

## 2. Action of $G$ on $S \times S$ and $S \times S$

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2. ACTION OF  $G$  ON  $S \times S$  AND  $S \times S \times S$

We now study the action of  $G$  on pairs of points of  $S$ . The main notion to be introduced is *transversality*, a notion that could be defined for any bounded symmetric domain. We give several equivalent definitions for our case.

PROPOSITION 2.1. *Let  $\sigma$  and  $\xi$  be two elements of  $S$ . Then the following are equivalent:*

- (i)  $\det(\mathbf{1}_q - \xi^* \sigma) \neq 0$ ;
- (ii)  $\xi - \sigma$  injective;
- (iii)  $\det(\mathbf{1}_p - \xi \sigma^*) \neq 0$ .

*If one of these equivalent conditions is satisfied, then  $\sigma$  and  $\xi$  are said to be transverse.*

*Proof.* Assume (i). As  $\mathbf{1}_q = \xi^* \xi$ , this condition amounts to  $\det(\xi^*(\xi - \sigma)) \neq 0$ , which in particular shows that  $\xi - \sigma$  is injective. Conversely, assume  $\xi - \sigma$  is injective and let  $v \in \mathbf{C}^q$  be such that  $v = \xi^* \sigma v$ . Now

$$\|v\| = \|\xi^* \sigma v\| \leq \|\sigma v\| \leq \|v\|,$$

and hence  $\|\xi^* \sigma v\| = \|\sigma v\|$ , which is possible only if  $\sigma v \in \text{Im } \xi$ . So there exists  $w \in \mathbf{C}^q$ , such that  $\sigma v = \xi w$ . But taking the image of both sides by  $\xi^*$  yields  $v = w$ , and hence  $\sigma v = \xi v$ , so that  $v = 0$ . So  $\mathbf{1}_q - \xi^* \sigma$  is injective and hence (ii)  $\implies$  (i). Under the same assumption (ii), let us prove that  $\xi \sigma^*$  cannot have 1 as an eigenvalue. Suppose  $v \in \mathbf{C}^p$  is such that  $\xi \sigma^* v = v$ . As  $\xi$  is a partial isometry, this forces  $\|\sigma^* v\| = \|v\|$ , and hence  $v$  belongs to the image of the map  $\sigma$ , so there exists  $w \in \mathbf{C}^q$  such that  $v = \sigma w$ . But then we also have  $v = \xi \sigma^* \sigma w = \xi w$  and hence  $(\sigma - \xi)w = 0$  which forces  $w = 0$ . Hence (iii) follows from (ii). Finally assume (iii). Then as  $\sigma$  is injective,  $(\mathbf{1}_p - \xi \sigma^*) \circ \sigma = \sigma - \xi$  is also injective. Hence (iii)  $\implies$  (ii).  $\square$

We will use the notation  $\sigma \top \xi$  to denote transversality. It is a symmetric condition. It is invariant under the action of  $G$ , as can easily be concluded from (6). For  $\sigma \in S$ , let

$$S_{\top}^{\sigma} = \{\xi \mid \sigma \top \xi\}.$$

Observe that the set  $S_{\top}^{ie}$  is exactly the subset in  $S$  where the Cayley transform is defined.

Let

$$(14) \quad S_{\top}^2 = \{(\sigma, \xi) \in S \times S \mid \sigma \top \xi\}.$$

As base point in  $S_{\top}^2$  we choose  $(ie, -ie)$ . Observe that  $c(-ie) = 0$ .

**THEOREM 2.2.** *The group  $G$  acts transitively on  $S_{\top}^2$ .*

*Proof.* Let  $(\sigma, \xi) \in S_{\top}^2$  and let us show that there exists an element of  $G$  which maps  $(\sigma, \xi)$  to  $(ie, -ie)$ . As  $G$  is transitive on  $S$ , we may assume that  $\sigma = ie$ . Then the transversality condition shows that  $\xi$  belongs to the domain of the Cayley transform. The element  $c(\xi)$  belongs to  ${}^cS$ , and we have already noticed that  ${}^cB$  is transitive on  ${}^cS$ . Hence  $c(\xi)$  can be mapped to  $0 = c(-ie)$ . Taking the image under the inverse Cayley transform gives the result.  $\square$

Denote by  $L$  the stabilizer of the base point  $(ie, -ie)$  in  $B$ . Under a Cayley transform, the group  ${}^cL = c \circ L \circ c^{-1}$  is the stabilizer in  ${}^cB$  of the element 0. Hence it is the subgroup of linear transformations given by

$$\begin{aligned} w_q &\longmapsto h^* w_q h \\ w' &\longmapsto u w h \end{aligned}$$

where  $h \in \text{GL}(q, \mathbf{C})$ ,  $u \in \text{U}(p - q)$  and  $\det h = (\det u)^{-1}$ .

