6. COHOMOLOGICAL INTERPRETATION

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Proof. Follows immediately from Corollary 5.11, equation (28) and Part (i) of Theorem 5.10. □

6. Cohomological Interpretation

Let $\mathbf{G}_m$ be the multiplicative group regarded as an affine group scheme over $X := \operatorname{Spec} C$ and let $\mu_3 \subset \mathbf{G}_m$ be the kernel of multiplication by 3. All the cohomology groups below are with respect to the flat topology on X.

THEOREM 6.1. *Suppose $[C^*]$ is divisible by 3 in* $\operatorname{Pic}(C)$. *Then the group $H^1_{\mathrm{fl}}(X, \mu_3)$ acts simply transitively on the set $\mathcal{S}(C)$ of C-equivalence classes of cubic C-forms with primitive determining mapping.*

Proof. Recall that the group $H^1_{\mathrm{fl}}(X, \mu_3)$ can be interpreted concretely as the set of isomorphism classes of pairs (L, ψ), where L is an invertible C-module and where $\psi\colon L_C^{\otimes 3} \to C$ is an isomorphism (see Milne [14, Chap. III, §4]). Let $[L, \psi]$ be an element of $H^1_{\mathrm{fl}}(X, \mu_3)$ and let (M, F) be a cubic C-form. By Theorem 5.1, Part (i), we can assume $F = F_\phi$, where $\phi\colon M^{\otimes 3} \to C^*$ is an isomorphism. We define an action of $H^1_{\mathrm{fl}}(X, \mu_3)$ on $\mathcal{S}(C)$ by

(30) $$[L, \psi] \cdot [M, F_\phi] = [L \otimes M, F_{\psi \otimes \phi}],$$

noting that

$$(L \otimes M)_C^{\otimes 3} = L_C^{\otimes 3} \otimes M_C^{\otimes 3} \xrightarrow{\psi \otimes \phi} C \otimes C^* = C^*$$

is an isomorphism. Let us show first that this action is simple. Suppose $[L \otimes M, F_{\psi \otimes \phi}] = [M, F_\phi]$. Then, $L \cong C$. Choosing an isomorphism $L \to C$, we have $\psi(x \otimes y \otimes z) = uxyz$, where $u \in C^\times$. Hence $[M, F_\phi] = [M, F_{u\phi}]$, and by Part (iii) of Theorem 5.1 we conclude that $u = c^3$ for some $c \in C^\times$. But then $c\colon C \to C$ provides an isomorphism of (C, ψ) with $(C, 1)$, thus $[L, \psi] = [C, 1]$.

We show now that the action is transitive. Let $[M_i, F_{\phi_i}]$ $(i = 1, 2)$ be elements of $\mathcal{S}(C)$. Let $M_2^\bullet = \operatorname{Hom}_C(M_2, C)$ and let $\phi_2^\bullet : (C^*)^\bullet \to (M_2^{\otimes 3})^\bullet$ be the dual of ϕ_2. Let $L = M_1 \otimes M_2^\bullet$ and let $\psi = \phi_1 \otimes {\phi_2^\bullet}^{-1}$. One verifies immediately that $[L, \psi] \cdot [M_2, F_{\phi_2}] = [M_1, F_{\phi_1}]$, which proves that the action is transitive. □

Note that, under the hypothesis of (6.1),

$$\mathcal{S}(C) = \coprod_{L \in \mathrm{Pic}\,(C)[3]} \mathrm{Cubic}_C(M_0 \otimes L)$$

where M_0 is any invertible module such that $M_0^{\otimes 3} \cong C^*$. Each $\mathrm{Cubic}_C(M)$ is a torsor for $C^\times/(C^\times)^3$ by Theorem 5.2.

Consider now the short exact sequence of group schemes over X

$$1 \longrightarrow \mu_3 \xrightarrow{\ i\ } \mathbf{G}_m \xrightarrow{\ 3\ } \mathbf{G}_m \longrightarrow 1$$

and the associated Kummer long exact sequence in flat cohomology

$$H^0_{\mathrm{fl}}(X, \mathbf{G}_m) \xrightarrow{\ 3\ } H^0_{\mathrm{fl}}(X, \mathbf{G}_m) \xrightarrow{\ \partial\ } H^1_{\mathrm{fl}}(X, \mu_3)$$
$$\xrightarrow{\ i_*\ } H^1_{\mathrm{fl}}(X, \mathbf{G}_m) \xrightarrow{\ 3\ } H^1_{\mathrm{fl}}(X, \mathbf{G}_m)\,.$$

