Proof of the lemma. Write $\alpha = \gamma + \delta t^N$, where δ is constant and γ of degree less than N. Assume that N is at least 2. Since δ is ϵ-hermitian and 2 is invertible in A we can write $\delta = \sigma + \epsilon\sigma^*$. Then

$$\begin{pmatrix} 1 & t & -\sigma^* t^{N-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma + \sigma t^N + \epsilon\sigma^* t^N & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \epsilon & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ -\sigma t^{N-1} & 0 & 1 \end{pmatrix}$$

is of degree $\leq N-1$ and after $N-1$ such transformations we get a linear matrix. □

Writing $\alpha = \alpha_0 + t\alpha_1$ as $\alpha_0(1+\nu t)$ we see immediately that, α being invertible, ν is nilpotent. The formal power series

$$\tau = (1+\nu t)^{-1/2} = \sum \binom{-1/2}{k} (\nu t)^k$$

is a polynomial. From $\alpha = \epsilon\alpha^*$ we get $\alpha_0^* = \epsilon\alpha_0$ and $\nu^*\alpha_0^* = \epsilon\alpha_0\nu$. This implies that $\tau^*\alpha_0^* = \epsilon\alpha_0\tau$ and therefore

$$\tau^*\alpha\tau = \tau^*\alpha_0(1+\nu t)\tau = \alpha_0\tau(1+\nu t)\tau = \alpha_0 .$$

This proves that (P, α) is Witt equivalent to $(P(0), \alpha(0))$ and is, therefore, hyperbolic. □

4. The Witt group of torsion modules

Let M be a finitely generated right $A[t]$-module and suppose that it is a t-torsion module and that it is projective as an A-module. Obviously, it will be finitely generated over A. We denote by $M^\sharp$ the left $A[t]$-module $\mathrm{Hom}_{A[t]}(M, A[t,t^{-1}]/A[t])$ and we consider it as a right module through the involution on $A[t]$.

Recall that, as an A-module, the quotient $A[t,t^{-1}]/A[t]$ can be written as a direct sum

$$A[t,t^{-1}]/A[t] = At^{-1} \oplus At^{-2} \oplus \cdots .$$

Thus, to any $f \in \mathrm{Hom}_{A[t]}(M, A[t,t^{-1}]/A[t])$ we can associate an A-linear map $f_{-1}\colon M \to A$, which is defined as the composite of f with the projection onto At^{-1}.

PROPOSITION 4.1. *The map*

$$\partial = \partial_M \colon M^\sharp = \mathrm{Hom}_{A[t]}(M, A[t,t^{-1}]/A[t]) \longrightarrow \mathrm{Hom}_A(M,A) = M^*$$

obtained by associating f_{-1} to f is a functorial A-linear isomorphism.

Proof. It is clear that ∂ is A-linear. To show that it is bijective we construct its inverse. Given any $g \in M^*$ define $\widetilde{g}$ by the (finite !) sum

$$\widetilde{g}(x) = t^{-1}g(x) + t^{-2}g(tx) + t^{-3}g(t^2x) + \cdots .$$

It is easy to check that $\widetilde{g} \in M^\sharp$, $(\widetilde{g})_{-1} = g$ and $\widetilde{f_{-1}} = f$. Functoriality is clear. $\square$

COROLLARY 4.2. *For any finitely generated t-torsion module M which is projective as an A-module the canonical homomorphism $M \longrightarrow M^{\sharp\sharp}$ is an isomorphism.*

Proof. It suffices to remark that the diagram

$$\begin{tikzcd} & M \arrow[dl, "can"'] \arrow[dr, "can"] & \\ M^{\sharp\sharp} \arrow[rr, "(\partial_M^*)^{-1}\circ\partial_{M^\sharp}"'] & & M^{**} \end{tikzcd}$$

commutes and that $M \xrightarrow{can} M^{**}$ is an isomorphism. $\square$

An *ϵ-hermitian t-torsion space* (or, briefly, a *t-torsion space*) is a pair $(M, \langle,\rangle)$ consisting of a finitely generated t-torsion right $A[t]$-module M which is projective as an A-module, and a perfect ϵ-hermitian pairing $\langle\ ,\ \rangle\colon M \times M \to A[t,t^{-1}]/A[t]$. Giving $\langle,\rangle$ is the same, of course, as giving its adjoint $\varphi\colon M \to M^\sharp$ defined by $\varphi(a)(b) = \langle a, b\rangle$.

