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THE NONAMENABILITY OF SCHREIER GRAPHS FOR INFINITE INDEX QUASICONVEX SUBGROUPS OF HYPERBOLIC GROUPS

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ABSTRACT. We show that if H is a quasiconvex subgroup of infinite index in a nonelementary hyperbolic group G then the Schreier coset graph for G relative to H is nonamenable.

1. INTRODUCTION

A connected graph of bounded degree X is *nonamenable* if X has nonzero Cheeger constant or, equivalently, if the spectral radius of the simple random walk on X is less than one (see Section 2 below for more precise definitions). Nonamenable graphs play an increasingly important role in the study of various probabilistic phenomena, such as random walks, harmonic analysis, Brownian motion, and percolations on graphs and manifolds (see for example [2, 5, 6, 7, 15, 17, 18, 24, 30, 43, 44, 62, 71, 72]), as well as in the study of expander families of finite graphs (see for example [52, 66, 67]).

It is well-known that a finitely generated group G is nonamenable if and only if the Cayley graph of G with respect to some (any) finite generating set is nonamenable. The notion of a *word-hyperbolic group* was introduced by M. Gromov [40] and has played a central role in Geometric Group Theory for the last fifteen years. Word-hyperbolic groups are nonamenable unless they are virtually cyclic. Thus the Cayley graphs of word-hyperbolic groups provide a large and interesting class of nonamenable graphs. In this paper we investigate nonamenability of Schreier coset graphs corresponding to subgroups of hyperbolic groups.

We recall the definition of a Schreier coset graph:

DEFINITION 1.1. Let G be a group and let $\pi: A \to G$ be a map where A is a finite alphabet such that $\pi(A)$ generates G (we refer to such an A as a *marked finite generating set* or just *a finite generating set* of G). Let $H \le G$ be a subgroup of G. The *Schreier coset graph* (or the *relative Cayley graph*) $\Gamma(G,H,A)$ for G relative to H with respect to A is an oriented labeled graph defined as follows:

1. The vertices of $\Gamma = \Gamma(G,H,A)$ are precisely the cosets of H in G, that is $V\Gamma := \{Hg \mid g \in G\}$.
2. The set of positively oriented edges of $\Gamma(G,H,A)$ is in one-to-one correspondence with the set $V\Gamma \times A$. For each pair $(Hg, a) \in V\Gamma \times A$ there is a positively oriented edge in $\Gamma(G,H,A)$ from Hg to $Hg\pi(a)$ labeled by the letter a.

Thus the label of every path in $\Gamma(G,H,A)$ is a word in the alphabet $A \cup A^{-1}$. The graph $\Gamma(G,H,A)$ is connected since $\pi(A)$ generates G. Moreover, $\Gamma(G,H,A)$ comes equipped with a natural simplicial metric obtained by giving every edge length one.

We can identify the Schreier graph $\Gamma(G,H,A)$ with the 1-skeleton of the covering corresponding to $H \le G$ of the presentation complex of G based on any presentation of the form $G = \langle A \mid R \rangle$. If M is a closed Riemannian manifold and $H \le G = \pi_1(M)$, then the Schreier graph $\Gamma(G,H,A)$ is quasi-isometric to the covering space of M corresponding to H. If H is normal in G and $G_1 = G/H$ is the quotient group, then $\Gamma(G,H,A)$ is exactly the Cayley graph of the group G_1 with respect to A. In particular, if $H = 1$ then $\Gamma(G,1,A)$ is the standard *Cayley graph of G with respect to A*, denoted $\Gamma(G,A)$.

A subgroup H of a word-hyperbolic group G is said to be *quasiconvex* in G if for any finite generating set A of G there is $\epsilon \ge 0$ such that every geodesic in $\Gamma(G,A)$ with both endpoints in H is contained in the ϵ-neighborhood of H in $\Gamma(G,A)$. Quasiconvex subgroups are closely related to geometric finiteness in the Kleinian group context [69]. They enjoy a number of particularly good properties and play an important role in hyperbolic group theory and its applications (see for example [3, 4, 8, 31, 34, 35, 36, 37, 38, 42, 45, 46, 48, 51, 53, 55, 61, 70]).

