

4.3 Classification of rational forms for some 6-dimensional Lie ALGEBRAS

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **48 (2002)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **19.04.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Thus, the case of a free nilpotent Lie algebra $f_c(p, \mathbf{C})$ (as a Lie algebra over $\mathbf{R}$) on p generators differs from the case 2.2.

REMARK. All rational forms of $f_2(2, \mathbf{C}) = \mathfrak{hei}_3(\mathbf{C})$ and $f_2(2, \mathbf{R}) \oplus f_2(2, \mathbf{R}) = \mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R})$ are listed in Theorem 3.

COROLLARY 4.2. *There are infinitely many non-commensurable (in any sense) lattices in the Lie groups of type $F_c(p, \mathbf{R}) \times F_c(p, \mathbf{R})$ where $F_c(p, \mathbf{R})$ is the free nilpotent Lie group on p free generators.*

4.3 CLASSIFICATION OF RATIONAL FORMS FOR SOME 6-DIMENSIONAL LIE ALGEBRAS

Let m be a rational number and $A_m = \mathbf{Q}[x]/(x^2 - m)$. A_m is a 2-dimensional commutative algebra over $\mathbf{Q}$ which depends only on m modulo square factors. Thus there are four types of A_m :

- 1) if $m = 1$ then $A_m \cong \mathbf{Q} \oplus \mathbf{Q}$;
- 2) if $m > 1$ is a positive square-free integer then $A_m \cong \mathbf{Q}(\sqrt{m})$ is a real quadratic field over $\mathbf{Q}$;
- 3) if $m = 0$ then A_0 is the algebra of dual numbers over $\mathbf{Q}$;
- 4) if m is a negative square-free integer then $A_m \cong \mathbf{Q}(\sqrt{m})$ is an imaginary quadratic field over $\mathbf{Q}$.

Let $\mathfrak{hei}_3(A_m)$ be a Heisenberg algebra over A_m considered over $\mathbf{Q}$. Then $\mathfrak{hei}_3(A_m)$ is a rational form of either $\mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R})$, or $\mathfrak{hei}_3(\mathbf{R}[x]/(x^2))$, or $\mathfrak{hei}_3(\mathbf{C})$. More precisely,

THEOREM 3. *Let $\mathfrak{h}$ be a 6-dimensional nilpotent Lie algebra of class 2 over $\mathbf{Q}$. Suppose that $[\mathfrak{h}, \mathfrak{h}]$ coincides with the 2-dimensional centre of $\mathfrak{h}$. Then $\mathfrak{h} \cong \mathfrak{hei}_3(A_m)$ for some $m \in \mathbf{Q}$ as above.*

Moreover,

- 1) $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R}) = \mathfrak{g}_+$ iff $m > 0$,
- 2) $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{hei}_3(\mathbf{R}[x]/(x^2)) = \mathfrak{g}_0$ iff $m = 0$,
- 3) $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{hei}_3(\mathbf{C}) = \mathfrak{g}_-$ iff $m < 0$,

and up to isomorphism there are no more rational forms for $\mathfrak{g}_-$, $\mathfrak{g}_0$, $\mathfrak{g}_+$. The Lie algebras $\mathfrak{hei}_3(A_m)$ and $\mathfrak{hei}_3(A_n)$ are isomorphic over $\mathbf{Q}$ if and only if A_m and A_n are isomorphic.

Proof. Take some $\mathbf{Q}$ -basis $x_1, \dots, x_6$ of $\mathfrak{h}$. First of all, we may suppose that $[x_1, x_2] = x_5$ (possibly after a change of basis). Thus x_5 is central. We have to deal with two cases.

CASE 1. All brackets $[x_1, x_j]$, $[x_2, x_j]$ ($j \geq 3$) are multiples of x_5 . If $[x_1, x_j] = a_j x_5$, $[x_2, x_j] = b_j x_5$ then we set $X_j = x_j - a_j x_2 + b_j x_1$ whence $[x_1, X_j] = [x_2, X_j] = 0$.

Since $[\mathfrak{h}, \mathfrak{h}]$ is 2-dimensional we conclude that some commutator, say $[x_3, x_4]$, is not a multiple of x_5 (for convenience, we use lower-case 'x' instead of 'X'). Consider

$$(4.1) \quad [x_3, x_4] = ax_1 + bx_2 + cx_3 + dx_4 + ex_5 + fx_6.$$

Commuting $[x_3, x_4]$ with x_1, x_2 we obtain that $a = b = 0$. Let us suppose that $f = 0$. Then

$$(4.2) \quad [x_3, x_4] = cx_3 + dx_4 + ex_5.$$

Recall that x_5 and $[x_3, x_4]$ in the form (4.2) span the 2-dimensional centre. Commuting $cx_3 + dx_4 + ex_5$ from (4.2) with x_3, x_4 we get $c = d = 0$ and a contradiction. Thus $f \neq 0$. We may assume that $[x_3, x_4] = x_6$ where x_6 is central. Hence, we have the following multiplication table for $\mathfrak{h}$: $[x_1, x_2] = x_5$, $[x_3, x_4] = x_6$, other brackets being equal to 0. Consequently,

$$\mathfrak{h} = \langle x_1, x_2, x_5 \rangle \oplus \langle x_3, x_4, x_6 \rangle \cong \mathfrak{hei}_3(\mathbf{Q}) \oplus \mathfrak{hei}_3(\mathbf{Q}).$$

