2.5 Dunkl operators and symmetric quantum integrals

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In general, m is not a homomorphism. However:

PROPOSITION 2.4. *Let $\mathcal{A}^W \subset \mathcal{A}$ denote the subalgebra of elements invariant under conjugation by W. Then the restriction of m to $\mathcal{A}^W$ is an algebra homomorphism.*

Proof. If $A \in \mathcal{A}^W$, then clearly $m(A)$ is W-invariant. Now if we take $A, B \in \mathcal{A}^W$ and f a W-invariant function we have that $B(f)$ is also W-invariant. So

$$m(AB)(f) = (AB)(f) = A(B(f)) = A(m(B)(f)) = m(A)(m(B)(f)).$$

Thus $m(AB)$ and $m(A)m(B)$ coincide on W-invariant functions and hence coincide. □

2.5 DUNKL OPERATORS AND SYMMETRIC QUANTUM INTEGRALS

In this subsection we will construct quantum integrals of the Calogero-Moser operator. This construction is due to Heckman [He] and is based on the Dunkl operators, introduced in [Du].

Fix a W-invariant function $c \colon \Sigma \to \mathbf{C}$ such that $\beta_s = c_s(c_s + 1)$ for each $s \in \Sigma$. Set $\delta_c := \prod_{s\in\Sigma} \alpha_s(x)^{c_s}$ and define

$$L = \delta_c(x) H \delta_c(x)^{-1}.$$

Then an easy computation shows that

$$L = \Delta - \sum_{s\in\Sigma} \frac{2c_s}{\alpha_s(x)} \partial_{\alpha_s},$$

where, for a vector $y \in \mathfrak{h}$, the symbol ∂_y denotes, as usual, the partial derivative in the y direction (notice that using the scalar product we are viewing α_s as a vector in $\mathfrak{h}$ orthogonal to the hyperplane fixed by s).

From now on we will work with L instead of H and study the eigenvalue problem

$$L\psi = \lambda\psi. \tag{4}$$

It is clear that ψ is a solution of this equation if and only if $\delta_c(x)^{-1}\psi$ is a solution of (3).

Since for any $s \in \Sigma$ and $f \in \mathbf{C}[\mathfrak{h}]$ we have that $f(sx) - f(x)$ is divisible by $\alpha_s(x)$, the operator

$$\frac{1}{\alpha_s(x)}(s-1) \in \mathcal{A}$$

maps $\mathbf{C}[\mathfrak{h}]$ to itself.

DEFINITION 2.5. Given $y \in \mathfrak{h}$, we define the *Dunkl operator* D_y on $\mathbf{C}[\mathfrak{h}]$ by

$$D_y := \partial_y + \sum_{s\in\Sigma} c_s \frac{(\alpha_s, y)}{\alpha_s(x)}(s-1).$$

We have the following very important theorem.

THEOREM 2.6 ([Du]). *Let $y, z \in \mathfrak{h}$. Then*

$$[D_y, D_z] = 0.$$

Proof. See [Du], [Op]. □

PROPOSITION 2.7 (Heckman [He]). *Let $\{y_1, \ldots, y_n\}$ be an orthonormal basis of $\mathfrak{h}$. Then we have*

$$m(\sum_{i=1}^{n} D_{y_i}^2) = L.$$

Proof. Observe that $m(\sum_{i=1}^n D_{y_i}^2) = \sum_{i=1}^n m(D_{y_i}^2)$, so we need to compute $m(D_y^2)$ for $y \in \mathfrak{h}$. We have $m(D_y^2) = m(D_y m(D_y)) = m(D_y \partial_y)$. A simple computation shows that

$$D_y\partial_y = \partial_y^2 + \sum_{s\in\Sigma} c_s \frac{(\alpha_s, y)}{\alpha_s(x)}(\partial_y(s-1) - \frac{2(\alpha_s, y)}{(\alpha_s, \alpha_s)}\partial_{\alpha_s} s).$$

Thus

$$m(D_y^2) = \partial_y^2 - 2\sum_{s\in\Sigma} c_s \frac{(\alpha_s, y)^2}{(\alpha_s, \alpha_s)\alpha_s(x)}\partial_{\alpha_s}.$$