Our main result is the following:

THEOREM 1.2. *Let G be a nonelementary word-hyperbolic group with a marked finite generating set A. Let $H \leq G$ be a quasiconvex subgroup of infinite index in G. Then the Schreier coset graph $\Gamma(G,H,A)$ is nonamenable.*

The study of Schreier graphs arises naturally in various generalizations of J. Stallings' theory of ends of groups [23, 29, 60, 61, 63]. The case of virtually cyclic (and hence quasiconvex) subgroups of hyperbolic groups is particularly important to understand in the theory of JSJ-decomposition for hyperbolic groups originally developed by Z. Sela [65] and later by B. Bowditch [11] (see also [59, 23, 28, 64] for various generalizations of the JSJ-theory). A variation of the Følner criterion of nonamenability (see Proposition 2.3 below), when the Cheeger constant is defined by taking the infimum over all finite subsets containing no more than a half of all the vertices, is used to define an important notion of *expander families* of finite graphs. Most known sources of expander families involve taking Schreier coset graphs corresponding to subgroups of finite index in a group with the Kazhdan property (T) (see [52, 66, 67] for a detailed exposition on expander families and their connections with nonamenability).

Since nonamenable graphs of bounded degree are well-known to be *transient* with respect to the simple random walk, Theorem 1.2 implies that $\Gamma(G,H,A)$ is also transient. M. Gromov [40] stated (see R. Foord [27] and I. Kapovich [49] for the proofs) that for any quasiconvex subgroup H in a hyperbolic group G with a finite generating set A, the coset graph $\Gamma(G,H,A)$ is a hyperbolic metric space. A great deal is known about random walks on hyperbolic graphs, but most of these results assume some kind of nonamenability. Thus Theorem 1.2 together with hyperbolicity of $\Gamma(G,H,A)$ and a result of A. Ancona [2] (see also [72]) immediately imply:

COROLLARY 1.3. *Let G be a nonelementary word-hyperbolic group with a finite generating set A. Let $H \leq G$ be a quasiconvex subgroup of infinite index in G and let Y be the Schreier coset graph $\Gamma(G,H,A)$. Then:*

1. *The trajectory of almost every simple random walk on Y converges in the topology of $Y \cup \partial Y$ to some point in ∂Y (where ∂Y is the hyperbolic boundary).*
2. *The Martin boundary of a simple random walk on Y is homeomorphic to the hyperbolic boundary ∂Y, and the Martin compactification of Y corresponding to the simple random walk on Y is homeomorphic to the hyperbolic compactification $Y \cup \partial Y$.*

Let us illustrate Theorem 1.2 for the case of a free group. Let $F = F(a,b)$ be free of rank two and let $H \leq F$ be a finitely generated subgroup of infinite index (which is therefore quasiconvex [68]). Set $A = \{a,b\}$. Then the Schreier graph $Y = \Gamma(F,H,A)$ looks like a finite graph with several infinite tree-branches attached to it (the "branches" are 4-regular trees except for the attaching vertices). In this situation it is easy to see that Y has positive Cheeger constant and so Y is nonamenable. Alex Lubotzky and Andrzej Zuk pointed out to the author that if G is a group with the Kazhdan property (T), then for any subgroup H of infinite index in G the Schreier coset graph for G relative to H is nonamenable. There are many examples of word-hyperbolic groups with Kazhdan property (T) (see for instance [73]) and in view of Theorem 1.2 it would be particularly interesting to investigate if they can possess non-quasiconvex finitely generated subgroups.