Isometries and orthogonal sums are defined in the obvious way. For any subset $X \subset M$ we define its orthogonal as

$$X^\perp = \{y \in M \mid \langle x,y\rangle = 0 \quad \forall x \in X\}.$$

A *sublagrangian* of (M, φ) is an $A[t]$-submodule L of M which satisfies the following two conditions:

(1) It is contained in its own orthogonal: $L \subseteq L^\perp$.

(2) The quotient M/L is projective over A (which is the same as saying that L, as an A-module, is a direct factor of M).

A sublagrangian L is a *lagrangian* if $L = L^{\perp}$. A t-torsion space is *metabolic* if it has a lagrangian. The Witt group of t-torsion spaces is the quotient of the Grothendieck group of t-torsion spaces with respect to orthogonal sums, modulo the subgroup generated by the metabolic spaces. We will denote it by $W_{tors}(A[t])$. Lemma 4.6 below will show that the opposite of the class of (M, φ) is the class of $(M, -\varphi)$.

LEMMA 4.3. *Let M and N be finitely generated t-torsion modules and $i\colon N \to M$ an $A[t]$-linear homomorphism. Assume that as A-modules M and N are projective. Then the map $i^{\sharp}\colon M^{\sharp} \to N^{\sharp}$ is surjective (respectively injective) if and only if $i^*\colon M^* \to N^*$ is surjective (respectively injective).*

Proof. Look:

$$\begin{array}{ccc} M^{\sharp} & \xrightarrow{i^{\sharp}} & N^{\sharp} \\ {\scriptstyle \partial_M}\big\downarrow & & \big\downarrow{\scriptstyle \partial_N} \\ M^* & \xrightarrow{i^*} & N^* \end{array}$$

□

PROPOSITION 4.4. *Let (M, φ) be a t-torsion space and L an $A[t]$-submodule of M. If M/L is projective over A, then $L = L^{\perp\perp}$ and $L^{\perp}$ is a direct factor of M as an A-module.*

Proof. First observe that as an A-module L is finitely generated and projective. Let $i\colon L \to M$ be the natural injection. By Lemma 4.3 the map $i^{\sharp} \circ \varphi$ is surjective, thus the sequence

$$0 \longrightarrow L^{\perp} \xrightarrow{j} M \xrightarrow{i^{\sharp}\circ\varphi} L^{\sharp} \longrightarrow 0$$

is exact. Hence $L^{\perp}$ is a direct factor of M as an A-module; in particular it is A-projective. Identifying L with $L^{\sharp\sharp}$ we can write the dual sequence as

$$0 \longrightarrow L \xrightarrow{i} M \xrightarrow{j^{\sharp}\circ\varphi^{\sharp}} (L^{\perp})^{\sharp} \longrightarrow 0.$$

Notice that it is exact by Lemma 4.3. Again by Lemma 4.3 the sequence

$$0 \longrightarrow L^{\perp\perp} \longrightarrow M \xrightarrow{j^{\sharp}\circ\varphi} (L^{\perp})^{\sharp} \longrightarrow 0$$

is exact because $L^{\perp}$ is a direct factor of M as an A-module. Since $\varphi^{\sharp} = \pm\varphi$, comparing the last two sequences we get the result. □

We now prove a fundamental result on the equivalence of t-torsion spaces.

THEOREM 4.5. *Let* (M,φ) *be an* ϵ*-hermitian* t*-torsion space and* L *a sublagrangian of* (M,φ)*. The quotient* $L^{\perp}/L$ *carries a natural structure of* t*-torsion* ϵ*-hermitian space and its class in* $W_{tors}(A[t])$ *is the same as that of* (M,φ).

Proof. We first prove the following lemma.

LEMMA 4.6. *Let* (M,φ) *be any* ϵ*-hermitian* t*-torsion space. The space* $(M,\varphi)\perp(M,-\varphi)$ *is metabolic.*

Proof of Lemma 4.6. We show that the image $L=\Delta(M)$ of the diagonal map $M\xrightarrow{\Delta}M\oplus M$ is a lagrangian. The condition $L\subseteq L^{\perp}$ is immediately verified. The quotient $(M\oplus M)/L$ is isomorphic to M, hence it is projective over A. It remains to see that $L^{\perp}\subseteq L$. If $(a,b)\in L^{\perp}$ we have $0=\langle(a,b),(x,x)\rangle=\langle a-b,x\rangle$ for any $x\in M$. Since the pairing $\langle,\rangle$ is perfect, this implies $a=b$, i.e. $(a,b)\in L$. □

