

## 2.5 Dunkl operators and symmetric quantum integrals

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In general,  $m$  is not a homomorphism. However:

PROPOSITION 2.4. *Let  $\mathcal{A}^W \subset \mathcal{A}$  denote the subalgebra of elements invariant under conjugation by  $W$ . Then the restriction of  $m$  to  $\mathcal{A}^W$  is an algebra homomorphism.*

*Proof.* If  $A \in \mathcal{A}^W$ , then clearly  $m(A)$  is  $W$ -invariant. Now if we take  $A, B \in \mathcal{A}^W$  and  $f$  a  $W$ -invariant function we have that  $B(f)$  is also  $W$ -invariant. So

$$m(AB)(f) = (AB)(f) = A(B(f)) = A(m(B)(f)) = m(A)(m(B)(f)).$$

Thus  $m(AB)$  and  $m(A)m(B)$  coincide on  $W$ -invariant functions and hence coincide.  $\square$

## 2.5 DUNKL OPERATORS AND SYMMETRIC QUANTUM INTEGRALS

In this subsection we will construct quantum integrals of the Calogero-Moser operator. This construction is due to Heckman [He] and is based on the Dunkl operators, introduced in [Du].

Fix a  $W$ -invariant function  $c: \Sigma \rightarrow \mathbf{C}$  such that  $\beta_s = c_s(c_s + 1)$  for each  $s \in \Sigma$ . Set  $\delta_c := \prod_{s \in \Sigma} \alpha_s(x)^{c_s}$  and define

$$L = \delta_c(x)H\delta_c(x)^{-1}.$$

Then an easy computation shows that

$$L = \Delta - \sum_{s \in \Sigma} \frac{2c_s}{\alpha_s(x)} \partial_{\alpha_s},$$

where, for a vector  $y \in \mathfrak{h}$ , the symbol  $\partial_y$  denotes, as usual, the partial derivative in the  $y$  direction (notice that using the scalar product we are viewing  $\alpha_s$  as a vector in  $\mathfrak{h}$  orthogonal to the hyperplane fixed by  $s$ ).

From now on we will work with  $L$  instead of  $H$  and study the eigenvalue problem

$$(4) \quad L\psi = \lambda\psi.$$

It is clear that  $\psi$  is a solution of this equation if and only if  $\delta_c(x)^{-1}\psi$  is a solution of (3).

Since for any  $s \in \Sigma$  and  $f \in \mathbf{C}[\mathfrak{h}]$  we have that  $f(sx) - f(x)$  is divisible by  $\alpha_s(x)$ , the operator

$$\frac{1}{\alpha_s(x)}(s - 1) \in \mathcal{A}$$

maps  $\mathbf{C}[\mathfrak{h}]$  to itself.

DEFINITION 2.5. Given  $y \in \mathfrak{h}$ , we define the Dunkl operator  $D_y$  on  $\mathbf{C}[\mathfrak{h}]$  by

$$D_y := \partial_y + \sum_{s \in \Sigma} c_s \frac{(\alpha_s, y)}{\alpha_s(x)} (s - 1).$$

We have the following very important theorem.

THEOREM 2.6 ([Du]). Let  $y, z \in \mathfrak{h}$ . Then

$$[D_y, D_z] = 0.$$

*Proof.* See [Du], [Op].  $\square$

PROPOSITION 2.7 (Heckman [He]). Let  $\{y_1, \dots, y_n\}$  be an orthonormal basis of  $\mathfrak{h}$ . Then we have

$$m\left(\sum_{i=1}^n D_{y_i}^2\right) = L.$$

*Proof.* Observe that  $m(\sum_{i=1}^n D_{y_i}^2) = \sum_{i=1}^n m(D_{y_i}^2)$ , so we need to compute  $m(D_y^2)$  for  $y \in \mathfrak{h}$ . We have  $m(D_y^2) = m(D_y m(D_y)) = m(D_y \partial_y)$ . A simple computation shows that

$$D_y \partial_y = \partial_y^2 + \sum_{s \in \Sigma} c_s \frac{(\alpha_s, y)}{\alpha_s(x)} (\partial_y (s - 1) - \frac{2(\alpha_s, y)}{(\alpha_s, \alpha_s)} \partial_{\alpha_s}).$$

Thus

$$m(D_y^2) = \partial_y^2 - 2 \sum_{s \in \Sigma} c_s \frac{(\alpha_s, y)^2}{(\alpha_s, \alpha_s) \alpha_s(x)} \partial_{\alpha_s}.$$

