

# 4. CONSEQUENCES OF GROMOV HYPERBOLICITY FOR THE SHAPE OF THE BOUNDARY

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is bounded in view of the estimate  $\kappa^{-1}(x^2 + y^2) < f(x, y) < \kappa(x^2 + y^2)$  for some universal  $\kappa > 0$  and of the fact that  $f(x, y) = \varepsilon$  on  $C_\varepsilon$ .  $\square$

#### 4. CONSEQUENCES OF GROMOV HYPERBOLICITY FOR THE SHAPE OF THE BOUNDARY

**PROPOSITION 4.1.** *Let  $D$  be a bounded convex domain in  $\mathbf{R}^n$  and let  $h$  be a Hilbert metric on  $D$ . If  $h$  is Gromov hyperbolic then the boundary  $\partial D$  is strictly convex, that is, it does not contain a line segment.*

This can be proven following the proof of N. Ivanov [Iv97] of Masur-Wolf's theorem [MW95] that the Teichmüller spaces (genus  $\geq 2$ ) are not Gromov hyperbolic. The proof makes use of Gromov's exponential divergence criterion, see [BH99, p.412]. For another proof of the above proposition, see [SM00].

**THEOREM 4.2.** *Let  $D$  be a bounded convex domain in  $\mathbf{R}^n$  and let  $h$  be the Hilbert metric on  $D$ . If  $h$  is Gromov hyperbolic then the boundary  $\partial D$  is smooth of class  $C^1$ .*

*Proof.* *2-dimensional case:* First, by the previous result,  $D$  is strictly convex. Let  $y = f(x)$ ,  $x \in (-a, a)$  be an equation of  $\partial D$  near some point. Then  $f$  is strictly convex and hence the one-sided derivatives  $f'_-(x)$ ,  $f'_+(x)$  exist and are strictly increasing on  $(\varepsilon, \varepsilon)$ , [RV73, §11].

We prove that  $f'_-(0) = f'_+(0)$ . Suppose not, then by choosing appropriate Cartesian coordinates we may assume that  $f'_-(0) < 0$  and  $f'_+(0) > 0$ . For each sufficiently small  $\varepsilon$  construct an ideal triangle  $\Delta = \Delta(\varepsilon)$  in  $D$  with one vertex  $0$  and two other vertices corresponding to the intersection of the line  $y = \varepsilon$  with  $\partial D$ . We assert that the slimness of  $\Delta(\varepsilon)$  tends to  $\infty$  when  $\varepsilon$  tends to zero. Namely we show that the Hilbert distance between the point  $P = (0, \varepsilon)$  and any point  $Q$  of the side  $[0, B]$  tends to  $\infty$ . Let  $f'_+(0) = \tan \alpha$ ,  $0 < \alpha < \pi/2$ . Let  $x_1 < x_2$  be the points such that  $f(x_1) = \varepsilon$  and  $f'_+(0)x_2 = \varepsilon$ . Then

$$PQ \geq \varepsilon \cos \alpha = f(x_1) \cos \alpha.$$

Let  $O, R$  be the intersection points of the line  $PQ$  with  $\partial D$ . We have therefore

$$QR \leq x_2 - x_1 = \frac{f(x_1)}{f'_+(0)} - x_1 = \frac{f(x_1) - f'_+(0)x_1}{f'_+(0)}$$

and hence, combining the last two inequalities,

$$\begin{aligned} \frac{PQ}{QR} &\geq \frac{f'_+(0)f(x_1)\cos\alpha}{f(x_1)-f'_+(0)x_1} \\ &= \frac{f'_+(0)\cos\alpha}{1-f'_+(0)\frac{x_1}{f(x_1)}} \rightarrow \infty \text{ when } x_1 \rightarrow 0. \end{aligned}$$

It follows that

$$h(P, Q) = \ln \left( 1 + \frac{PQ}{OP} \right) \left( 1 + \frac{PQ}{QR} \right) \rightarrow \infty \text{ when } x_1 \rightarrow 0$$

and hence the slimness of  $\Delta(\varepsilon)$  tends to  $\infty$  when  $\varepsilon$  tends to zero.

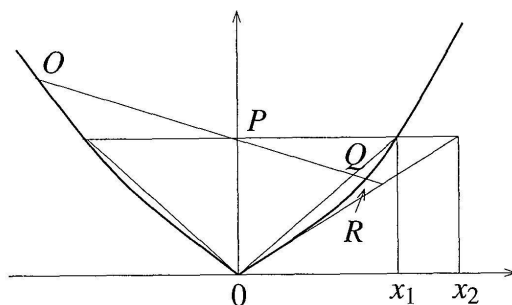


FIGURE 4

Hyperbolicity implies  $C^1$

It remains to show that  $f'$  is continuous. By [RV73, §14] we have

$$\begin{aligned} \lim_{x \rightarrow x_0^+} f'_+(x) &= f'_+(x_0), \\ \lim_{x \rightarrow x_0^-} f'_-(x) &= f'_-(x_0). \end{aligned}$$

From this we conclude that  $f'_+$  is continuous at  $x_0$  since  $f'_+(x_0) = f'_-(x_0)$ . But  $f'_-(x_0) = f'_+(x_0)$  hence  $f'$  is also continuous at  $x_0$ .

