Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 48 (2002)

Heft: 3-4: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: TORSION NUMBERS OF AUGMENTED GROUPS WITH APPLICATIONS TO KNOTS AND LINKS

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Kapitel: 3. EXTENDED FOX FORMULA AND RECURRENCE

DOI: https://doi.org/10.5169/seals-66079

EXAMPLE 2.11. Consider the augmented group (G,χ) such that

$$G = \langle x, a \mid x^{-2}a^2xa^{-6}xa^2,\ x^{-3}axa^{-4}xa^4xa^{-1}\rangle\,,$$

and $\chi\colon G \to \mathbf{Z}$ maps $x \mapsto 1$ and $a \mapsto 0$. A straightforward calculation shows that $\mathcal{M} \cong \mathcal{R}_1/(2f,(t-1)f)$, where $f(t) = t^2-3t+1$. The Alexander polynomial Δ is $\gcd(2f,(t-1)f) = f$, and it has Mahler measure greater than 1. However, the topological entropy of the homeomorphism σ is zero by Corollary 18.5 of [Sc95]. As in the proof of the theorem above, it follows that the torsion numbers b_r have trivial exponential growth rate; that is, $\limsup_{r\to\infty} b_r^{1/r} = 1$.

3. EXTENDED FOX FORMULA AND RECURRENCE

Let (G,χ) be an augmented group, and $\mathcal{A}$ the $N\times M$ presentation matrix for the $\mathcal{R}_1$-module $\mathcal{M}$ as in (2.1). For any positive integer r we can obtain a presentation matrix for the finitely generated abelian group $\mathcal{M}_r$ by replacing each entry $q(t)$ of $\mathcal{A}$ by the $r\times r$ block $q(C_r)$, where C_r is the companion matrix of t^r-1,

$$C_r = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

We call the resulting $rN\times rM$ matrix $\mathcal{A}(C_r)$. The proof is not difficult. The torsion number b_r is equal to the absolute value of the product of the nonzero elementary divisors of $\mathcal{A}(C_r)$.

Assume first that $\mathcal{M}$ is a cyclic module. Then $\mathcal{A}$ is the 1×1 matrix $(\Delta(t))$, and the $r\times r$ matrix $(\Delta(C_r))$ presents $\mathcal{M}_r$. The Betti number β_r is the number of zeros of Δ that are r^{th} roots of unity. When it vanishes the matrix $(\Delta(C_r))$ is nonsingular. Then all elementary divisors of the matrix are nonzero, and their product is equal (up to sign) to the product of the eigenvalues, which is the determinant. Fox's formula (Proposition 2.5) follows by choosing a basis for $\mathbf{C}^r$ that diagonalizes the companion matrix C_r; we then see that the eigenvalues of $\Delta(C_r)$ are $\Delta(\zeta)$, where ζ ranges over the r^{th} roots of unity. In general, β_r is equal to

$$s = \sum_{\substack{d|r \\ \Phi_d|\Delta}} \deg \Phi_d = \sum_{\substack{d|r \\ \Phi_d|\Delta}} \phi(d),$$

where Φ_d is, as before, the d^{th} cyclotomic polynomial, and ϕ is Euler's phi function. We appeal to the following result, a special case of Theorem 2.1 of [MM82].

LEMMA 3.1. *Let A be an integral $r \times r$ matrix with rank $r - s$. Suppose that R is an integral $s \times r$ matrix with an $s \times s$ minor invertible over $\mathbf{Z}$ such that $RA = 0$ and $AR^T = 0$ (where R^T denotes the transpose matrix). Then the product of the nonzero eigenvalues of A is equal to $\pm \det(RR^T)$ times the product of the nonzero elementary divisors of A.*

EXAMPLE 3.2. Suppose that we have a factorization $t^r - 1 = \Phi \cdot \Psi$ in $\mathbf{Z}[t]$. Set $A = \Phi(C_r)$. Then we can construct a matrix R satisfying the hypotheses of Lemma 3.1. We regard $\mathcal{R}_1/(t^r - 1)$ as a free abelian group with generators $1, t, \ldots, t^{r-1}$. Then the rows of A represent the polynomials $\Phi, t\Phi, \ldots, t^{r-1}\Phi$ (modulo $t^r - 1$). The rank of A is $r - s$, where $s = \deg \Phi$. We take R to be the $s \times r$ matrix with rows representing $\Psi, t\Psi, \ldots, t^{s-1}\Psi$. Consider first the product RA. Regarding the product of the i^{th} row of R with A as a linear combination of the rows of A, we see that it represents the polynomial $t^{i-1}\Psi \cdot \Phi \equiv 0$ (modulo $t^r - 1$). Hence $RA = 0$.

