# 11. Geodesic triangles are thin

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of $a$, such that both paths $\mathcal{G}$ and $\mathcal{G}'$ have one point $A$-close to $h'$, for some constant $A$. Since $\mathcal{G}$ and $\mathcal{G}'$ both end or begin at the point $a$, this implies that $\mathcal{G}'$ admits a point $B$-close to each point of the orbit-segment between $a$ and $h'$. In particular there exists $Q \in \mathcal{G}'$ which is $B + L_1$-close to $P \in \mathcal{G}$.

It remains to consider the case where no horizontal geodesic connects the past orbits of the endpoints of the considered $(J, J')$-quasi geodesic bigon. Then, in the future orbit of the initial endpoint there exists a point $z$ whose past orbit can be connected to the past orbit of the terminal endpoint, and this property is not satisfied by the point $w$ with $f(z) - f(w) = t_0$, which is either in the future or past orbit of the initial endpoint. The strong hyperbolicity of the semi-flow and Proposition 8.1 then give a constant $C_{8.1}(M, J, J')$ such that initial subpaths of both sides of the bigon are $C_{8.1}(M, J, J') + t_0$-close to the orbit-segment connecting the initial endpoint of the bigon to $z$. From what precedes, any $(R, R')$-quasi geodesic bigon between $z$ and the terminal endpoint of the considered bigon is $X(R, R')$-thin, for some constant $X(R, R')$. This easily implies that the given bigon is $2(C_{8.1}(M, J, J') + t_0) + X(R, R' + C_{8.1}(M, J, J') + t_0)$-thin. $\square$

## 11. GEODESIC TRIANGLES ARE THIN

The following lemma was suggested to the author by I. Kapovich, and allows us to simplify the conclusion. Let us recall that, in the context of quasi geodesic metric spaces, an $(r', s')$-chain bigon is a bigon whose sides are $(r', s')$-chains. Still with this terminology, an $(r, s)$-chain triangle is a triangle whose sides are $(r, s)$-chains.

LEMMA 11.1. *Let $X$ be an $(r, s)$-quasi geodesic metric space. If $(r', s')$-chain bigons are $\delta(r', s')$-thin, $r' \geq r$, $s' \geq s$, then $X$ is $2\delta(r, 3s)$-hyperbolic.*

*Proof.* We consider an $(r, s)$-chain triangle with vertices $a$, $b$, $c$ and sides $[ab]$, $[ac]$ and $[bc]$. We consider a point $x$ in the $(r, s)$-chain $[ab]$ which is closest to $c$. We claim that $[cx] \cup [xb]$ is an $(r, 3s)$-chain, where $[cx]$ and $[xb]$ denote $(r, s)$-chains from $c$ to $x$ and from $x$ to $b$. Indeed, for any points $u, v$ in $[xb]$ or $[cx]$, one obviously has $rd_X(u, v) \geq |[uv]|_X$. Let us thus assume that $u \in [cx]$ and $v \in [xb]$. Since $x$ is a point in $[ab]$ closest to $c$, $x$ is a point in $[ab]$ closest to $u$. Thus $|[ux]|_X \leq |[uv]|_X$. Moreover $|[xv]|_X \leq |[xu]|_X + |[uv]|_X$. Therefore $|[ux]|_X + |[xv]|_X \leq 3|[uv]|_X$. Whence the claim. The given $(r,s)$-chain triangle can be decomposed into two $(r,3s)$-chain bigons. Therefore this triangle is $2\delta(r,3s)$-thin. $\square$

LEMMA 11.2. *Let $(\widetilde{X}, f, \sigma_t, \mathcal{H})$ be a forest-stack. There exists a constant $C_{11.2}(r,s)$ such that any $(r,s)$-chain in $(\widetilde{X}, d_{(\widetilde{X},\mathcal{H})})$ is contained in a $(C_{11.2}(r,s), C_{11.2}(r,s))$-quasi geodesic.*

*Proof.* Any pair of consecutive points $x_{i-1}, x_i$, $i = 1, \ldots, k$, in an $(r,s)$-chain $c = x_0, \ldots, x_k$ can be connected by a telescopic path $p_i$ which is the concatenation of exactly one vertical and one horizontal geodesic. The vertical length of the vertical geodesic is bounded above by $d_{(\widetilde{X},\mathcal{H})}(x_{i-1}, x_i)$. By the bounded-dilatation property, the horizontal length of the horizontal geodesic is bounded above by $\lambda_+^{d_{(\widetilde{X},\mathcal{H})}(x_{i-1},x_i)} d_{(\widetilde{X},\mathcal{H})}(x_{i-1}, x_i)$. If $p$ is the concatenation of the $p_i$'s then $p$ is a telescopic path containing the chain $c$, whose telescopic length satisfies

$$|p|_{(\widetilde{X},\mathcal{H})} \leq \sum_{i=1}^{k} (1 + \lambda_+^{d_{(\widetilde{X},\mathcal{H})}(x_{i-1},x_i)})\, d_{(\widetilde{X},\mathcal{H})}(x_{i-1}, x_i)\,.$$