CASE 2. Among the brackets $[x_1, x_j]$, $[x_2, x_j]$ ($j \geq 3$) there is at least one which is not a multiple of x_5 . In this case we may suppose (changing indices if necessary) that this bracket is $[x_1, x_3]$. Let

$$(4.3) \quad [x_1, x_3] = ax_1 + bx_2 + cx_3 + dx_4 + ex_5 + fx_6.$$

and let us suppose that $d = f = 0$. Then

$$(4.4) \quad [x_1, x_3] = ax_1 + bx_2 + cx_3 + ex_5.$$

Commuting the right-hand term of (4.4) with x_1 we get

$$0 = [x_1, [x_1, x_3]] = bx_5 + c[x_1, x_3] = cax_1 + cbx_2 + c^2x_3 + (ce + b)x_5.$$

Hence $c = b = 0$. By virtue of this $a = 0$ and we obtain a contradiction if we commute both sides of (4.4) with x_2 . It follows that either $d \neq 0$ or $f \neq 0$. In other words, we may suppose that $[x_1, x_3]$ is equal to x_6 .

Now

$$(4.5) \quad [x_1, x_2] = x_5, \quad [x_1, x_3] = x_6$$

where x_5, x_6 span $[\mathfrak{h}, \mathfrak{h}]$. Suppose that $[x_2, x_3] = ax_5 + bx_6$. Adding if necessary some multiples of x_1 to x_2 and x_3 we obtain $[x_2, x_3] = 0$. In the same way we may suppose that $[x_1, x_4] = 0$. Adding to x_4 some multiple of x_1 we also obtain a relation $[x_2, x_4] = Cx_6$. Moreover, after scaling x_4 we get $C = 0$ or $C = 1$. Thus, $\mathfrak{h}$ has a basis in which the *non-trivial* brackets are the following:

$$(4.6) \quad \begin{aligned} [x_1, x_2] &= x_5, & [x_1, x_3] &= x_6, \\ [x_2, x_4] &= Cx_6 \quad (C = 0 \text{ or } C = 1), & [x_3, x_4] &= Ax_5 + Bx_6. \end{aligned}$$

In any case $A^2 + B^2 + C^2 \neq 0$ because x_4 cannot belong to the 2-dimensional centre of $\mathfrak{h}$.

We will show that we can always make $C = 1$ and $B = 0$ in (4.6).

SUBCASE 2.1. If $C = 0$ then the following basis transformation

$$(4.7) \quad \begin{aligned} X_1 &= x_1, & X_2 &= ax_2 + x_3, \\ X_3 &= Ax_2 + Bx_3, & X_4 &= x_4, \end{aligned}$$

yields (a is any constant such that $aB \neq A$)

$$(4.8) \quad \begin{aligned} [X_1, X_2] &= ax_5 + x_6 = X_5, & [X_1, X_3] &= Ax_5 + Bx_6 = X_6, \\ [X_2, X_4] &= Ax_5 + Bx_6 = X_6, & [X_3, X_4] &= B(Ax_5 + Bx_6) = BX_6. \end{aligned}$$

From now on we may suppose that $C = 1$ in (4.6) and we arrive at

SUBCASE 2.2: $C = 1, A = 0$. Let

$$(4.9) \quad \begin{aligned} X_1 &= x_1 + ax_4, & X_2 &= x_2 - ax_3, \\ X_3 &= x_2 + dx_3, & X_4 &= -x_1 + dx_4, \end{aligned}$$

where $a, d, a + d \neq 0, aB \neq 1, dB \neq -1$. Hence

$$(4.10) \quad \begin{aligned} [X_1, X_2] &= x_5 + (a^2B - 2a)x_6 = X_5, \\ [X_1, X_3] &= x_5 + (d - a - adB)x_6 = X_6, \\ [X_2, X_4] &= x_5 + (d - a - adB)x_6 = X_6, \\ [X_3, X_4] &= x_5 + (d^2B + 2d)x_6 = \lambda X_5 + (1 - \lambda)X_6. \end{aligned}$$

Since a, d and $a + d$ are all non-zero, X_5 and X_6 are linearly independent. Straightforward computations yield

$$\lambda = \frac{dB + 1}{aB - 1} \neq 0, 1.$$

Thus we have the following alternative.