Nonamenability of graphs is closely related to cogrowth:

COROLLARY 1.4. *Let $G = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_m \rangle$ be a nonelementary word-hyperbolic group and let $H \leq G$ be a quasiconvex subgroup of infinite index. Let a_n be the number of freely reduced words in $A = \{x_1, \ldots, x_k\}^{\pm 1}$ of length n representing elements of H. Let b_n be the number of all words in A of length n that represent elements of H. Then*

$$\limsup_{n\to\infty} \sqrt[n]{a_n} < 2k-1$$

and

$$\limsup_{n\to\infty} \sqrt[n]{b_n} < 2k.$$

In [10, 50] Theorem 1.2 and Corollary 1.4 play a useful role in obtaining results about "generic-case" complexity of the membership problem as well as about some interesting measures on free groups.

It is easy to see that the statement of Theorem 1.2 need not hold for finitely generated subgroups which are not quasiconvex. For example, a remarkable construction of E. Rips [58] states that for any finitely presented group Q there is a short exact sequence

$$1 \to K \to G \to Q \to 1,$$

where G is nonelementary torsion-free word-hyperbolic and where K can be generated by two elements (but K is usually not finitely presentable). If Q is chosen to be infinite amenable, then $[G:K] = \infty$ and the Schreier graph for G relative to K is amenable. Finitely presentable and even hyperbolic examples of such subgroups are also possible. For instance, if F is a free group of finite rank and $\phi\colon F \to F$ is an atoroidal automorphism, then the mapping torus group of ϕ

$$M_\phi = \langle F, t \mid t^{-1}ft = \phi(f) \text{ for all } f \in F \rangle$$

is word-hyperbolic [8, 13]. In this case $M_\phi / F \simeq \mathbf{Z}$ and hence the Schreier graph for M_ϕ relative to F is amenable.

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2. Nonamenability for graphs

Let X be a connected graph of bounded degree. We define the *spectral radius* $\rho(X)$ of X as

$$\rho(X) := \limsup_{n\to\infty} \sqrt[n]{p^{(n)}(x,y)}$$

where x,y are two vertices of X and $p^{(n)}(x,y)$ is the probability that an n-step simple random walk starting at x will end up at y. It is well-known that $\rho(X) \leq 1$ and that the definition of $\rho(X)$ does not depend on the choice of x,y.

DEFINITION 2.1 (Amenability for graphs). A connected graph X of bounded degree is said to be *amenable* if $\rho(X) = 1$ and *nonamenable* if $\rho(X) < 1$.

It is also well-known that nonamenability of X implies that X is *transient*, that is that for a simple random walk on X the probability of ever returning to the basepoint is less than 1 (see for example Theorem 51 of [16]). We refer the reader to [16, 71, 72] for comprehensive background information about random walks on graphs and for further references on this topic.

CONVENTION 2.2. Let X be a connected graph of bounded degree with the simplicial metric d. For a finite nonempty subset $S \subset VX$ we will denote by $|S|$ the number of elements in S.

If S is a finite subset of the vertex set of X and $k \geq 1$ is an integer, we will denote by $\mathcal{N}_k^X(S) = \mathcal{N}_k(S)$ the set of all vertices v of X such that $d(v, S) \leq k$. Also, we will denote $\eth^X S = \eth S := \mathcal{N}_1(S) - S$.

The number

$$\iota(X) := \inf\{\frac{|\eth S|}{|S|} \mid S \text{ is a finite nonempty subset of the vertex set of } X\}$$

is called the *Cheeger constant* or the *isoperimetric constant* of X.