We get

$$m\left(\sum_{i=1}^n D_{y_i}^2\right) = \sum_i \partial_{y_i}^2 - 2 \sum_{s \in \Sigma} c_s \frac{\sum_{i=1}^n (\alpha_s, y_i)^2}{(\alpha_s, \alpha_s) \alpha_s(x)} \partial_{\alpha_s} = L,$$

since  $\sum_{i=1}^n (\alpha_s, y_i)^2 = (\alpha_s, \alpha_s)$ .  $\square$

We are now ready to give the construction of quantum integrals of  $L$ . Consider the symmetric algebra  $S\mathfrak{h} = \mathbf{C}[y_1, \dots, y_n]$  which we can identify, using the fact that the Dunkl operators commute, with the polynomial ring  $\mathbf{C}[D_{y_1}, \dots, D_{y_n}] \subset \mathcal{A}$ . The restriction of  $m$  to  $S\mathfrak{h}^W$  is an algebra homomorphism into the ring  $\mathcal{D}(U)$  (and in fact into  $\mathcal{D}(U/W)$ ). Since  $S\mathfrak{h}^W$  is itself a polynomial ring  $\mathbf{C}[q_1, \dots, q_n]$ , with  $q_1, \dots, q_n$  of degree  $d_1, \dots, d_n$ ,

$d_i$  being the degrees of basic  $W$ -invariants, we obtain a polynomial ring of commuting differential operators in  $\mathcal{D}(U)$ . Given  $q \in \mathbf{C}[q_1, \dots, q_n]$  we will denote by  $L_q$  the corresponding differential operator. We may assume that  $q_1 = \sum_{i=1}^n y_i^2$  so that  $L = L_{q_1}$ . Thus for every  $q \in \mathbf{C}[q_1, \dots, q_n]$ ,  $L_q$  is a quantum integral of the quantum Calogero-Moser system. In particular, the operators  $L_{q_1}, \dots, L_{q_n}$  are  $n$  algebraically independent pairwise commuting quantum integrals.

Now the eigenvalue problem (4) may be replaced by

$$L_p \psi = \lambda_p \psi$$

for  $p \in \mathbf{C}[q_1, \dots, q_n]$ , where the assignment  $p \rightarrow \lambda_p$  is an algebra homomorphism  $\mathbf{C}[q_1, \dots, q_n] \rightarrow \mathbf{C}$ .

In other words, we may say that since  $\mathbf{C}[q_1, \dots, q_n] = \mathbf{C}[\mathfrak{h}^*/W] = \mathbf{C}[\mathfrak{h}/W]$ , for every point  $k \in \mathfrak{h}/W$ , we have the eigenvalue problem

$$(5) \quad L_p \psi = p(k) \psi.$$

**PROPOSITION 2.8.** *Near a generic point  $x_0 \in \mathfrak{h}$ , the system  $L_p \psi = p(k) \psi$  has a space of solutions of dimension  $|W|$ .*

*Proof.* The proposition follows easily from the fact that the symbols of  $L_{q_i}$  are  $q_i(\partial)$ , and that  $\mathbf{C}[y_1, \dots, y_n]$  is a free module over  $\mathbf{C}[q_1, \dots, q_n]$  of rank  $|W|$ .  $\square$

## 2.6 ADDITIONAL INTEGRALS FOR INTEGER VALUED $c$

If  $c_s \notin \mathbf{Z}$ , the analysis of the solutions of the equations  $L_p \psi = p(k) \psi$  is rather difficult (see [HO]). However, in the case  $c: \Sigma \rightarrow \mathbf{Z}$ , the system can be simplified. Let us consider this case. First remark that, since  $\beta_s = c_s(c_s + 1)$ , by changing  $c_s$  to  $-1 - c_s$  if necessary, we may assume that  $c$  is non-negative. So we will assume that  $c$  takes non-negative integral values and we will denote it by  $m$ .

System (5) can be further simplified, if we can find a differential operator  $M$  (not a polynomial of  $L_{q_1}, \dots, L_{q_n}$ ) such that  $[M, L_p] = 0$  for all  $p \in \mathbf{C}[q_1, \dots, q_n]$ . Then the operator  $M$  will act on the space of solutions of (5), hopefully with distinct eigenvalues. So if  $\mu$  is such an eigenvalue, the system

$$\begin{cases} L_p \psi = p(k) \psi \\ M \psi = \mu \psi \end{cases}$$