Using the canonical isomorphisms (see [14, Chap. III, §4])

$$H^0_{\mathrm{fl}}(X, \mathbf{G}_m) \simeq C^\times, \quad H^1_{\mathrm{fl}}(X, \mathbf{G}_m) \simeq \mathrm{Pic}\,(C)$$

we obtain a short exact sequence

$$1 \to C^\times/C^{\times 3} \xrightarrow{\partial} H^1_{\mathrm{fl}}(X, \mu_3) \xrightarrow{i_*} \mathrm{Pic}\,(C)[3] \to 1\,. \tag{31}$$

By what we have proved, $\mathcal{S}(C)$ will be empty unless $[C^*]$ is divisible by 3 in $\mathrm{Pic}\,(C)$. By the Kummer sequence, $[C^*]$ is divisible by 3 if and only if

$$\partial[C^*] = 0 \in H^2_{\mathrm{fl}}(X, \mu_3)\,.$$

Assume that this holds and consider the group $H(C)$ of binary quadratic mappings as defined by Kneser in [11]. The determining form construction (14) gives a well-defined map

$$e\colon \mathcal{S}(C) \longrightarrow H(C)$$
$$[M, F] \longmapsto [M, q_F, \mathcal{D}(M)]\,.$$

We fix a "base point" $[M_0, F_0] \in \mathcal{S}(C)$ and we modify the map e slightly so that it becomes a map of pointed sets. We define

$$e'\colon \mathcal{S}(C) \longrightarrow H(C)$$
$$[M, F] \longmapsto e[M, F] - e[M_0, F_0]\,.$$

We also define a map $f\colon H^1_{\mathrm{fl}}(X, \mu_3) \to \mathcal{S}(C)$ by $f(x) = x \cdot [M_0, F_0]$, where $\cdot$ is the action defined in (30). Note that by virtue of Theorem 6.1, the map f is bijective.

With this notation we have a commutative square

$$\begin{array}{ccc} H^1_{\mathrm{fl}}(X, \mu_3) & \xrightarrow{i_*} & \mathrm{Pic}(C)[3] \\ f\big\downarrow & & \big\uparrow j \\ \mathcal{S}(C) & \xrightarrow{e'} & H(C)[3] \end{array} \tag{32}$$

where $j\colon H(C) \to \mathrm{Pic}(C)$ is the natural homomorphism $[M,q,N] \mapsto [M]$. Kneser [11, §6] has shown that j is an isomorphism (see also Section 2), so the two vertical maps in (32) are bijections and the horizontal maps are surjections.

Note that because of the exact sequence (31), the fibers of e' are in one-to-one correspondence with the elements of the group $C^\times / C^{\times^3}$. This is, of course, equivalent to Theorem 5.2, Part (ii).

7. Explicit computations and cubic trace forms

In this section we assume that $A := C \otimes K$ is a quadratic étale algebra over K. In this case the trace form $(x, y) \to \mathrm{Tr}_{A/K}(xy)$ is nondegenerate and gives rise to a natural isomorphism between the codifferent

$$C' = \{x \in A : \mathrm{Tr}_{A/K}(xC) \subset R\}$$

and the dual C^*. If M is a fractional C-ideal with $M^3 \simeq C'$, then, by Theorem 5.1, the cubic forms on M with primitive determining form are given by

$$F_u(\mathbf{x}) = \mathrm{Tr}_{A/K}(ua\mathbf{x}^3), \tag{33}$$

where $a \in A$ is a fixed element with $aM^3 = C'$, and u is a unit of C. Moreover, by Theorem 5.1, two such forms F_u and F_v are C-isomorphic if and only if u and v represent the same element of $C^\times/(C^\times)^3$.

We shall compute explicitly some examples for $R = \mathbf{Z}$ using (33). In this case we have $C = \mathbf{Z}[t]/(f(t))$, where f is a monic degree-two polynomial with distinct roots and coefficients in $\mathbf{Z}$.

Let ω be the class of t in C. It is well-known, and easy to prove, that the codifferent C' is a principal fractional C-ideal generated by $f'(\omega)^{-1}$, where f' is the derivative of f. Hence, $[C^*]$ is trivial in $\mathrm{Pic}(C)$ (note that this holds more generally provided $\mathrm{Pic}(R) = 0$).