7.2 Actions of higher rank lattices

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **47 (2001)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **24.09.2024**

7.2 Actions of higher rank lattices

We now study actions of the most general higher rank lattices on the circle. Most of this section is an expansion (and a translation) of a small part of [26] to which we refer for more information.

THEOREM 7.4 ([26]). *Let Γ be a lattice in a simple Lie group G with real rank greater than or equal to 2. Then any action of Γ on the circle has a finite orbit.*

Of course, in such a situation a subgroup of finite index in Γ acts with a fixed point so that, deleting this fixed point, we get an action of a subgroup of finite index acting on the line. Recall our question 7.3 concerning ordering on lattices; it can be reformulated in the following way:

PROBLEM 7.5. *Let Γ be a lattice in a simple Lie group G with real rank greater than or equal to 2. Is it true that any homomorphism from Γ to* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *has a finite image?*

These notes only deal with actions by homeomorphisms and we decided not to discuss properties connected with smooth diffeomorphisms. However, we mention that the previous question has a positive answer assuming some smoothness.

THEOREM 7.6 ([26]). *Let Γ be a lattice in a simple Lie group G with real rank greater than or equal to 2. Then any homomorphism from Γ to the group of C^1-diffeomorphisms of the circle has a finite image.*

This theorem is an immediate consequence of 7.4 and of two important results. The first one, due to Kazhdan, states that a lattice like the one in the theorem is finitely generated and admits no non trivial homomorphism into $\mathbf{R}$ (see [48]). The second, due to Thurston, states that if a finitely generated group Γ has no non trivial homomorphism to $\mathbf{R}$ then any homomorphism from Γ to the group of germs of C^1-diffeomorphisms of $\mathbf{R}$ in the neighbourhood of the fixed point 0 is trivial (see [66]).

If we add more smoothness assumptions (but this is not the goal of this paper...), A. Navas, following earlier ideas of Segal and Reznikov, recently proved a remarkable theorem which applies to groups with Kazhdan's property (T) (see [57]). Note that lattices in higher rank semi-simple Lie groups have this property (see [32]).

THEOREM 7.7 (Navas). *Let Γ be a finitely generated subgroup of the group of diffeomorphisms of the circle of class $C^{1+\alpha}$ with $\alpha > 1/2$. If Γ satisfies Kazhdan's property (T), then Γ is finite.*

When the Lie group G is not simple but only semi-simple, the situation is more complicated since there are some interesting examples of irreducible higher rank lattices that do act. We have already described some examples of irreducible lattices in $\mathrm{SL}(2,\mathbf{R}) \times \mathrm{SL}(2,\mathbf{R})$ which act on the circle via their projection on the first factor (which is a dense subgroup in $\mathrm{SL}(2,\mathbf{R})$). As a matter of fact, the next result shows that these examples are basically the only ones.

If ϕ_1 and $\phi_2 \colon \Gamma \to \mathrm{Homeo}_+(\mathbf{S}^1)$ are homomorphisms, we say that ϕ_1 is *semi-conjugate to a finite cover of* ϕ_2 if there is a continuous map $h \colon \mathbf{S}^1 \to \mathbf{S}^1$ which is onto and locally monotonous, such that for every $\gamma \in \Gamma$ we have $\phi_2(\gamma)h = h\phi_1(\gamma)$.

THEOREM 7.8 ([26]). *Let Γ be an irreducible lattice in a semi-simple Lie group G with real rank greater than or equal to 2. Let ϕ be a homomorphism from Γ to the group of orientation preserving homeomorphisms of the circle. Then either $\phi(\Gamma)$ has a finite orbit or ϕ is semi-conjugate to a finite cover of a homomorphism which is the composition of:*

i) *the embedding of Γ in G,*

ii) *a surjection from G to* $\mathrm{PSL}(2,\mathbf{R})$,

iii) *the projective action of* $\mathrm{PSL}(2,\mathbf{R})$ *on the circle.*

These theorems show that higher rank lattices have very few actions on the circle. Hence, according to Section 6.15, the second bounded cohomology groups of lattices should be small. This is indeed what Burger and Monod showed in [12]:

THEOREM 7.9 (Burger, Monod). *Let Γ be a cocompact irreducible lattice in a semi-simple Lie group G with real rank greater than or equal to 2. Then the second bounded cohomology group $H_b^2(\Gamma,\mathbf{R})$ injects in the usual cohomology group $H^2(\Gamma,\mathbf{R})$.*

The assumption that the lattice is cocompact is important in the proof but the theorem probably generalizes to non-cocompact lattices. Note also that for many lattices in semi-simple Lie groups, it turns out that the usual cohomology group $H^2(\Gamma, \mathbf{R})$ vanishes. This is the case for instance for cocompact torsion free lattices in $\mathrm{SL}(n, \mathbf{R})$ for $n \geq 4$ but more generally for cocompact torsion free lattices in the group of isometries of an irreducible symmetric space of non compact type of rank at least 3 which is not hermitian symmetric (see [7]). In these cases, Theorem 7.9 means that $H_b^2(\Gamma, \mathbf{R})$ vanishes. Hence, using 6.6, we deduce that every action Γ on the circle has a finite orbit. In other words, Theorems 7.4 and 7.9 are closely related and, indeed they have been proved simultaneously (and independently). It would be very useful to compare the two proofs.

As we have already noticed, the vanishing of the second bounded cohomology group is closely related to the notion of commutator length. If Γ is any group and γ is in the first commutator subgroup Γ', we denote by $|\gamma|$ the least integer k such that γ can be written as a product of k commutators. We "stabilize" this number and define $\|\gamma\|$ as $\lim_{n\to\infty} |\gamma^n|/n$ (which always exists by sub-additivity). It turns out that for a finitely generated group Γ it is equivalent to say that the second bounded cohomology group $H_b^2(\Gamma, \mathbf{R})$ injects in the usual cohomology group $H^2(\Gamma, \mathbf{R})$, and to say that this "stable commutator norm" $\| \ \|$ vanishes identically [5]. Theorem 7.9 therefore implies that for cocompact higher rank lattices, this stable norm vanishes. The following question is natural:

PROBLEM 7.10. *Let Γ be an irreducible lattice as in Theorem 7.4. Does there exist an integer $k \geq 1$ such that every element of the first commutator subgroup of Γ is a product of k commutators ?*

Recall that by a theorem of Kazhdan, there is no non trivial homomorphism from Γ to $\mathbf{R}$; this is equivalent to the fact that the first commutator group of Γ has finite index in Γ. A positive answer to the previous question would be a strengthening of this fact.

7.3 LATTICES IN LINEAR GROUPS

In this section, we prove Theorem 7.4 for lattices in $\mathrm{SL}(n, \mathbf{R})$ $(n \geq 3)$. The general case of a semi-simple Lie group is much harder but the proof that we present here contains the main ideas. As a matter of fact, we shall