The columns of A represent the polynomials $\Phi(t^{-1})$, $t\Phi(t^{-1}), \ldots$, $t^{r-1}\Phi(t^{-1})$, and so the i^{th} column of AR^T represents $\Phi(t^{-1}) \cdot t^i\Psi(t)$ (modulo $t^r - 1$). Since Φ is a product of cyclotomic polynomials, we have $t^{\deg \Phi}\Phi(t^{-1}) = \pm\Phi(t)$. (A cyclotomic polynomial has this property since its set of roots is preserved by inversion, and its leading and constant coefficients are ± 1.) So AR^T is also zero.

Since the degree of $t^i\Psi$ is less than r for $i < s$, the $s \times s$ minor consisting of the first s columns of R is upper triangular. The diagonal entries are the constant term of Ψ, which must be ± 1. Hence this minor is invertible over $\mathbf{Z}$.

The matrix $\mathcal{A}$ presents $\mathcal{R}_1/(\Phi, t^r - 1) \cong \mathcal{R}_1/(\Phi)$, a free abelian group, so the product of its elementary divisors is 1. Lemma 3.1 implies that $\det(RR^T)$

is equal up to sign to the product of the nonzero eigenvalues of $\Phi(C_r)$; that is,

$$\det(RR^T) = \pm \prod_{\substack{\zeta^r=1 \\ \Phi(\zeta)\neq 0}} \Phi(\zeta) . \tag{3.1}$$

THEOREM 3.3. *Suppose that the $\mathcal{R}_1$-module $\mathcal{M}$ is isomorphic to $\mathcal{R}_1/(\Delta)$. For any positive integer r, let Φ be the product of the distinct cyclotomic polynomials Φ_d such that $d \mid r$ and $\Phi_d \mid \Delta$. Then*

$$b_r = \left| \prod_{\substack{\zeta^r=1 \\ \Delta(\zeta)\neq 0}} \left(\frac{\Delta}{\Phi}\right)(\zeta) \right| . \tag{3.2}$$

REMARKS 3.4.

(i) We follow the convention that if no cyclotomic polynomial divides Δ, then $\Phi = 1$. Clearly b_r is a pure torsion number if and only if $\Phi = 1$. In this case (3.2) reduces to Fox's formula (2.2).

(ii) See [Sa95] and [HS97] for more calculations and estimations of torsion numbers b_r arising from link groups.

Proof of Theorem 3.3. We write Δ as $\Phi \cdot g$, for some $g \in \mathbf{Z}[t]$. The matrix $\Delta(C_r)$, which presents $\mathcal{M}_r = \mathcal{R}_1/(\Delta, t^r - 1)$, has rank $r - \deg \Phi$. The rank is the same as that of $\Phi(C_r)$. Consider the matrix R of Example 3.2. We have $R\Delta(C_r) = (R\Phi(C_r))g(C_r) = 0$ and also $\Delta(C_r)R^T = (\Phi(C_r)g(C_r))R^T = g(C_r)(\Phi(C_r)R^T) = 0$. Formula (3.2) now follows from Lemma 3.1 together with (3.1). □

If $\mathcal{M}$ is a direct sum of cyclic modules, then Theorem 3.3 can be applied to each summand and the terms produced by (3.2) multiplied together in order to compute b_r.

When $\mathcal{M}$ is not necessarily a direct sum of cyclic modules, but it is torsion-free as an abelian group, then it is "virtually" a direct sum of cyclic modules by the following lemma, which appears as Lemma 9.1 in [Sc95]. The main idea of the proof is to consider the natural injection of $\mathcal{M} \hookrightarrow \mathcal{M} \otimes_{\mathbf{Z}} \mathbf{Q}$, and use the fact that $\mathcal{M} \otimes_{\mathbf{Z}} \mathbf{Q}$ is a finitely generated module over the ring $\mathbf{Q}[t^{\pm 1}]$, which is a principal ideal domain.