SUBCASE 2.3.1: $C = 1$; $A, B, 4A + B^2 \neq 0$. Let now

$$(4.11) \quad \begin{aligned} X_1 &= x_1 + tx_4, & X_2 &= x_2 - tx_3, \\ X_3 &= x_3, & X_4 &= x_4, \end{aligned}$$

where $t = -B/2A$. Hence

$$(4.12) \quad \begin{aligned} [X_1, X_2] &= (1 + t^2A)x_5 + (t^2B - 2t)x_6 = X_5, \\ [X_1, X_3] &= -tAx_5 + (1 - tB)x_6 = X_6, \\ [X_2, X_4] &= -tAx_5 + (1 - tB)x_6 = X_6, \\ [X_3, X_4] &= Ax_5 + Bx_6 = \alpha X_5 = \frac{4A^2}{4A + B^2} X_5. \end{aligned}$$

SUBCASE 2.3.2: $C = 1$; $A, B \neq 0, 4A + B^2 = 0$. The same transformation (4.11) yields

$$(4.13) \quad \begin{aligned} [X_1, X_2] &= 0, \\ [X_1, X_3] &= -tAx_5 + (1 - tB)x_6 = X_6, \\ [X_2, X_4] &= -tAx_5 + (1 - tB)x_6 = X_6, \\ [X_3, X_4] &= Ax_5 + Bx_6 = X_5 \end{aligned}$$

and, after the transformation $x_1 = X_3, x_2 = X_4, x_3 = X_1, x_4 = X_2, x_5 = X_5, x_6 = -X_6$, we obtain (4.12) with $\alpha = 0$. Anyway, we obtain the desired form of $\mathfrak{h}$

$$(4.14) \quad [x_1, x_2] = x_5, \quad [x_1, x_3] = x_6, \quad [x_2, x_4] = x_6, \quad [x_3, x_4] = Ax_5.$$

Scaling x_3, x_4 by $\lambda \neq 0$ we may suppose that $A = m$ where m is a square-free integer as above.

In order to conclude the proof of the first part of the theorem we point out an isomorphism $\rho: \mathfrak{h} \rightarrow \mathfrak{h} \text{ei}_3(A_m)$. Recall that A_m has a basis $1, x$ over $\mathbf{Q}$ such that $x^2 = m$. Here are the matrices representing $\rho(x_i)$ if $m \neq 1$ (the case $m = 1$ is left to the reader as an easy exercise):

$$(4.15) \quad \begin{aligned} \rho(x_1) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \rho(x_2) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, & \rho(x_5) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \rho(x_3) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}, & \rho(x_4) &= \begin{pmatrix} 0 & -x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \rho(x_6) &= \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Now it is evident that $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R}$ is isomorphic to either $\mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R})$, or $\mathfrak{hei}_3(\mathbf{R}[x]/(x^2))$, or $\mathfrak{hei}_3(\mathbf{C})$ depending on the sign of m . Thus, we have classified up to $\mathbf{Q}$ -isomorphism all rational forms for these 3 real Lie algebras. By Theorem 2 these forms are non-isomorphic. The proof of the theorem is complete. $\square$

REMARK. It is worth mentioning that the above three real Lie algebras are not pairwise isomorphic over $\mathbf{R}$. Indeed, the centralizer of any element in $\mathfrak{g}_- = \mathfrak{hei}_3(\mathbf{C})$ is even dimensional over $\mathbf{R}$ since this algebra can be viewed as a complex Lie algebra, whereas in both $\mathfrak{g}_+ = \mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R})$ and $\mathfrak{g}_0 = \mathfrak{hei}_3(\mathbf{R}[x]/(x^2))$ there are elements with 5-dimensional centralizers. In order to show that the last two algebras are not isomorphic we need some more information about elements with 5-dimensional centralizers.

The centralizer $C(x)$ will not be changed if we scale x by any $\lambda \neq 0$ or add to x any central element. This means that dimension of the centralizer is a well-defined function on the projective space $\mathbf{P}(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$ where $\mathfrak{g}$ is either $\mathfrak{g}_+$ or $\mathfrak{g}_0$. Straightforward computations show that in $\mathbf{P}(\mathfrak{g}_0/[\mathfrak{g}_0, \mathfrak{g}_0])$ all points with 5-dimensional centralizer belong to a unique line whereas in $\mathbf{P}(\mathfrak{g}_+/[\mathfrak{g}_+, \mathfrak{g}_+])$ the points under consideration form two disjoint lines.

REFERENCES

- [1] BOURBAKI, N. *Groupes et algèbres de Lie. Chap. I, 3*. Hermann, Paris, 1972.
- [2] CORWIN, L. J. and F. P. GREENLEAF. *Representations of Nilpotent Lie Groups and their Applications. Part I*. Cambridge Univ. Press, 1989.
- [3] DIXMIER, J. Sur les représentations unitaires des groupes de Lie nilpotents III. *Canad. J. Math.* 10 (1958), 321–348.
- [4] RAGHUNATHAN, M. *Discrete Subgroups of Lie Groups*. Springer-Verlag, Berlin, 1972.
- [5] MALCEV, A. I. On a class of homogeneous spaces. *Amer. Math. Soc. Translation* 39 (1951); *Izvestiya Akad. Nauk SSSR. Ser. Mat.* 13 (1949), 9–32.

(Reçu le 25 juillet 2001)

Yu. S. Semenov

MIIT, division 'Applied Mathematics – 1'
 Obraztsova 15
 127994 Moscow
 Russia
 e-mail: yury_semenov@hotmail.com