2	x_3
19	95	482
45	158	844
64	74	132
131	321	2207
186	208	421
245	283	494
368	404	898
491	549	1103
920	1063	1841

Since each $k \geq 6$ is of the form $4s$, $4s+2$, $4s+3$ or $4s+5$, we obtain from the above computations and parametric solution in Theorem 2

THEOREM 3. *For $l = 4$ and any fixed $k \geq 6$, equation* (1) *has infinitely many solutions satisfying* (2).

I firmly believe that in the remaining case $k = 5$ there are infinitely many solutions. In the case $l \geq 5$ the following conjecture seems to be true.

CONJECTURE 4. *For each $l \geq 5$ there exists an integer $n = n(l)$ such that for each $k \geq n$ equation* (1) *has infinitely many solutions satisfying condition* (2).

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