We now prove the theorem. By Proposition 4.4, $L^{\perp}$ is a direct factor of M as an A-module. Since $L\subseteq L^{\perp}$ is also a direct factor of M, the quotient $L^{\perp}/L$ is projective. Denoting by $\bar{a},\bar{b}$ the classes modulo L of two elements $a,b\in L$, we define the hermitian structure of $L^{\perp}/L$ by $\langle\bar{a},\bar{b}\rangle=\langle a,b\rangle$. It is clear that $\langle a,b\rangle$ only depends on $\bar{a}$ and $\bar{b}$. We first check that this pairing defines a t-torsion space. It is clearly ϵ-hermitian. The injectivity of the adjoint map $L^{\perp}/L\to(L^{\perp}/L)^{\sharp}$ follows immediately from Proposition 4.4. To show surjectivity consider any $A[t]$-linear map $f\colon L^{\perp}\to A[t,t^{-1}]/A[t]$. Since $L^{\perp}$ is a direct factor of M as an A-module, f, by Lemma 4.3, extends to an $A[t]$-linear map $\tilde{f}\colon M\to A[t,t^{-1}]/A[t]$. Choose an $m\in M$ for which $\tilde{f}=\langle m,\cdot\rangle$. If $\tilde{f}$ vanishes on L, then m is in $L^{\perp}$. This proves that $L^{\perp}/L$ is a t-torsion space.

To show that $L^{\perp}/L$ is equivalent to (M,φ) we check that the image of the diagonal map $\Delta\colon L^{\perp}\to M\oplus L^{\perp}/L$ is a lagrangian of $(M,-\varphi)\perp L^{\perp}/L$ which is, therefore, metabolic. It is easy to check that $\Delta(L^{\perp})$ is contained in its own orthogonal. Conversely, if $(a,\bar{b})\in M\oplus L^{\perp}/L$ is orthogonal to every $(x,\bar{x})$, then $\langle a-b,x\rangle=0$ for every $x\in L^{\perp}$. This means that $a-b$ is in $L^{\perp\perp}$, which by Proposition 4.4 coincides with L. We thus have $(a,\bar{b})=(a,\bar{a})\in\Delta(L^{\perp})$. □

The next proposition connects the Witt group of t-torsion spaces with the Witt group of A.

PROPOSITION 4.7. *The isomorphisms*

$$\partial_M \colon \mathrm{Hom}_{A[t]}(M, A[t,t^{-1}]/A[t]) \to \mathrm{Hom}_A(M,A)$$

induce a surjective homomorphism

$$\partial^W \colon W_{tors}(A[t]) \to W(A)\,.$$

Proof. Associating to any t-torsion space (M,φ) the hermitian space $(M, \partial_M \circ \varphi)$ preserves isometries and orthogonal sums and, by Lemma 4.3, transforms metabolic t-torsion spaces into hyperbolic spaces (with the same lagrangian). Therefore it induces a homomorphism

$$\partial^W \colon W_{tors}(A[t]) \to W(A)\,.$$

To find a preimage (M,φ) of a space (M,α) over A consider M as an $A[t]$-module annihilated by t and replace $\alpha \colon M \to M^*$ by $\varphi = \partial_M^{-1} \circ \alpha$. □

5. THE WITT GROUP OF EXTENDED SPACES

Let $W'(A[t,t^{-1}])$ be the group defined in the introduction.

THEOREM 5.1. *Let A be an associative ring with involution, in which 2 is invertible. The homomorphism*

$$\psi \colon W(A) \oplus W(A) \to W'(A[t,t^{-1}])$$

mapping (ξ,η) to $\xi + t\eta$ is an isomorphism.

Proof. The injectivity of ψ is based on the following result, whose proof will be given in §6.

PROPOSITION 5.2. *There exists a homomorphism*

$$Res \colon W'(A[t,t^{-1}]) \to W(A)$$

with the following properties:

R_1 : *For any constant space $\xi \in W(A) \subset W'(A[t,t^{-1}])$, $Res(\xi) = 0$.*

R_2 : *For any constant space $\xi \in W(A) \subset W'(A[t,t^{-1}])$, $Res(t \cdot \xi) = \xi$.*

Proof. See Theorem 6.7. □