There are many alternative definitions of nonamenability:

PROPOSITION 2.3. *Let X be a connected graph of bounded degree with the simplicial metric d. Then the following conditions are equivalent:*

1. *The graph X is nonamenable.*
2. (Følner Criterion) *We have $\iota(X) > 0$.*
3. (Gromov's Doubling Condition) *There is some $k \geq 1$ such that for any finite nonempty subset $S \subseteq VX$ we have*
$$|\mathcal{N}_k(S)| \geq 2|S|.$$
4. *For any integer $q > 1$ there is some $k \geq 1$ such that for any finite nonempty subset $S \subseteq VX$ we have*
$$|\mathcal{N}_k(S)| \geq q|S|.$$
5. *For some $0 < \sigma < 1$ we have $p^{(n)}(x, y) = o(\sigma^n)$ for any $x, y \in VX$.*
6. *Let $W(X)$ be the pseudogroup of "bounded perturbations of the identity", that is $W(X)$ consists of all bijections ϕ between subsets of VX such that*
$$\sup_{x \in dom(\phi)} d(x, \phi(x)) < \infty.$$
Then $W(X)$ admits a "paradoxical decomposition", that is there exist nonempty subsets Y_1, Y_2 of VX and $\phi_1 \colon Y_1 \to VX$, $\phi_2 \colon Y_2 \to VX$ such that $\phi_1, \phi_2 \in W(X)$, $VX = Y_1 \sqcup Y_2$ and $\phi_1(Y_1) = \phi_1(Y_2) = VX$.
7. ("Grasshopper Criterion") *There exists a map $\phi \colon VX \to VX$ such that*
$$\sup_{x \in VX} d(x, \phi(x)) < \infty$$
and such that for any $x \in VX$ we have $\left|\phi^{-1}(x)\right| \geq 2$.
8. *There exists a map $\phi \colon VX \to VX$ such that*
$$\sup_{x \in VX} d(x, \phi(x)) < \infty$$
and such that for any $x \in VX$ we have $\left|\phi^{-1}(x)\right| = 2$.

9. *The bottom of the spectrum for the combinatorial Laplacian operator on* X *is* > 0 *(see* [21] *for the precise definitions).*
10. *We have* $H_0^{uf}(X) = 0$ *(see* [9] *for the precise definition of the* uniformly finite *homology groups* H_i^{uf} *).*
11. *We have* $H_0^{(l_p)}(X) = 0$ *for any* $1 < p < \infty$ *(see* [24] *for the precise definition of* $H_i^{(l_p)}$ *).*

All of the above statements are well-known, but we will still provide some sample references. The fact that (1), (2), (5) and (6) are equivalent is stated in Theorem 51 of [16]. The fact that (3), (4), (6), (7) and (8) are equivalent follows from Theorem 32 of [16]. The equivalence of (2) and (9) is due to J. Dodziuk [21]. J. Block and S. Weinberger [9] established the equivalence of (2) and (10). Finally, G. Elek [24] proved that (2) is equivalent to (11).

One can characterize amenability of regular graphs in terms of cogrowth.

DEFINITION 2.4. Let X be a connected graph of bounded degree with a base-vertex x_0. Let $a_n = a_n(X, x_0)$ be the number of reduced edge-paths of length n from x_0 to x_0. Let $b_n = b_n(X, x_0)$ be the number of all edge-paths of length n from x_0 to x_0. Set

$$\alpha(X) := \limsup_{n\to\infty} \sqrt[n]{a_n} \quad \text{and} \quad \beta(X) := \limsup_{n\to\infty} \sqrt[n]{b_n}\,.$$

Then we will call $\alpha(X)$ the *cogrowth rate* of X and we will call $\beta(X)$ the *non-reduced cogrowth rate* of X. These definitions are independent of the choice of x_0.

It is easy to see that for a d-regular connected graph X we have $\alpha(X) \le d-1$ and $\beta(X) \le d$. Moreover, $\rho(X) = \frac{\beta(X)}{d}$. The following result was originally proved by R. Grigorchuk [39] and J. Cohen [19] for the Cayley graphs of finitely generated groups and by L. Bartholdi [5] for arbitrary regular graphs.