We recall that a polynomial in $\mathbf{Z}[t]$ is said to be primitive if the only constants that divide it are ± 1.

LEMMA 3.5. *Assume that* $\mathcal{M}$ *is a finitely generated* $\mathcal{R}_1$*-module that is torsion-free as an abelian group. Then there exist primitive polynomials* $\pi_1, \ldots, \pi_n \in \mathbf{Z}[t]$ *such that* $\pi_i \mid \pi_{i+1}$ *for all* $i = 1, \ldots, n-1$, *and an* $\mathcal{R}_1$*-module injection* $i \colon \mathcal{M} \to \mathcal{M}' = \mathcal{R}_1/(\pi_1) \oplus \cdots \oplus \mathcal{R}_1/(\pi_n)$ *such that* $\mathcal{M}'/i(\mathcal{M})$ *is finite.*

For notational convenience we identify $\mathcal{M}$ with its image in $\mathcal{M}'$. Consider the mappings $\mu \colon \mathcal{M} \to \mathcal{M}$ and $\mu' \colon \mathcal{M}' \to \mathcal{M}'$ given by $a \mapsto (t^r - 1)a$. Clearly $\ker \mu$ is a submodule of $\ker \mu'$. We define $\kappa(r)$ to be the index $|\ker \mu' : \ker \mu|$. Let b_r' denote the order of the torsion subgroup of $\mathcal{M}'/(t^r - 1)\mathcal{M}'$. The proof of the following theorem extends techniques of [We80].

THEOREM 3.6. *If the finitely generated* $\mathcal{R}_1$*-module* $\mathcal{M}$ *is torsion-free as an abelian group, then for any positive integer* r,

$$b_r = \frac{b_r'}{\kappa(r)}. \tag{3.3}$$

Moreover, if γ *is the cyclotomic order of* Δ, *then* $\kappa(r + \gamma) = \kappa(r)$ *for all* r.

LEMMA 3.7. *Let* $0 \to A_1 \to A_2 \to \cdots \to A_m \to 0$ *be an exact sequence of finite abelian groups. Then*

$$\prod |A_{\text{even}}| = \prod |A_{\text{odd}}|.$$

Lemma 3.7 is easily proved using induction on m. We leave the details to the reader.

Proof of Theorem 3.6. Consider the finite quotient $p \colon \mathcal{M}' \to \mathcal{M}'/\mathcal{M}$ and mapping $\bar{\mu} \colon \mathcal{M}'/\mathcal{M} \to \mathcal{M}'/\mathcal{M}$ given by $a \mapsto (t^r - 1)a$. The exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{M} & \xrightarrow{\ i\ } & \mathcal{M}' & \xrightarrow{\ p\ } & \mathcal{M}'/\mathcal{M} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu'} & & \downarrow{\scriptstyle \bar{\mu}} & & \\
0 & \longrightarrow & \mathcal{M} & \xrightarrow{\ i\ } & \mathcal{M}' & \xrightarrow{\ p\ } & \mathcal{M}'/\mathcal{M} & \longrightarrow & 0
\end{array}$$

induces a second exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker\mu & \xrightarrow{i} & \ker\mu' & \xrightarrow{p} & \ker\overline{\mu} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{M} & \xrightarrow{i} & \mathcal{M}' & \xrightarrow{p} & \mathcal{M}'/\mathcal{M} & \longrightarrow & 0 \\
 & & \mu\downarrow & & \mu'\downarrow & & \overline{\mu}\downarrow & & \\
0 & \longrightarrow & \mathcal{M} & \xrightarrow{i} & \mathcal{M}' & \xrightarrow{p} & \mathcal{M}'/\mathcal{M} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{M}_r & \xrightarrow{\overline{i}} & \mathcal{M}'_r & \xrightarrow{\overline{p}} & \operatorname{coker}\overline{\mu} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

and hence by the Snake Lemma we obtain a long exact sequence

$$(3.4)\qquad 0 \to \ker\mu \xrightarrow{i} \ker\mu' \xrightarrow{p} \ker\overline{\mu} \xrightarrow{d} \mathcal{M}_r \xrightarrow{\overline{i}} \mathcal{M}'_r \xrightarrow{\overline{p}} \operatorname{coker}\overline{\mu} \to 0\,.$$