Since we consider $(r,s)$-chains, we have $d_{(\widetilde{X},\mathcal{H})}(x_{i-1}, x_i) \leq r$. Thus $|p|_{(\widetilde{X},\mathcal{H})} \leq (1 + \lambda_+^r) \sum_{i=1}^{k} d_{(\widetilde{X},\mathcal{H})}(x_{i-1}, x_i)$. By definition of an $(r,s)$-chain $\sum_{i=1}^{k} d_{(\widetilde{X},\mathcal{H})}(x_{i-1}, x_i) \leq s\, d_{(\widetilde{X},\mathcal{H})}(x_0, x_k)$. Thus $|p|_{(\widetilde{X},\mathcal{H})} \leq s(1 + \lambda_+^r) d_{(\widetilde{X},\mathcal{H})}(x_0, x_k)$. Any subpath $p'$ of $p$ decomposes as a concatenation $q p_i p_{i+1} \cdots p_m q'$ where $q$, $q'$ are proper subpaths respectively of $p_{i-1}$ and $p_{m+1}$. The same arguments as above prove that $|p_i p_{i+1} \cdots p_m|_{(\widetilde{X},\mathcal{H})} \leq s(1 + \lambda_+^r) d_{(\widetilde{X},\mathcal{H})}(i(p_i), t(p_m))$. Furthermore $|q|_{(\widetilde{X},\mathcal{H})} \leq (1 + \lambda_+^r) r$ and $|q'|_{(\widetilde{X},\mathcal{H})} \leq (1 + \lambda_+^r) r$.

This implies that $|p'|_{(\widetilde{X},\mathcal{H})} \leq |p_i p_{i+1} \cdots p_m|_{(\widetilde{X},\mathcal{H})} + 2r(1 + \lambda_+^r)$ and $d_{(\widetilde{X},\mathcal{H})}(i(p_i), t(p_m)) \leq d_{(\widetilde{X},\mathcal{H})}(i(p'), t(p')) + 2r$. We conclude that

$$|p'|_{(\widetilde{X},\mathcal{H})} \leq s(1 + \lambda_+^r) d_{(\widetilde{X},\mathcal{H})}(i(p'), t(p')) + 2r(1 + s)(1 + \lambda_+^r)\,.$$

Setting $C_{11.2}(r,s) = \max(s, 2r(1+s))(1 + \lambda_+^r)$, we get Lemma 11.2. $\square$

LEMMA 11.3. *There exists a constant $C_{11.3}(J,J')$ such that any $(J,J')$-quasi geodesic $\mathcal{G}$ is $C_{11.3}(J,J')$-close to a straight $(C_{11.3}(J,J'),C_{11.3}(J,J'))$-quasi geodesic.*

*Proof.* Let us call *bad subpath* of $\mathcal{G}$ any 'maximal' subpath $p$ of $\mathcal{G}$ whose endpoints lie in a same orbit-segment of the semi-flow, where 'maximal' means that, if $p_0$ (resp. $p_1$) are arbitrarily small, non trivial subpaths preceding (resp. following) $p$ in $\mathcal{G}$, then the endpoints of $p_0$ and $p_1$ do not lie in a same orbit-segment. We consider a bad subpath $p$. It might happen that $p$ contains other bad subpaths $p_\alpha$. In this case, we choose one of them, denoted by $q$, and we replace all the other bad subpaths in $p$ by the orbit-segment between their endpoints. Since orbit-segments are telescopic geodesics, the resulting path, denoted by $p'$, is a $(J,J')$-quasi geodesic. Since $p'$ does not contain any bad subpath other than $q$, there exists a point $a \in q \subset p'$ such that $p'$ is the concatenation of two straight $(J,J')$-quasi geodesics $g_0$, $g_1$, where $g_0$ goes from its initial point $i(p')$ to $a$, and $g_1$ goes from $a$ to its terminal point $t(p')$. We now consider the $(J,J')$-quasi geodesic triangle of vertices $i(p')$, $t(p')$, $a$, and with sides $g_0$, $g_1$ and the orbit-segment $O$ between $i(p')$ and $t(p')$. We consider any point $z \in g_1$ which minimizes the telescopic distance between $i(p')$ and $g_1$. We choose a telescopic geodesic $g_2$ between $i(p')$ and $g_1$.