THEOREM 2.5 ([5]). *Let* X *be a connected* d*-regular graph with* $d \ge 3$. *Set* $\alpha = \alpha(X)$, $\beta = \beta(X)$ *and* $\rho = \rho(X)$. *Then*

$$\rho = \begin{cases} \frac{2\sqrt{d-1}}{d} & \text{if } 1 \le \alpha \le \sqrt{d-1} \\ \frac{\sqrt{d-1}}{d}\left(\frac{\sqrt{d-1}}{\alpha} + \frac{\alpha}{\sqrt{d-1}}\right) & \text{if } \sqrt{d-1} \le \alpha \le d-1\,. \end{cases}$$

In particular $\rho < 1 \iff \alpha < d-1 \iff \beta < d$.

3. Hyperbolic metric spaces

We refer the reader to [1, 4, 14, 20, 25, 32, 40] for the basic information about Gromov-hyperbolic metric spaces. We briefly recall the main definitions.

If (X,d) is a geodesic metric space and $x,y \in X$, we shall denote by $[x,y]$ a geodesic segment from x to y in X.

Definition 3.1 (Gromov product). Let (X,d) be a metric space and suppose $x,y,z \in X$. We set

$$(x,y)_z := \frac{1}{2}[d(z,x) + d(z,y) - d(x,y)].$$

Note that $(x,y)_z = (y,x)_z$.

Definition 3.2 (Hyperbolic metric space [1]). Let (X,d) be a geodesic metric space. We say that (X,d) is *δ-hyperbolic* (where $\delta \geq 0$) if for any $p,x,y,z \in X$ we have:

$$(x,y)_p \geq \min\{(x,z)_p, (y,z)_p\} - \delta.$$

The space X is said to be *hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$.

There are many equivalent definitions of hyperbolicity, for example:

Proposition 3.3 ([1, 20, 32]). *Let (X,d) be a geodesic metric space. Then the following conditions are equivalent.*

1. *The space X is hyperbolic.*
2. *There exists a constant $\delta' \geq 0$ such that if $x,y,z \in X$ and $y' \in [x,y]$, $z' \in [x,z]$ are such that $d(x,y') = d(x,z') \leq (y,z)_x$ then $d(y',z') \leq \delta'$.*
3. (Thin Triangles Condition) *There exists $\delta'' \geq 0$ such that for any $x,y,z \in X$, for any geodesic segments $[x,y]$, $[x,z]$ and $[y,z]$ and for any point $p \in [x,y]$ there is a point $q \in [x,z] \cup [y,z]$ such that $d(p,q) \leq \delta''$.*

Definition 3.4 (Word-hyperbolic group). A finitely generated group G is said to be *word-hyperbolic* if for some (and hence for any) finite generating set A of G the Cayley graph $\Gamma(G,A)$ is hyperbolic.

Definition 3.5 (Gromov product for sets). Let (X,d) be a metric space. Let $x \in X$ and $Q, Q' \subseteq X$. Define $(Q,Q')_x := \sup\{(q,q')_x \mid q \in Q,\ q' \in Q'\}$.

4. Quasiconvex subgroups of hyperbolic groups

Detailed background information on quasiconvex subgroups of hyperbolic groups can be found in [1, 4, 20, 31, 38, 34, 32, 51, 54, 68] and other sources.

Convention 4.1. Suppose G is a finitely generated group with a fixed finite generating set A. Let $X = \Gamma(G,A)$ be the Cayley graph of G with respect to A. We will denote the word-metric corresponding to A on X by d_A. Also, for $g \in G$ we will denote $|g|_A := d_A(1,g)$. For a word w in the alphabet $A \cup A^{-1}$ we will denote by $\overline{w}$ the element of G represented by w.

Definition 4.2 (Quasiconvexity). For $\epsilon \geq 0$ a subset Z of a metric space (X,d) is *ϵ-quasiconvex* if, for any $z_1, z_2 \in Z$ and any geodesic $[z_1,z_2]$ in X, the segment $[z_1,z_2]$ is contained in the closed ϵ-neighborhood of Z. A subset $Z \subseteq X$ is *quasiconvex* if it is ϵ-quasiconvex for some $\epsilon \geq 0$.