**LEMMA 2.3.** *Let  $\begin{pmatrix} w_q \\ w' \end{pmatrix}, \begin{pmatrix} v_q \\ v' \end{pmatrix} \in {}^cS$ . Then they belong to the same orbit under the action of  ${}^cL$  if and only if  $w_q$  and  $v_q$  belong to the same orbit under the action of  $\text{GL}(q, \mathbf{C})$ .*

*Proof.* One implication being trivial, we only have to prove the other one. So assume there exists  $h \in \text{GL}(q, \mathbf{C})$  such that  $v_q = h^* w_q h$ . Let  $\mu$  be a complex number such that  $\mu^{p-q} = \det h$  and let  $u = \mu^{-1} \mathbf{1}_{p-q}$ . Clearly  $(\det u)^{-1} = \det h$ . Using the action of  $(h, u)$  we may assume that  $v_q = w_q$ . Let  $s_q = \frac{1}{2i}(w_q - w_q^*)$ . This is an Hermitian matrix and as  $w_q$  and  $v_q$  belong to  ${}^cS$ , we get

$$s_q = w'^* w' = v'^* v'.$$

Looking to the columns of  $w'$  (or  $v'$ ), we may think of  $w'$  as a family of  $q$  vectors in  $\mathbf{C}^{p-q}$ . Then the matrix  $s_q$  is the Gram matrix of these vectors. But two sets of vectors in  $\mathbf{C}^{p-q}$  are conjugate under the action of the unitary group  $\text{U}(p - q)$  if and only if they have the same Gram matrix. Hence there exists  $u \in \text{U}(p - q)$  such that  $v' = u w'$ . Let  $\lambda$  be a complex number such that  $\lambda^q = \det u$ . Then using the action of  $(\lambda^{-1} \mathbf{1}_q, u)$ , we get the result.  $\square$

Let us denote by  $H_q$  the real vector space of  $q \times q$  Hermitian matrices, and let  $\Omega_q$  be the subset of all positive-definite matrices. For any integer  $r$  such that  $0 \leq r \leq q$  let  $\Omega_q^{(r)}$  be the set of all positive semi-definite  $q \times q$  Hermitian matrices of rank less than  $r$ . For  $r < q$ , the set  $\Omega_q^{(r)}$  is contained in the boundary of  $\Omega_q$ , whereas for  $r = q$ ,  $\Omega_q^{(q)} = \overline{\Omega}_q$ .

Let

$$T_q^{(r)} = \{x + iy \mid x \in H_q, y \in \Omega_q^{(r)}\}.$$

The group  $GL(q, \mathbf{C})$  acts on  $T_q^{(r)}$  by the action  $(h, w) \mapsto hwh^*$ .

Finally let

$$\tilde{T}_q^{(r)} = \{z \in T_q^{(r)} \mid z \text{ invertible}\}.$$

Clearly the action of  $GL(q, \mathbf{C})$  preserves  $\tilde{T}_q^{(r)}$ .

Let  $\begin{pmatrix} w_q \\ w' \end{pmatrix}$  be in  ${}^cS$ . Then  $w_q = x_q + iw'^*w'$ , with  $x_q \in H_q$ . Let

$$r = \inf(q, p - q).$$

The rank of the matrix  $w'^*w'$  is at most  $r$ . Hence  $w_q$  belongs to  $T_q^{(r)}$ . Conversely, it is easily seen that any positive semi-definite Hermitian matrix of rank at most  $r$  can be written as  $w'^*w'$  for some  $w' \in \text{Mat}((p - q) \times q, \mathbf{C})$ .

Let

$$(15) \quad S_{\top}^3 = \{(\sigma_1, \sigma_2, \sigma_3) \in S \times S \times S \mid \sigma_1 \top \sigma_2, \sigma_2 \top \sigma_3, \sigma_3 \top \sigma_1\}.$$

**THEOREM 2.4.** *The  $G$ -orbits in  $S_{\top}^3$  are in one-to-one correspondance with the orbits of  $GL(q, \mathbf{C})$  in  $\tilde{T}_q^{(r)}$ .*

*Proof.* From Theorem 2.2 we already know that any orbit contains an element of the form  $(ie, -ie, \sigma)$  with  $\sigma \in S$ . Now use the Cayley transform. The element  $w = c(\sigma)$  is in  ${}^cS$ , and the transversality condition is equivalent to the condition  $\det(w_q) \neq 0$ . In other words,  $w_q \in \tilde{T}_q^{(r)}$ . The result now follows from Lemma 2.3.  $\square$