Introduction

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **48 (2002)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **22.09.2024**

L'Enseignement Mathématique, t. 48 (2002), p. 237–258

A TOPOLOGICAL PROOF OF THE GROTHENDIECK FORMULA IN REAL ALGEBRAIC GEOMETRY

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INTRODUCTION

In 1973 A. Tognoli [28], improving upon earlier work of J. Nash [25], demonstrated that every closed (compact without boundary) smooth manifold M is diffeomorphic to a nonsingular algebraic subset X of $\mathbf{R}^n$ for some n. The reader can also consult [11, Theorem 14.1.10] for a proof that requires the reading of only a few pages of [11]. This remarkable result of Nash-Tognoli soon gave rise to a larger program. By carefully choosing X one wanted to realize algebraically not only M alone, but also some objects such as submanifolds, vector bundles, homology or cohomology classes, etc. attached to it.

Examples of successes include the relative Nash-Tognoli theorem dealing with finite collections of smooth submanifolds of M [1] and a theorem asserting that X can be selected in such a way that every topological real vector bundle on X is isomorphic to an algebraic vector bundle [9]. A special case of the relative Nash-Tognoli theorem was used in [2] to obtain an elegant topological characterization of real algebraic sets with isolated singularities. A conjecture was put forward that X can be chosen with each homology class in $H_*(X; \mathbf{Z}/2)$ represented by an algebraic subset of X [3]; this would have simplified many constructions and facilitated a topological characterization of all real algebraic sets.

However, the conjecture was refuted in [7] by the following argument. For any integer $m \geq 11$, there exist an m-dimensional closed smooth manifold M

*) Both authors were partially supported by the Volkswagen Stiftung (Research in Pairs at Oberwolfach) and European grant RAAG 2002-2006.

and a cohomology class v in $H^2(M;\mathbf{Z}/2)$ such that v cannot be represented as the second Stiefel-Whitney class of a real vector bundle on M (it is now known that $m \geq 11$ can be replaced by $m \geq 6$, which is sharp [27]). On the other hand, for any compact nonsingular real algebraic set X, each cohomology class in $H^2(X;\mathbf{Z}/2)$, whose Poincaré dual homology class can be represented by an algebraic subset of X, is the second Stiefel-Whitney class of some algebraic vector bundle on X. Therefore the conjecture has to be false.

We call the latter part of the argument the *Grothendieck formula in real algebraic geometry*. This was proved in [7] in two steps. First a proof of the Grothendieck formula relating vector bundles and algebraic cycles on schemes over $\mathbf{R}$ was sketched (an analog of the formula from earlier papers [18, 19] for varieties over an algebraically closed field); this sketch contains some flaws. Then a connection, established in [15], between the Chern classes with values in the Chow ring and the Stiefel-Whitney classes yielded the conclusion. The appearance of [17] allowed for a shorter proof [12], based on the same principles and free from the flaws mentioned above. According to the authors' experience such proofs still present considerable difficulty for many topologically inclined mathematicians. The goal of this paper is to give a self-contained topological proof that uses only the simplest facts from algebra. Several applications of the Grothendieck formula in real algebraic geometry, besides the one discussed above, are contained in [12, 13, 22].

The paper assumes knowledge of singular homology and cohomology with coefficients in $\mathbf{Z}/2$ at the level of [26]. Real vector bundles and their Stiefel-Whitney classes, for which a good reference is [24], are also used. All smooth (of class C^∞) manifolds are assumed to be paracompact and without boundary. From real algebraic geometry we require only a few notions, recalled here and elucidated in detail in just a few pages of [5], [8], or [11]. Basic and generally well-known facts from commutative algebra that are needed can all be found in [23].

1. The Grothendieck formula

Real algebraic varieties

The Zariski topology on $\mathbf{R}^n$ is the topology for which the closed sets are precisely the algebraic subsets of $\mathbf{R}^n$. Let V be a nonempty Zariski locally closed subset of $\mathbf{R}^n$ (that is, V is the difference of two algebraic