Let $T\mathcal{M}_r$ and $T\mathcal{M}'_r$ be the torsion subgroups of $\mathcal{M}_r$ and $\mathcal{M}'_r$, respectively. Since $\ker\overline{\mu}$ is finite, its image under the connecting homomorphism d is contained in $T\mathcal{M}_r$. Also, $\overline{i}$ maps $T\mathcal{M}_r$ into $T\mathcal{M}'_r$. Hence we have an induced sequence

$$(3.5)\qquad 0 \to \ker\mu \xrightarrow{i} \ker\mu' \xrightarrow{p} \ker\overline{\mu} \xrightarrow{d} T\mathcal{M}_r \xrightarrow{\overline{i}} T\mathcal{M}'_r \xrightarrow{\overline{p}} \operatorname{coker}\overline{\mu} \to 0\,.$$

It is not difficult to verify that (3.5) is exact. The only nonobvious thing to check is that the kernel of $\overline{p}$ is contained in the image of $\overline{i}$. To see this, assume that $\overline{p}(y) = 0$. By the exactness of (3.4) there exists an element $x \in \mathcal{M}_r$ such that $\overline{i}(x) = y$. If $x \notin T\mathcal{M}_r$, then the multiples of x are distinct in $\mathcal{M}_r$ and each maps by $\overline{i}$ into the finite group $T\mathcal{M}'_r$, contradicting the fact that $\ker\overline{i} = d(\ker\overline{\mu})$ is finite.

The following sequence is exact.

$$(3.6)\qquad 0 \to \ker\mu'/i(\ker\mu) \to \ker\overline{\mu} \to T\mathcal{M}_r \to T\mathcal{M}'_r \to \operatorname{coker}\overline{\mu} \to 0\,.$$

Since $\mathcal{M}'_r/\mathcal{M}_r$ is finite, $\ker\overline{\mu}$ and $\operatorname{coker}\overline{\mu}$ have the same order. Lemma 3.7 now completes the proof of (3.3), $\kappa(r)$ being the order of $\ker\mu'/i(\ker\mu)$.

The modules $\mathcal{M}$ and $\mathcal{M}'$ have characteristic polynomial π_n. Since $\mathcal{M}$ embeds in $\mathcal{M}'$ with finite index, a prime polynomial annihilates a nonzero element of $\mathcal{M}$ if and only if it annihilates a nonzero element of $\mathcal{M}'$. Such polynomials are exactly the prime divisors of π_n. It follows that $\ker \mu$ and $\ker \mu'$ are both periodic, with period equal to the least common multiple γ of the positive integers d such that Φ_d divides Δ. Hence the same is true for $\kappa(r)$. □

THEOREM 3.8. *Assume that the finitely generated $\mathcal{R}_1$-module $\mathcal{M}$ is a direct sum of cyclic modules or is torsion free as an abelian group. Then the set of torsion numbers b_r satisfies a linear homogeneous recurrence relation with constant coefficients.*

Proof. Write

$$\Delta = \left(\prod_{d\in D} \Phi_d^{e_d}\right) \cdot g\,,$$

where $D = \{d \;:\; \Phi_d \mid \Delta\}$, and let γ be the cyclotomic order of Δ. We will show that for each $R \in \{0, \dots, \gamma - 1\}$, the subsequence of b_r with r congruent to R modulo γ satisfies

$$b_r = C_R\, r^{M_R} |\mathrm{Res}(g, t^r - 1)|\,, \tag{3.7}$$

where C_R, M_R are constants,

$$M_R = \sum_{\substack{d\in D\\ d\mid R}} \phi(d)(e_d - 1) \le M = \sum_{d\in D} \phi(d)(e_d - 1)\,.$$

