

5. Remarks and questions

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **46 (2000)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **24.09.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern. Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden. Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Fix a generic functional ω on \mathbf{R}^{H-B} which maps a hive to a linear combination with positive coefficients of the labels at non-border hive vertices. For each $b \in \rho(C)$, let $\ell(b)$ be the unique hive in $\rho^{-1}(b) \cap C$ where ω is maximal. Then $\ell: \rho(C) \rightarrow C$ is a continuous piece-wise linear map [11, §1]. Notice that since w has positive coefficients, $\ell(b)$ has no increasable subsets.

We want to prove that the labels of $\ell(b)$ are \mathbf{Z} -linear combinations of the labels of b . In particular $\ell(b)$ is an integral hive if b is integral. For a regular border $b \in \rho(C)$, Proposition 1 implies that the flatspaces of $\ell(b)$ consist of small triangles and rhombi; by Proposition 2 this implies that all labels of $\ell(b)$ are \mathbf{Z} -linear combinations of the labels of b . Finally, since the regular borders are dense in each maximal subcone of $\rho(C)$ where ℓ is linear, ℓ must be integrally defined everywhere. \square

5. REMARKS AND QUESTIONS

Knutson and Tao's proof of the saturation conjecture implies that Klyachko's inequalities for T_n can be produced by a simple recursive algorithm, which uses the inequalities for T_k , $1 \leq k \leq n-1$ ([9], [10], [12], [6]). A triple of partitions (λ, μ, ν) with $|\nu| = |\lambda| + |\mu|$ is in T_n if and only if

$$\sum_{i=1}^k \nu_{\gamma_i+k+1-i} \leq \sum_{i=1}^k \lambda_{\alpha_i+k+1-i} + \sum_{i=1}^k \mu_{\beta_i+k+1-i}$$

for all triples $(\alpha, \beta, \gamma) \in T_k$ with $\gamma_1 \leq n-k$. Another important consequence is Horn's conjecture, which says that the same inequalities describe which sets of eigenvalues can arise from two Hermitian matrices and their sum [8].

P. Belkale has shown that the inequality produced by a triple (α, β, γ) with Littlewood-Richardson coefficient $c_{\alpha\beta}^{\gamma} \geq 2$ follows from the other inequalities. Knutson, Tao, and Woodward have announced a proof that the remaining inequalities are independent, i.e. they describe the facets of the cone $\rho(C)$. Their proof uses an interesting operation of overlaying two hives, which is defined in terms of Knutson and Tao's honeycomb model [10].

These results have made it very interesting to determine which triples (λ, μ, ν) have coefficient $c_{\lambda\mu}^{\nu}$ equal to one. Fulton has conjectured that this is equivalent to $c_{N\lambda, N\mu}^{N\nu}$ being one for any $N \in \mathbf{N}$. This has been verified in all cases with $N|\nu| \leq 68$. (Recently Knutson and Tao have reported that they can prove this as well.)

For $n = 3$ it is easy to show that a triple of partitions has Littlewood-Richardson coefficient one if and only if it corresponds to a point on the

boundary of the cone $\rho(C)$. In general, Fulton's conjecture implies that the triples with coefficient one are exactly those corresponding to points in a collection of faces of $\rho(C)$. For $n \geq 3$ this means that all triples corresponding to interior points in $\rho(C)$ have coefficient at least two.

One approach for proving Fulton's conjecture is to show that if $b \in \rho(C) \cap \mathbf{Z}^B$, then any generic positive functional ω on \mathbf{R}^{H-B} must be minimized (as well as maximized) at an integral hive in $\rho^{-1}(b) \cap C$. In fact, by Proposition 2 it is enough to prove:

If $b \in \rho(C)$ is a generic border and if a generic positive functional ω is minimized at $h \in \rho^{-1}(b) \cap C$, then the flatspaces of h consist of small triangles and rhombi.

Part of proving this is to specify when a border b is generic. We believe the statement is true if b avoids finitely many hyperplanes in \mathbf{R}^B .

The Littlewood-Richardson coefficients $c_{\lambda\mu}^\nu$ have the following natural generalization. Given decreasing sequences of integers ν , and $\lambda(1), \dots, \lambda(r)$, let $c_{\lambda(1), \dots, \lambda(r)}^\nu$ denote the multiplicity of V_ν in the holomorphic representation $V_{\lambda(1)} \otimes \dots \otimes V_{\lambda(r)}$. When $\nu = (0, \dots, 0)$, this specializes to the symmetric Littlewood-Richardson coefficient $c_{\lambda(1), \dots, \lambda(r)}$ which is the dimension of the $\mathrm{GL}_n(\mathbf{C})$ -invariant subspace of $V_{\lambda(1)} \otimes \dots \otimes V_{\lambda(r)}$. Postnikov and Zelevinsky have pointed out that the saturation conjecture as stated in the introduction implies a similar result for these generalized coefficients, i.e.

$$(5.1) \quad c_{\lambda(1), \dots, \lambda(r)}^\nu \neq 0 \iff c_{N\lambda(1), \dots, N\lambda(r)}^{N\nu} \neq 0.$$

Knutson has shown us that, by combining several hive triangles, one obtains a polytope whose integral points count these more general coefficients. This gives rise to another proof of (5.1).

In [3] other generalized Littlewood-Richardson coefficients related to quiver varieties are described. A different generalization related to Hecke algebras is defined in [7], and quantum Littlewood-Richardson coefficients are studied in [2]. It would be very interesting if these coefficients can be realized as the number of integral points in some polytopes.