We get

$$m(\sum_{i=1}^{n} D_{y_i}^2) = \sum_i \partial_{y_i}^2 - 2\sum_{s\in\Sigma} c_s \frac{\sum_{i=1}^n (\alpha_s, y_i)^2}{(\alpha_s, \alpha_s)\alpha_s(x)}\partial_{\alpha_s} = L,$$

since $\sum_{i=1}^n (\alpha_s, y_i)^2 = (\alpha_s, \alpha_s)$. □

We are now ready to give the construction of quantum integrals of L. Consider the symmetric algebra $S\mathfrak{h} = \mathbf{C}[y_1, \ldots, y_n]$ which we can identify, using the fact that the Dunkl operators commute, with the polynomial ring $\mathbf{C}[D_{y_1}, \ldots, D_{y_n}] \subset \mathcal{A}$. The restriction of m to $S\mathfrak{h}^W$ is an algebra homomorphism into the ring $\mathcal{D}(U)$ (and in fact into $\mathcal{D}(U/W)$). Since $S\mathfrak{h}^W$ is itself a polynomial ring $\mathbf{C}[q_1, \ldots, q_n]$, with $q_1, \ldots, q_n$ of degree $d_1, \ldots, d_n$,

d_i being the degrees of basic W-invariants, we obtain a polynomial ring of commuting differential operators in $\mathcal{D}(U)$. Given $q \in \mathbf{C}[q_1, \dots, q_n]$ we will denote by L_q the corresponding differential operator. We may assume that $q_1 = \sum_{i=1}^n y_1^2$ so that $L = L_{q_1}$. Thus for every $q \in \mathbf{C}[q_1, \dots, q_n]$, L_q is a quantum integral of the quantum Calogero-Moser system. In particular, the operators $L_{q_1}, \dots, L_{q_n}$ are n algebraically independent pairwise commuting quantum integrals.

Now the eigenvalue problem (4) may be replaced by

$$L_p \psi = \lambda_p \psi$$

for $p \in \mathbf{C}[q_1, \dots, q_n]$, where the assignment $p \to \lambda_p$ is an algebra homomorphism $\mathbf{C}[q_1, \dots, q_n] \to \mathbf{C}$.

In other words, we may say that since $\mathbf{C}[q_1, \dots, q_n] = \mathbf{C}[\mathfrak{h}^*/W] = \mathbf{C}[\mathfrak{h}/W]$, for every point $k \in \mathfrak{h}/W$, we have the eigenvalue problem

$$L_p \psi = p(k) \psi \,. \tag{5}$$

PROPOSITION 2.8. *Near a generic point $x_0 \in \mathfrak{h}$, the system $L_p \psi = p(k)\psi$ has a space of solutions of dimension $|W|$.*

Proof. The proposition follows easily from the fact that the symbols of L_{q_i} are $q_i(\partial)$, and that $\mathbf{C}[y_1, \dots, y_n]$ is a free module over $\mathbf{C}[q_1, \dots, q_n]$ of rank $|W|$. □

2.6 ADDITIONAL INTEGRALS FOR INTEGER VALUED c

If $c_s \notin \mathbf{Z}$, the analysis of the solutions of the equations $L_p \psi = p(k)\psi$ is rather difficult (see [HO]). However, in the case $c \colon \Sigma \to \mathbf{Z}$, the system can be simplified. Let us consider this case. First remark that, since $\beta_s = c_s(c_s+1)$, by changing c_s to $-1-c_s$ if necessary, we may assume that c is non-negative. So we will assume that c takes non-negative integral values and we will denote it by m.

System (5) can be further simplified, if we can find a differential operator M (not a polynomial of $L_{q_1}, \dots, L_{q_n}$) such that $[M, L_p] = 0$ for all $p \in \mathbf{C}[q_1, \dots, q_n]$. Then the operator M will act on the space of solutions of (5), hopefully with distinct eigenvalues. So if μ is such an eigenvalue, the system

$$\begin{cases} L_p \psi = p(k)\, \psi \\ M\psi = \mu\, \psi \end{cases}$$