§1. HOMOTOPIC STABILITY

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THREE REMARKS ON GEODESIC DYNAMICS AND FUNDAMENTAL GROUP

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§ 1. HOMOTOPIC STABILITY

For a Riemannian manifold V we denote by $S = S(V)$ the space of all unit tangent vectors. We denote by $G(V)$ the geodesic foliation on S: leaves are orbits of the geodesic flow (i.e. liftings to S of geodesics from V).

GEODESIC RIGIDITY. If V, W are closed manifolds of negative curvature with isomorphic fundamental groups, then the spaces $S(V)$ and $S(W)$ are homeomorphic. Moreover the geodesic foliations $G(V)$ and $G(W)$ are homeomorphic (i.e. there is a homeomorphism $S(V) \to S(W)$ sending leaves from $G(V)$ into leaves from $G(W)$).

It is unknown whether V and W are homeomorphic. The last question was discussed several times by Cheeger, Gromoll, Meyer and myself. In the end Cheeger constructed (see below) a homeomorphism between the Stiefel fiberings over V and W. Later Veech suggested to me that geodesic rigidity would be a better geometric substitute for the Mostow rigidity theorem than existence of a homeomorphism $V \to W$. Unfortunately the geodesic rigidity theorem is too simple and superficial and it does not lead to deep corollaries of the Mostow theorem. For example, finiteness of the group $\mathrm{Aut}(\pi_1(V))/\mathrm{Conj}(\pi_1(V))$ can not be derived (at least directly) from geodesic rigidity. ("Aut" means the group of automorphisms of the fundamental group, "Conj" means the group of inner automorphisms. The case $\dim V = 2$ is excluded.)

THE CHEEGER HOMEOMORPHISM. Denote by $\mathrm{St}_2(V)$ the space of all tangential orthonormal 2-frames in V. If V and W are as before, then the spaces $\mathrm{St}_2(V)$ and $\mathrm{St}_2(W)$ are homeomorphic.

The Cheeger construction is more canonical than our geodesic homeomorphism. In particular, $\mathrm{St}_2(V)$ can be viewed as a functorial object over $\pi_1(V)$.

STABLE HOMEOMORPHISM. As we have already mentioned, existence of a homeomorphism $V \to W$ is still a problem, but existence of a stable homeomorphism (homeomorphism $V \times \mathbf{R}^N \to W \times \mathbf{R}^N$ with large N) follows immediately from the topological equivalence between unit tangent bundles of V and W (see below).

CONSTRUCTIONS AND PROOFS

Ideas and notions involved in the constructions below are well known and due to M. Morse (see Appendix 2 for details).

For a complete simply connected manifold X of negative curvature we denote by $\mathrm{Cl}(X)$ its compactification (closure) and by $\partial(X)$ the complement $\mathrm{Cl}(X) \setminus X$. The space ∂X is homeomorphic to the $(n-1)$-sphere, $n = \dim X$, and it can be viewed as the set of all asymptotic classes of geodesic rays in X.

Consider a group Γ of isometries acting on X. Such an action can be continuously extended to $\partial(X)$. When Γ is discrete and the factor X/Γ is compact, the space $\partial(X)$ and the action of Γ in $\partial(X)$ depend (functorially) only on Γ. When $\Gamma = \pi_1(V)$ and X is the universal covering of V, then the unit tangent bundle of V is topologically equivalent to the bundle associated to the covering $X \to V$ with fiber $\partial(X)$. This immediately yields the topological equivalence of tangent bundles and so the stable homeomorphism theorem.

For a geodesic ray $r \subset X$ we denote by $\partial(r) \in \partial(X)$ the asymptotic class it represents. For an oriented geodesic g we denote by $\partial^+(g) \in \partial(X)$ and $\partial^-(g) \in \partial(X)$ the asymptotic classes of its positive and negative directions. When X has strictly negative curvature (i.e. the upper limit of sectional curvature is negative), the map $g \mapsto \big(\partial^+(g), \partial^-(g)\big) \in \partial(X) \times \partial(X)$ establishes a homeomorphism between the set of all oriented geodesics in X and the complement $\partial^2(X) = \big(\partial(X) \times \partial(X)\big) \setminus \Delta$, where Δ is the diagonal.