*n-dimensional case:* Recall the known result that if  $f$  is a differentiable convex function defined on an open convex set  $S$  in  $\mathbf{R}^{n+1}$ , then it is  $C^1$  on  $S$ , see for example [RV73]. Let  $D$  be a bounded convex domain in  $\mathbf{R}^{n+1}$ ,  $n \geq 2$ . It is enough to prove that  $\partial D$  is differentiable at any point. Given a point  $p \in \partial D$ , we can choose the coordinate axis of  $\mathbf{R}^{n+1}$  so that the origin  $O$  of the coordinates is at  $p$ , all of  $D$  lies in the halfspace  $x_0 \geq 0$  and in a neighbourhood of  $p$  the surface  $\partial D$  can be represented as the graph of a nonpositive convex function  $x_0 = f(x_1, x_2, \dots, x_n)$ ,  $x = (x_1, x_2, \dots, x_n)$ ,  $f(0) = 0$ . Considering the 2-dimensional sections in the planes  $x_0, x_i$ ,  $i = 1, \dots, n$ , we obtain that the partial derivatives of  $f$  at 0 exist and  $f_{x_i}(0) = 0$ ,  $i = 1, \dots, n$ . We have to prove that for each  $\varepsilon > 0$  there is a

neighbourhood  $U_\varepsilon$  of 0 such that  $f(x) < \varepsilon|x|$  in this neighbourhood. But in view of  $f_{x_i}(0) = 0$ ,  $i = 1, \dots, n$ , we have  $f(0, \dots, 0, x_i, 0, \dots, 0) < \varepsilon|x_i|$  for sufficiently small  $x_i$  and hence by convexity  $f(x) < \varepsilon|x|$  for sufficiently small  $|x|$ .  $\square$

REMARK 4.3. The following was announced in [B00]: *If a strictly convex domain  $D$  is divisible, that is, if it admits a proper cocompact group of isometries  $\Gamma$ , then  $D$  is Gromov hyperbolic if and only if  $\partial D$  is  $C^1$ .* Our Theorem 4.2 shows that in the implication (Gromov hyperbolicity + divisibility  $\Rightarrow C^1$ ) the condition of divisibility is superfluous.

## 5. NON-STRICTLY CONVEX DOMAINS

This section owes much of its existence to [Be97] and [Be99]. Using a different argument, we prove certain extensions to arbitrary convex bounded domains of some of the results obtained in those papers.

LEMMA 5.1. *Let  $D$  be a bounded convex domain in  $\mathbf{R}^n$ . Let  $\{x_n\}, \{y_n\}$  be two sequences of points in  $D$ . Assume that  $x_n \rightarrow \bar{x} \in \partial D$ ,  $y_n \rightarrow \bar{y} \in \bar{D}$  and  $[\bar{x}, \bar{y}] \not\subseteq \partial D$ . Let  $x'_n$  and  $y'_n$  denote the endpoints of the chord through  $x_n$  and  $y_n$  as usual. Then  $x'_n$  converges to  $\bar{x}$  and  $y'_n$  converges to the endpoint  $\bar{y}'$  of the chord defined by  $\bar{x}$  and  $\bar{y}$  different from  $\bar{x}$ .*

*Proof.* Compare with Lemma 5.3. in [Be97]. Every limit point of chord endpoints must belong to the line through  $\bar{x}$  and  $\bar{y}$ . In addition, in the case of  $x'_n$  for example, any limit point must lie on the halfline from  $\bar{x}$  not containing  $\bar{y}$ . At the same time each limit point must belong to the boundary of  $D$ , and the statement follows since the line through  $\bar{x}$  and  $\bar{y}$  intersects  $\partial D$  only in  $\bar{x}$  and  $\bar{y}'$ .  $\square$

THEOREM 5.2. *Let  $D$  be a bounded convex domain. Let  $\{x_n\}$  and  $\{z_n\}$  be two sequences of points in  $D$ . Assume that  $x_n \rightarrow \bar{x} \in \partial D$ ,  $z_n \rightarrow \bar{z} \in \partial D$  and  $[\bar{x}, \bar{z}] \not\subseteq \partial D$ . Then there is a constant  $K = K(\bar{x}, \bar{z})$  such that for the Gromov product  $(x_n | z_n)_y$  in Hilbert distances relative to some fixed point  $y$  in  $D$  we have*

$$\limsup_{n \rightarrow \infty} (x_n | z_n)_y \leq K.$$