9. PUTTING PATHS IN FINE POSITION

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geodesic, with the same endpoints, in fine position with respect to h, which is $C_{7.3}(A(|h|_r, J, J'), J, J')$-close to g and which is a stair or the concatenation of two stairs. Lemma 6.4, together with Lemma 5.4 applied as above, then provide $C_{6.4}(B(|h|_r, J, J'), B(|h|_r, J, J'))$ and

$$D(|h|_r, J, J') = C_{5.4}(1, 3, C_{6.4}(B(|h|_r, J, J'), B(|h|_r, J, J'))$$

such that this, or these, stair(s) are $D(|h|_r, J, J')$-close to the orbit-segments between h and their endpoints. We conclude that g is $C_{7.3}(A(|h|_r, J, J'), J, J')+D(|h|_r, J, J')$-close to these orbit-segments. The last point of the proposition is obvious. □

9. Putting paths in fine position

PROPOSITION 9.1. *Let h be a horizontal geodesic. Let g be a straight (J, J')-quasi geodesic, which joins the future or past orbits of the endpoints of h. There exist a constant $C_{9.1}(J, J')$ and a $(C_{9.1}(J, J'), C_{9.1}(J, J'))$-quasi geodesic $\mathcal{G}$ which is $C_{9.1}(J, J')$-close to g, which has the same endpoints as g, and which is in fine position with respect to h.*

Proof. We consider a maximal subpath g' of g whose endpoints lie in the future or past orbits of some points in h, and such that no other point of g' satisfies this property. Consider any maximal $-$-hole b in g', and let I denote the horizontal geodesic between the endpoints of b.

CASE 1. Either I is contained in a cancellation or I is the concatenation of two horizontal geodesics, each contained in a cancellation.

Lemma 6.7 gives $C_{6.7}(J, J')$ such that, if $|I|_{f(I)} \geq C_{6.7}(J, J')$ then I is dilated in the future after $C_{6.7}(J, J')t_0$. Lemma 5.3 gives $C_{5.3}(C_{6.7}(J, J'))$ such that the horizontal length of any horizontal geodesic contained in a cancellation and dilated in the future after $C_{6.7}(J, J')t_0$ is at most $C_{5.3}(C_{6.7}(J, J'))$. By Lemma 5.4 we get an upper bound $C_{5.4}(C_{6.7}(J, J'), 2, C_{5.3}(C_{6.7}(J, J')))$ on the horizontal length of I.

CASE 2. There exists another horizontal geodesic in another connected component of the same stratum whose pulled-tight projection agrees with that of I after some finite time.

We consider the maximal geodesic preimage I' of I under $\sigma_{C_{6.7}(J,J')t_0}$ which connects two points of b. It admits a decomposition into subpaths I'_α

connecting points in b such that the subpath of b between the endpoints of each I'_α is a $-$-hole. The strong hyperbolicity of the semi-flow implies, by Lemma 6.7, that the horizontal length of each I'_α is bounded above by $C_{6.7}(J,J')$. Since g is a (J,J')-quasi geodesic, we get $\max_{x\in b}(f(I)-f(x)) \leq JC_{6.7}(J,J') + J' + C_{6.7}(J,J')$.

CASE 3. Some subpath of I connects the future or past orbits of points in h.

The only possibility is that I be a pulled-tight image of h, i.e. $g' = b$. Consider a geodesic preimage I' of I under $\sigma_{C_{6.7}(J,J')t_0}$ between two points in b. Then proceed as in Case 2, the only difference being that for each subpath I_α, *either* there exists a horizontal geodesic in another connected component of the same stratum, whose pulled-tight projection agrees with that of I_α after some finite time (this is exactly Case 2), *or* I_α is contained in a cancellation or in the union of two cancellations, and the arguments are exactly those of Case 1. The bounded-dilatation property then gives an upper bound on the horizontal length of I.

We denote by $A(J,J')$ the largest of the constants found in Cases 1, 2 and 3. We denote by $A'(J,J')$ the largest of the constants $A(J,J')$, $C_{7.3}(A(J,J'),J,J')$ and $C_{7.2}(A(J,J'),J,J')$. Lemmas 7.2, 7.3 and 7.1 then give $B(J,J') = C_{7.1}(A'(J,J'),A'(J,J'),\ J,J')$, such that replacing the maximal $-$-holes in g' by the horizontal geodesic between their endpoints yields a straight $(B(J,J'),B(J,J'))$-quasi geodesic stair S, with the same endpoints, which is $A'(J,J')$-close to g'. Let I' be a horizontal geodesic between S and a future or past orbit of some point in h, which is minimal in the sense of inclusion, i.e. does not contain any subpath connecting S to a future or past orbit of a point in h. This horizontal geodesic I' is a pulled-tight image of a subpath of S in the stratum considered. It is either contained in a cancellation, or is the union of two horizontal geodesics contained in a cancellation. Lemma 6.4 gives $C_{6.4}(B(J,J'),B(J,J'))$ such that, if $|I'|_{f(I')} \geq C_{6.4}(B(J,J'),B(J,J'))$ then I' is dilated in the futur after t_0. From Lemmas 5.3 and 5.4 we get $|I'|_{f(I')} \leq C_{5.4}(1,2,C_{5.3}(1))$. Therefore S is at horizontal distance at most $D(J,J') = \max(C_{6.4}(B(J,J'),B(J,J')),C_{5.4}(1,2,C_{5.3}(1)))$ from a straight stair $\mathcal{S}(g')$, with the same endpoints and in fine position with respect to h. Lemmas 7.4 and 7.1 then give $E(J,J') = C_{7.1}(C_{7.4}(D(J,J'),B(J,J'),B(J,J')),C_{7.4}(D(J,J'),B(J,J'),B(J,J')),J,J')$ such that replacing the maximal subpaths g' as above by the given stair $\mathcal{S}(g')$ gives a straight $(E(J,J'),E(J,J'))$-quasi geodesic, with the same endpoints as g, in fine position with respect to h, and which is $D(J,J')$-close to g. $\square$