4. Rational knots and their mirror images

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rational knots. This constitutes a unique use of topological mathematics as a theoretical underpinning for a problem in molecular biology.

4. Rational knots and their mirror images

In this section we give an application of Theorem 2. An unoriented knot or link K is said to be *achiral* if it is topologically equivalent to its mirror image $-K$. If a link is not equivalent to its mirror image then it is said be *chiral*. One then can speak of the *chirality* of a given knot or link, meaning whether it is chiral or achiral. Chirality plays an important role in the applications of knot theory to chemistry and molecular biology. In [8] the authors find an explicit formula for the number of achiral rational knots among all rational knots with n crossings. It is interesting to use the classification of rational knots and links to determine their chirality. Indeed, we have the following well-known result (for example see [35] and [16], p. 24, Exercise 2.1.4; compare also with [31]):

THEOREM 5. *Let $K = N(T)$ be an unoriented rational knot or link, presented as the numerator of a rational tangle T. Suppose that $F(T) = p/q$ with p and q relatively prime. Then K is achiral if and only if $q^2 \equiv -1 \mod p$. It follows that the tangle T has to be of the form $[[a_1], [a_2], \ldots, [a_k], [a_k], \ldots, [a_2], [a_1]]$ for any integers $a_1, \ldots, a_k$.*

Note that in this description we are using a representation of the tangle with an even number of terms. The leftmost twists $[a_1]$ are horizontal, thus $|p| > |q|$. The rightmost starting twists are then vertical.

Proof. With $-T$ the mirror image of the tangle T, we have that $-K = N(-T)$ and $F(-T) = p/(-q)$. If K is isotopic to $-K$, it follows from the classification theorem for rational knots that either $q(-q) \equiv 1 \mod p$ or $q \equiv -q \mod p$. Without loss of generality we can assume that $0 < q < p$. Hence $2q$ is not divisible by p and therefore it is not the case that $q \equiv -q \mod p$. Hence $q^2 \equiv -1 \mod p$.

Conversely, if $q^2 \equiv -1 \mod p$, then it follows from the Palindrome Theorem that *the continued fraction expansion of p/q has to be palindromic with an even number of terms.* To see this, let $p/q = [c_1, \ldots, c_n]$ with n even, and let $p'/q' = [c_n, \ldots, c_1]$. The Palindrome theorem tells us that $p' = p$ and that $qq' \equiv -1 \mod p$. Thus we have that both q and q' satisfy the equation $qx \equiv -1 \mod p$ and both q and q' are between 1 and $p-1$. Since this equation has a unique solution in this range, we conclude that $q = q'$. It follows at once that the continued fraction sequence for p/q is symmetric.

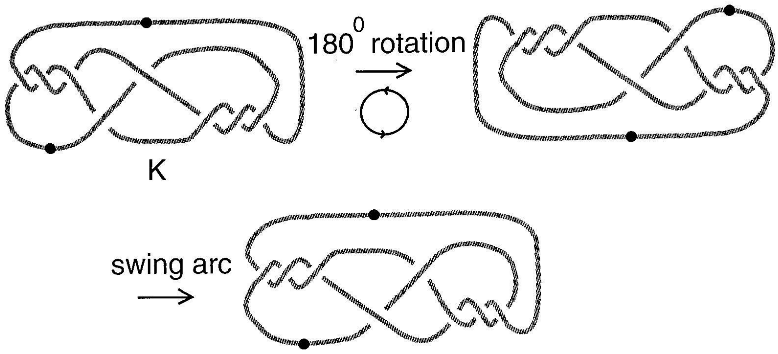

FIGURE 29
An achiral rational link

It is then easy to see that the corresponding rational knot or link $K = N(T)$ is equivalent to its mirror image. One rotates K by $180°$ in the plane and swings an arc, as Figure 29 illustrates. The point is that the crossings of the second row of the tangle T, that are seemingly crossings of opposite type than the crossings of the upper row, become after the turn crossings of the upper row, and so the types of crossings are switched. This completes the proof. $\square$

5. ON CONNECTIVITY

We shall now introduce the notion of *connectivity* and we shall relate it to the fraction of unoriented rational tangles. We shall say that an unoriented rational tangle has *connectivity type* [0] if the NW end arc is connected to the NE end arc and the SW end arc is connected to the SE end arc. These are the same connections as in the tangle $[0]$. Similarly, we say that the tangle has *connectivity type* $[\infty]$ *or* $[1]$ if the end arc connections are the same as in the tangles $[\infty]$ and $[+1]$ (or equivalently $[-1]$) respectively. The basic connectivity patterns of rational tangles are exemplified by the tangles $[0]$, $[\infty]$ and $[+1]$. We can represent them iconically by

$$[0] = \asymp$$
$$[\infty] = ><$$
$$[1] = \chi$$