THE CHEEGER HOMEOMORPHISM

Denote by $\partial^3(X)$ *the set of triples* (x_1, x_2, x_3), $x_1, x_2, x_3 \in \partial(X)$ *with* $x_i \neq x_j$ *for* $i \neq j$. *If* X *has strictly negative curvature, then* $\mathrm{St}_2(X)$ *is canonically homeomorphic to* $\partial^3(X)$.

Proof. Realize an $s \in \mathrm{St}_2(X)$ by a pair (g, r) where $g \subset X$ is an oriented geodesic and $r \subset X$ is a geodesic ray starting from a point $x \in g$ and normal to g. Set $\mathrm{Chee}(s) = \mathrm{Chee}(g, r) = \left(\partial^+(g), \partial^-(g), \partial(r)\right)$. This is a homeomorphism because the normal projection $P = P_g\colon X \to g$ can be continuously extended to $\mathrm{Cl}(X) \setminus \{\partial^+(g), \partial^-(g)\}$.

REMARK. The original construction of Cheeger is more symmetric: he realizes an $s \in \mathrm{St}_2(X)$ by a triple of rays (r_1, r_2, r_3) all starting from the same point $x \in X$ with angles $120°$ between every two of them.

Applying the above construction to the universal covering X of a compact manifold V we get a homeomorphism between $\mathrm{St}_2(V)$ and the factor of $\partial^3(X)$ by the diagonal action of $\Gamma = \pi_1(V)$. This proves the Cheeger homeomorphism theorem.

GEODESIC RIGIDITY

Realize points from $S(X)$ by pairs (g, x), where g is an oriented geodesic and $x \in g$. When X and Y are the universal coverings of V and W, an isomorphism $I\colon \pi_1(V) \to \pi_1(W)$ induces a homeomorphism $D\colon \partial^2(X) \to \partial^2(Y)$. View D as a homeomorphism between the sets of oriented geodesics in X and Y. Take a smooth equivariant map $f_0\colon X \to Y$ (i.e. the lifting of a smooth homotopy equivalence $V \to W$ corresponding to I) and define a map $F_0\colon S(X) \to S(Y)$ as follows: $F_0(g, x) = (h, y)$, $h = D(g)$, $y = y(x) = P_h \circ f_0(x)$. (We use in Y the same representation of points from $S(Y)$ as in X and P_h means the normal projection $Y \to h$.) The map F_0 preserves the foliations $G(X)$ and $G(Y)$ but it is not necessarily a homeomorphism: it can identify points lying on the same geodesic. Choose natural parameters (length) in all geodesics and average F_0 along geodesics by the formula:

$$F_c(g, x) = \left(h, \frac{1}{c}\int_x^{x+c} y(t)\, dt\right).$$

where $h = D(g)$, $t, x \in g$, $y(t) \in h$, $y(t) = P_h \circ f_0(t)$. When c is large enough, the map F_c is a homeomorphism and it is obviously equivariant. Returning to V and W we get the geodesic homeomorphism $S(V) \to S(W)$.

Generalizations

The hyperbolic ideas of Morse were successfully applied to discrete type systems by Shub (*expanding endomorphisms*, see [Sh]) and Franks (π_1*-diffeomorphisms*, see [Fr]). Their results are discussed (and slightly generalized) in Appendix 3.

From a global geometric point of view generalizations of totally hyperbolic systems must include manifolds of nonpositive curvature and correspondingly semihyperbolic systems. (See Appendix 4.)

In differential dynamics most attention has always been paid to "local" versions of hyperbolicity (stability, Anosov's systems, Axiom A diffeomorphisms of Smale). We do not touch here upon that more analytical line of development of Morse's ideas.

§2. Entropy

Take a closed Riemannian manifold V, consider its universal covering X and denote by $\mathrm{Vol}_x(R)$, $x \in X$, the volume of the ball of radius R centered at x. Set $H(V) = \lim_{R\to\infty} \log \mathrm{Vol}_x(R)$. The limit obviously exists and does not depend on x. Denote by $h(V)$ the topological entropy of the geodesic flow in $S(V)$.

Entropy estimate. *We have $h(V) \geq H(V)$.*

Corollary. *If the fundamental group $\pi_1(V)$ can be presented by k generators and one relation and* $\mathrm{Diam}(V) \leq 1$ (Diam means the *diameter* of V), *then $h(V) \geq \log(k-1)$.*

The entropy estimate immediately follows from the Covering lemma.