If G is a finitely generated group and A is a finite generating set of G, a subgroup $H \leq G$ is *quasiconvex in G with respect to A* if $H \subseteq \Gamma(G,A)$ is a quasiconvex subset.

It turns out [20, 32, 4, 31] that for subgroups of word-hyperbolic groups quasiconvexity is independent of the choice of a finite generating set for the ambient group. Thus a subgroup H of a hyperbolic group G is termed *quasiconvex* if $H \subseteq \Gamma(G,A)$ is quasiconvex for some finite generating set A of G.

We summarize some well-known basic facts regarding quasiconvex subgroups and provide some sample references:

Proposition 4.3. *Let G be a word-hyperbolic group with a finite generating set A. Let $X = \Gamma(G,A)$ be the Cayley graph of G with the word-metric d_A induced by A. Then:*

1. *If $H \leq G$ is a subgroup, then either H is virtually cyclic (in which case H is called* elementary*) or H contains a free subgroup F of rank two which is quasiconvex in G (in this case H is said to be* nonelementary*)* [20, 32].
2. *Every cyclic subgroup of G is quasiconvex in G* [1, 20, 32].
3. *If $H \leq G$ is quasiconvex then H is finitely presentable and word-hyperbolic* [1, 20, 32].

4. *Suppose* $H \leq G$ *is generated by a finite set* Q *inducing the word-metric* d_Q *on* H. *Then* H *is quasiconvex in* G *if and only if there is a* $C > 0$ *such that for any* $h_1, h_2 \in H$

$$d_Q(h_1, h_2) \leq C d_A(h_1, h_2)$$

(*see* [20, 32, 4, 31]).

5. *The set* $\mathcal{L}$ *of all* A*-geodesic words is a regular language that provides a bi-automatic structure for* G. *Moreover, a subgroup* $H \leq G$ *is quasiconvex if and only if* H *is* $\mathcal{L}$*-rational, that is the set* $\mathcal{L}_H = \{w \in \mathcal{L} \mid \overline{w} \in H\}$ *is a regular language* [31].
6. *If* $H_1, H_2 \leq G$ *are quasiconvex, then* $H_1 \cap H_2 \leq G$ *is quasiconvex* [68].
7. [51, 46] *Let* $C \leq B \leq G$ *where* B *is quasiconvex in* G *(and hence* B *is hyperbolic) and* C *is quasiconvex in* B. *Then* C *is quasiconvex in* G [51, 46].
8. *Let* $C \leq B \leq G$ *where* C *is quasiconvex in* G *and where* B *is word-hyperbolic. Then* C *is quasiconvex in* B [51, 46].
9. *Suppose* $H \leq G$ *is an infinite quasiconvex subgroup. Then* H *has finite index in its* commensurator $Comm_G(H)$ (*see* [51]), *where* $Comm_G(H) := \{g \in G \mid [H : g^{-1}Hg \cap H] < \infty$ *and* $[g^{-1}Hg : g^{-1}Hg \cap H] < \infty\}$.

Part 1 of the above proposition implies that a nonelementary subgroup of a hyperbolic group is nonamenable.

5. Proof of the main result

Let G be a nonelementary word-hyperbolic group with a finite generating set A. Let $X = \Gamma(G, A)$ be the Cayley graph of G with the word metric d_A. Let $\delta \geq 1$ be an integer such that the space $(\Gamma(G, A), d_A)$ is δ-hyperbolic. Let $H \leq G$ be a quasiconvex subgroup of infinite index in G. These conventions, unless specified otherwise, will be fixed for the remainder of the paper.

We shall need the following useful fact:

LEMMA 5.1. *There exists an integer constant* $K = K(G, H, A) > 0$ *with the following properties.*

Assume $g \in G$ *is shortest with respect to* d_A *in the coset class* Hg. *Then for any* $h \in H$ *we have* $(g, h)_1 \leq K$ *(and hence* $(g, H)_1 \leq K$*).*

Proof. The conclusion of Lemma 5.1 follows directly from the proofs of Lemma 4.1 and Lemma 4.5 of [4]. We will present the argument for completeness. For the hyperbolic space $X = \Gamma(G, A)$ choose $\delta' \geq 0$ as in part 2 of Proposition 3.3. Let $\epsilon \geq 0$ be such that H is an ϵ-quasiconvex subset of X.

