

2. Some classical definitions

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2. SOME CLASSICAL DEFINITIONS

We begin with some very general definitions concerning group actions. For an introduction to this subject, we refer to [42].

Let Γ be any group and X be any topological space. An *action* of Γ on X is a homomorphism ϕ from Γ to the group $\text{Homeo}(X)$ of homeomorphisms of X . An element $\gamma \in \Gamma$ and a point $x \in X$ produce the point $\gamma \cdot x = \phi(\gamma)(x)$. Conversely a map

$$(\gamma, x) \in \Gamma \times X \mapsto \gamma \cdot x \in X$$

comes from an action if for every γ , the point $\gamma \cdot x$ depends continuously on x and if for every γ_1, γ_2 we have $\gamma_1 \cdot (\gamma_2 \cdot x) = (\gamma_1 \gamma_2) \cdot x$ and $e \cdot x = x$ (e denotes the identity element in Γ).

Two actions ϕ_1 and ϕ_2 of Γ on X_1 and X_2 are *conjugate* if there exists a homeomorphism h from X_1 to X_2 such that for every $\gamma \in \Gamma$, one has $\phi_2(\gamma) = h\phi_1(\gamma)h^{-1}$.

An action ϕ is *faithful* if it is injective, *i.e.* if non trivial elements in the group act non trivially on the space. This is a minor assumption since we can always consider the associated faithful action of the quotient group $\Gamma/\ker(\phi)$.

The *orbit* of a point x is the set $\mathcal{O}(x) = \{\phi(\gamma)(x) \mid \gamma \in \Gamma\} \subset X$. The main object of topological dynamics is to study the topological properties of the partition of X into orbits. An action is *transitive* if there is only one orbit. We say in this case that X is *homogeneous* under the action of Γ . Of course, these transitive actions are quite trivial from the topological dynamics point of view but this does not mean that the geometrical study of homogeneous spaces is not interesting!

The *stabilizer* of the point x is the subgroup

$$\text{Stab}(x) = \{\gamma \in \Gamma \mid \phi(\gamma)(x) = x\} \subset \Gamma.$$

There is a natural bijection between the quotient $\Gamma/\text{Stab}(x)$ and the orbit $\mathcal{O}(x)$. Note that the stabilizers of two points in the same orbit are conjugate subgroups in Γ . An action is *free* if the stabilizer of every point is trivial, *i.e.* if the action of a non trivial element of Γ has no fixed point.

In some cases, Γ might be a topological group. In these cases, we frequently consider *continuous actions* such that $\gamma \cdot x$ is a continuous function on $\Gamma \times X$. The orbit map bijection from $\Gamma/\text{Stab}(x)$ to $\mathcal{O}(x)$ is continuous but is usually not a homeomorphism when $\mathcal{O}(x)$ is equipped with the induced topology from X . The easiest non trivial example is the case where $\Gamma = \mathbf{R}$, *i.e.* of a topological flow: if the stabilizer of a point x is trivial, the orbit $\mathcal{O}(x)$ is the image of a continuous bijection $\mathbf{R} \rightarrow \mathcal{O}(x) \subset X$ but in many cases this orbit might be recurrent (for instance dense in X) and this bijection is not a homeomorphism. There is however a special case in which this map is indeed a homeomorphism and we use this fact constantly (and sometimes implicitly) in these notes. Consider a Lie group G acting continuously and transitively on a manifold M and denote by H the stabilizer of a point. Then H is a closed subgroup of G , hence a closed Lie subgroup, and the quotient space G/H is naturally a smooth manifold. In this case, the orbit map from G/H to M is a homeomorphism.