As we saw in section 2, the sequence $|\mathrm{Res}(g, t^r - 1)|$ satisfies a linear homogeneous recurrence relation with characteristic polynomial p of degree at most $2^{\deg g}$. We may normalize p to be monic, $p(t) = \prod_j (t - \lambda_j)^{n_j}$, with λ_j distinct. The general solution to this recurrence relation has the form $\sum_j q_j(r)\lambda_j^r$, where q_j is a polynomial of degree less than n_j (see [Br92], Theorem 7.2.2, for example). Each of the sequences $a_r^{(R)} = C_R r^{M_R} |\mathrm{Res}(g, t^r - 1)|$ satisfies the recurrence relation given by $\hat{p}(t) = \prod_j (t - \lambda_j)^{n_j + M}$. It also satisfies the recurrence relation given by $P(t) = \prod_j (t^\gamma - \lambda_j^\gamma)^{n_j + M}$, since $\hat{p}$ divides P. Because the powers of t occurring in P are all multiples of γ, the latter recurrence relation also describes the sequence $\{b_r\}$, which is composed of the subsequences $b_{R+\gamma n} = a_{R+\gamma n}^{(R)}$. We note that the degree of P is at most $\gamma(M+1)2^{\deg g}$.

First we consider the case when $\mathcal{M}$ is cyclic. Given R we set

$$\Phi = \prod_{\substack{d \in D \\ d|R}} \Phi_d .$$

By Theorem 3.3 we have

$$b_r = \Big| \prod_{\substack{\zeta^r=1 \\ \Delta(\zeta)\neq 0}} \Big(\frac{\Delta}{\Phi}\Big)(\zeta) \Big| = \Big|\operatorname{Res}\Big(\frac{\Delta}{\Phi}, \frac{t^r-1}{\Phi}\Big)\Big|$$

$$= \prod_{d\in D} \Big|\operatorname{Res}\Big(\Phi_d, \frac{t^r-1}{\Phi}\Big)\Big|^{e'_d} \Big|\operatorname{Res}\Big(g, \frac{t^r-1}{\Phi}\Big)\Big| ,$$

where

$$e'_d = \begin{cases} e_d - 1 & \text{if } d \mid R, \\ e_d & \text{if } d \nmid R . \end{cases}$$

For each d dividing R,

$$\operatorname{Res}\Big(\Phi_d, \frac{t^r-1}{\Phi}\Big) = \prod_{\Phi_d(\omega)=0} \frac{t^r-1}{\Phi(t)}\Big|_{t=\omega}$$

$$= \prod_{\Phi_d(\omega)=0} \frac{(t^d-1)(1+t^d+\cdots+t^{(r/d-1)d})}{\Phi_d(t)\widehat{\Phi}(t)}\Big|_{t=\omega}$$

$$= \prod_{\Phi_d(\omega)=0} \Bigg[\frac{t^d-1}{\Phi_d(t)}\Big|_{t=\omega} \cdot \frac{r/d}{\widehat{\Phi}(\omega)}\Bigg] = C_d \cdot r^{\phi(d)} ,$$

where $\widehat{\Phi} = \Phi/\Phi_d$ and C_d depends only on d and R. For $d \in D$ not dividing R,

$$\operatorname{Res}(\Phi_d, \frac{t^r-1}{\Phi}) = \prod_{\Phi_d(\omega)=0} \frac{\omega^r-1}{\Phi(\omega)}$$

is constant for r congruent to R modulo γ, since d divides γ. Finally,

$$\operatorname{Res}(g, \frac{t^r-1}{\Phi}) = c_0^{r-\deg \Phi} \prod_{g(\alpha)=0} \frac{\alpha^r-1}{\Phi(\alpha)} ,$$

where c_0 is the leading coefficient of g; the expression can be rewritten as $C \operatorname{Res}(g, t^r-1)$, where C depends only on R. Thus we can express b_r in the desired form (3.7) for all r congruent to R modulo γ.

For the case when $\mathcal{M}$ is a direct sum of cyclic modules $\mathcal{R}_1/(\pi_1) \oplus \cdots \oplus \mathcal{R}_1/(\pi_n)$ we apply the above result to each summand and use the facts that $\Delta = \pi_1 \ldots \pi_n$ and b_r is the product of the torsion numbers of the summands to see that equation (3.7) still holds. Finally, if $\mathcal{M}$ is torsion free as an abelian group, we use Theorem 3.6. □