Let $g \in G$ be a shortest element of Hg, so that for any $h \in H$ we have $|hg|_A \leq |g|_A$. We claim that $(h, g)_1 \leq \epsilon + \delta'$ for any $h \in H$.

Suppose not, that is $(h, g)_1 > \epsilon + \delta'$ for some $h \in H$. Consider two geodesic segments $[1, g]$ and $[1, h]$ in X and let $t \in [1, h]$, $s \in [1, g]$ be such that $d_A(1, s) = d_A(1, t) = (h, g)_1$. Thus $d_A(s, t) \leq \delta'$ by the choice of δ'. Since H is ϵ-quasiconvex in X, there is $h' \in H$ such that $d_A(t, h') \leq \epsilon$. Then

$$\begin{aligned} \left|(h')^{-1}g\right|_A = d_A(h', g) &\leq d_A(h', t) + d_A(t, s) + d_A(s, g) \\ &\leq \epsilon + \delta + |g|_A - (h, g)_1 < |g|_A , \end{aligned}$$

which contradicts the assumption that g is shortest in Hg.

LEMMA 5.2. *Let $T_1, T_2 > 0$ be some positive numbers. Let $g \in G$ be such that $(g, H)_1 \leq T_1$ and $|g|_A > T_1 + T_2 + \delta$. Let $f \in G$ be such that $|f|_A \leq T_2$. Then $(gf, H)_1 \leq T_1 + \delta$.*

Proof. Note that $|g|_A = (g, gf)_1 + (1, gf)_g$. Since $(1, gf)_g \leq d(g, gf) = |f|_A \leq T_2$, we conclude that

$$(g, gf)_1 = |g|_A - (1, gf)_g > T_1 + T_2 + \delta - T_2 = T_1 + \delta .$$

Therefore for any $h \in H$ we have

$$T_1 + \delta \geq (g, h)_1 + \delta \geq \min\{(g, gf)_1, (gf, h)_1\}$$

and hence $(gf, h)_1 \leq T_1 + \delta$ because $(g, gf)_1 > T_1 + \delta$. Since $h \in H$ was arbitrary, this means that $(gf, H)_1 \leq T_1 + \delta$.

LEMMA 5.3. *Suppose $g_1, g_2 \in G$ are such that $Hg_1 = Hg_2$. Then there is $h \in H$ such that $hg_1 = g_2$ and that*

$$|h|_A \leq (g_1, H)_1 + (g_2, H)_1 .$$

Proof. Since $Hg_1 = Hg_2$, there is $h \in H$ with $hg_1 = g_2$. Hence

$$|h|_A = (h, g_2)_1 + (1, hg_1)_h = (h, g_2)_1 + (h^{-1}, g_1)_1 \leq (g_2, H)_1 + (g_1, H)_1 .$$

Proof of Theorem 1.2. Let $K = K(G,H,A) > 0$ be the constant provided by Lemma 5.1. Put $Y = \Gamma(G,H,A)$. Thus Y is a connected $2m$-regular infinite graph, where m is the number of elements in A. Denote the simplicial metric on Y by d_Y.

Let N be the number of all elements $g \in G$ with $|g|_A \le 2K + 2\delta$. In particular Y has at most N vertices within distance $2K + 2\delta$ of the coset $H1 \in VY$.