We denote by $u$ (resp. $v$) the path from $i(p')$ to $a$ (resp. $t(p')$) which is the concatenation of $g_2$ with the subpath of $g_1$ between $z$ and $a$ (resp. $t(p')$). As in the proof of Lemma 11.1, we prove that the bigon of vertices $i(p')$ and $a$, with sides $g_0$ and $u$, and the bigon of vertices $i(p')$ and $t(p')$ with sides $v$ and $O$ are straight $(3J,3J')$-quasi geodesic bigons. By Proposition 10.1, these bigons are $Bi(3J,3J')$-thin. Thus there exist two points $x \in g_0$ and $y \in g_1$ which are $2Bi(3J,3J')$-close, and such that the subpaths of $g_0$ (resp. of $g_1$) between $i(p')$ and $x$ (resp. between $t(p')$ and $y$) are $2Bi(3J,3J')$-close to $O$. Since $p'$ is a $(J,J')$-quasi geodesic, we conclude that $p'$ is $(2J+2)Bi(3J,3J')+J'$-close to $O$. The same conclusion holds if one considers any bad subpath other than $q$ in $p$. Thus any point in $p$ is $(2J+2)Bi(3J,3J')+J'$-close to $O$. Since the choice of the bad subpath $p$ is arbitrary, the proof is complete. □

*Proof of Theorem 4.4.* Let $(\widetilde{X},f,\sigma_t,\mathcal{H})$ be a forest-stack equipped with some horizontal metric $\mathcal{H}$ such that $(\sigma_t)_{t\in\mathbf{R}^+}$ is strongly hyperbolic with respect to $\mathcal{H}$. By the Lemma-Definition of Section 3.2, this forest-stack is a $(1,2)$-quasi geodesic metric space. Let us consider any $(r,s)$-chain bigon, $r \geq 1$, $s \geq 2$. By Lemma 11.2, it is contained in a $(C_{11.2}(r,s),C_{11.2}(r,s))$-quasi geodesic bigon. By Lemma 11.3, this bigon is $A(r,s)$-close, with $A(r,s) = C_{11.3}(C_{11.2}(r,s),C_{11.2}(r,s))$, to a straight $(A(r,s),A(r,s))$-quasi geodesic bigon.

Proposition 10.1 provides a $\kappa(r,s) = Bi(A(r,s), A(r,s))$ such that this bigon is $\kappa(r,s)$-thin. Thus the given $(r,s)$-chain bigon is $\delta(r,s)$-thin, with $\delta(r,s) = \kappa(r,s) + 2A(r,s)$. By Lemma 11.1, the given forest-stack, which is a $(1,2)$-quasi geodesic metric space, is $2\delta(1,6)$-hyperbolic. □

## 12. Back to mapping-telescopes

In this section we elucidate the relationships between forest-stacks and mapping-telescopes.

### 12.1 Statement of the theorem

An $\mathbf{R}$-*tree* (see [9], [2] among many others) is a metric space such that any two points are joined by a unique arc and this arc is a geodesic for the metric. In particular an $\mathbf{R}$-tree is a topological tree. An $\mathbf{R}$-*forest* is a union of disjoint $\mathbf{R}$-trees.

LEMMA 12.1. *Let $(\Gamma, d_\Gamma)$ be an $\mathbf{R}$-forest and let $\psi\colon \Gamma \to \Gamma$ be a forest-map of $\Gamma$. Let $(K_\psi, f, \sigma_t)$ be the mapping-telescope of $(\psi, \Gamma)$ equipped with a structure of forest-stack as defined in Section 2. Then there is a horizontal metric $\mathcal{H} = (m_r)_{r \in \mathbf{R}}$ on $K_\psi$ such that*

1. *The $\mathbf{R}$-forests $(f^{-1}(r), m_r)$ and $(f^{-1}(r+1), m_{r+1})$ are isometric. Each stratum $(f^{-1}(n), m_n)$, $n \in \mathbf{Z}$, is isometric to $(\Gamma, d_\Gamma)$.*
2. *For any real $r$ and any horizontal geodesic $g \in f^{-1}(r)$, the map*

$$l_{r,g}\colon \begin{cases} +1 - r] \to \mathbf{R}^+ \\ \quad t \quad \mapsto |\sigma_t(g)|_{r+t} \end{cases} .$$

*is monotone.*

*Such a horizontal metric is called a* horizontal $d_\Gamma$-metric. *The telescopic metric associated to a horizontal $d_\Gamma$-metric is called a* mapping-telescope $d_\Gamma$-metric.

*Proof.* We make each $\Gamma \times \{n\}$, $n \in \mathbf{Z}$, an $\mathbf{R}$-forest isometric to $\Gamma$. We consider a cover of $\Gamma$ by geodesics of length 1 which intersect only at their endpoints. Each $\Gamma \times \{n\}$ inherits the same cover. There is a disc $D_{e,n}$ in $K_\psi$ for each such horizontal geodesic $e$ in $\Gamma \times \{n\}$. This disc is bounded by $e$, $\psi(e)$ and the orbit-segments between the endpoints of $e$ and those of $\psi(e)$.