Since G is nonelementary word-hyperbolic and thus nonamenable, the Cayley graph $X = \Gamma(G,A)$ is nonamenable. By part 4 of Proposition 2.3 there is a constant $k' > 0$ such that for any finite nonempty subset S of G the k'-neighborhood of S in X has at least $4N|S|$ vertices. Let N_1 be the number of elements of G of length at most $K+\delta+k'$. Choose $k'' > 1$ such that for any vertex $Hg \in VY$ with $d_Y(H1,Hg) \le K + \delta + k'$ the k''-neighborhood of Hg has at least $4N_1$ vertices. Such k'' exists since by assumption $[G:H] = \infty$ and hence the graph Y is infinite. Set $k := \max\{k', k''\}$.

Suppose now that $F \subset VY$ is a finite nonempy subset. Write $F = F_1 \sqcup F_2$ where F_1 is the intersection of F with the closed ball of radius $K + \delta + k'$ in Y.

If $|F_1| \ge |F|/2$, then $|F| \le 2N_1$ and the k-neighborhood of F in Y has at least $4N_1 \ge 2|F|$ vertices. Suppose now that $|F_1| < |F|/2$, so that $|F_2| \ge |F|/2$. Then

$$F_2 = \{Hg_1, \dots, Hg_t\}$$

where $|F_2| = t$ and where each $g_i \in G$ is shortest in Hg_i with $|g_i|_A > K + \delta + k'$. By Lemma 5.1 $(g_i, H)_1 \le K$. By Lemma 5.2 for any $f \in G$ with $|f|_A \le k'$ and for each $i = 1, \dots, t$ we have $(g_if, H)_1 \le K + \delta$.

Let $S := \{g_1, \dots, g_t\}$ and let S' be the set of all vertices of X contained in the k'-neighborhood of S in X. By the choice of k' we have $|S'| \ge 4N|S| = 4N|F_2|$. On the other hand, Lemma 5.3 implies that if $g, g' \in S'$ are such that $Hg = Hg'$ then $hg = g'$ for some $h \in H$ with $|h|_A \le 2K + 2\delta$. By the choice of N this means that the set $F' := \{Hg \mid g \in S'\}$ contains at least

$$|S'|/N = 4N|F_2|/N = 4|F_2| \ge 2|F|$$

distinct elements. However, F' is obviously contained in the k-neighborhood of F in Y.

We have verified that for any finite nonempy subset $F \subseteq VY$ the k-neighborhood of F in Y contains at least $2|F|$ vertices. By the Doubling Condition (part 3 of Proposition 2.3) this implies that Y is nonamenable.

We can now obtain Corollary 1.4 stated in the Introduction.

COROLLARY 5.4. *Let $G = \langle x_1, \dots, x_k \mid r_1, \dots, r_m \rangle$ be a nonelementary word-hyperbolic group and let $H \leq G$ be a quasiconvex subgroup of infinite index. Let a_n be the number of freely reduced words in $A = \{x_1, \dots, x_k\}^{\pm 1}$ of length n that represent elements of H. Let b_n be the number of all words in A of length n that represent elements of H. Then*

$$\limsup_{n \to \infty} \sqrt[n]{a_n} < 2k - 1$$

and

$$\limsup_{n \to \infty} \sqrt[n]{b_n} < 2k .$$

Proof. Note that $k \geq 2$ since G is nonelementary. Put $A = \{x_1, \dots, x_k\}$ and $Y = \Gamma(G, H, A)$. We choose $x_0 := H1 \in VY$ as the base-vertex of Y. Note that Y is $2k$-regular by construction. Also, for any vertex x of Y and any word w in $A \cup A^{-1}$ there is a unique path in Y with label w and origin x. The definition of Schreier coset graphs also implies that a word w represents an element of H if and only if the unique path in Y with origin x_0 and label w terminates at x_0. Therefore $a_n(Y)$ equals the number of freely reduced words in the alphabet $A = \{x_1, \dots, x_k\}^{\pm 1}$ of length n that represent elements of H. Similarly, $b_n(Y)$ equals the number of all words in A of length n representing elements of H. By Theorem 1.2, Y is nonamenable. Hence by Theorem 2.5, $\alpha(Y) < 2k - 1$ and $\beta(Y) < 2k$, as required.

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