

2.K-theoretic PRELIMINARIES

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **46 (2000)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **20.09.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern. Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden. Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

We now recall three elementary, well-known facts about hermitian spaces.

PROPOSITION 1.5. *Let (P, α) be any space. Then:*

1. *The space $(P, \alpha) \perp (P, -\alpha)$ is hyperbolic.*
2. *If L is a lagrangian of (P, α) , then (P, α) is isometric to $H(L)$.*
3. *If M is a sublagrangian of (P, α) , then the map α induces on M^\perp/M a natural structure of hermitian space that makes it Witt equivalent to (P, α) .*

2. K -THEORETIC PRELIMINARIES

We recall a few results proved in the twelfth chapter of Bass' book [1]. For any ring A we denote by $K_0(A)$ the Grothendieck group of finitely generated projective right A -modules and by $K_1(A)$ the abelianized general linear group of A : $K_1(A) = GL(A)/[GL(A), GL(A)]$. By Whitehead's lemma $K_1(A)$ is also the quotient of $GL(A)$ by the subgroup $E(A)$ generated by all elementary matrices over A .

For any functor F from rings to abelian groups we denote by $N_+F(A)$ the kernel of the map $F(A[t]) \rightarrow F(A)$ obtained by putting $t = 0$. Similarly, we denote by $N_-F(A)$ the kernel of $F(A[t^{-1}]) \rightarrow F(A)$ obtained by putting $t^{-1} = 0$. The inclusions of $A[t]$ and $A[t^{-1}]$ into $A[t, t^{-1}]$ define a map

$$N_+F(A) \oplus N_-F(A) \longrightarrow F(A[t, t^{-1}])$$

whose cokernel will be denoted by $LF(A)$. The functor LK_1 turns out to be naturally isomorphic to K_0 , hence we will denote LK_i by K_{i-1} for $i = 1$ and also for $i = 0$.

THEOREM 2.1. *Let A be any associative ring.*

(a) *For $i = 0$ or 1 there exists a natural embedding*

$$\lambda_i: K_{i-1}(A) \longrightarrow K_i(A[t, t^{-1}])$$

such that the composite

$$K_{i-1}(A) \xrightarrow{\lambda_i} K_i(A[t, t^{-1}]) \rightarrow LK_i(A) = K_{i-1}(A)$$

is the identity.

(b) *The embedding λ_i and the canonical homomorphism*

$$N_{\pm}K_i(A) \rightarrow K_i(A[t, t^{-1}])$$

yield canonical decompositions

$$K_1(A[t, t^{-1}]) = K_1(A) \oplus N_+K_1(A) \oplus N_-K_1(A) \oplus K_0(A)$$

and

$$K_0(A[t, t^{-1}]) = K_0(A) \oplus N_+K_0(A) \oplus N_-K_0(A) \oplus K_{-1}(A).$$

Proof. See [1], Theorem 7.4 of chapter XII. \square

We will also use the following well-known result.

PROPOSITION 2.2. *If 2 is invertible in A, the groups $N_{\pm}K_1(A)$ are uniquely divisible by 2.*

Proof. By [1], XII, 5.3, every element of $N_+K_1(A)$ can be represented by a matrix $\alpha = 1 + \nu t$, with ν a nilpotent matrix of $M_n(A)$. Let

$$P(X) = \sum_0^{\infty} \binom{1/2}{n} X^n \in \mathbf{Z}[1/2][X].$$

Then $P(\nu t) \in M_n(A[t])$ and $(P(\nu t))^2 = 1 + \nu t$. This shows that $N_+K_1(A)$ is divisible by 2. To show uniqueness it suffices to show that $N_+K_1(A)$ has no 2-torsion. Take $\alpha = 1 + \nu t$ as before and suppose that $\alpha^2 \in E(A[t])$. Put $s = t(2 + \nu t)$, so that $\alpha^2 = 1 + \nu s$. Since

$$t = \sum_1^{\infty} \binom{1/2}{n} \nu^{n-1} s^n$$

we have $M_n(A)[t] = M_n(A)[s]$. If $\alpha^2 = 1 + \nu s \in E(A[s]) = E(M_n(A)[s])$ we clearly also have $\alpha = 1 + \nu t \in E(M_n(A)[t])$. \square

COROLLARY 2.3. *If 2 is invertible in A, the groups $N_{\pm}K_0(A)$ are uniquely divisible by 2.*

Proof. $K_0(A)$ is a direct factor of $K_1(A[X, X^{-1}])$, hence $N_{\pm}K_0(A)$ is a direct factor of $N_{\pm}K_1(A[X, X^{-1}])$. \square

Assume now that A has an involution. Associating to any projective module its dual and to any matrix its conjugate transpose yields actions of $\mathbf{Z}/2$ on K_0 and K_1 which are compatible with the decompositions of Theorem 2.1. From Corollary 2.3 we immediately deduce

COROLLARY 2.4. *Suppose that A is a ring with involution, in which 2 is invertible. Then*

$$H^2(\mathbf{Z}/2, K_0(A[t, t^{-1}])/K_0(A)) = H^2(\mathbf{Z}/2, K_{-1}(A)).$$

3. THE WITT GROUP OF POLYNOMIAL RINGS

THEOREM 3.1. *Let A be an associative ring with involution, in which 2 is invertible. Let ϵ be 1 or -1 and let W be the Witt group functor of ϵ -hermitian spaces. The natural homomorphism*

$$W(A) \longrightarrow W(A[t])$$

is an isomorphism.

Proof. It suffices to show that the homomorphism $W(A[t]) \rightarrow W(A)$ given by the evaluation at $t = 0$ is an isomorphism. Surjectivity is obvious. To prove injectivity let (P, α) be a space over $A[t]$ and $(P(0), \alpha(0))$ its reduction modulo t . Suppose that $(P(0), \alpha(0))$ is isometric to some hyperbolic space $H(Q)$. Choosing a projective module Q' such that $Q \oplus Q'$ is free and adding to (P, α) the space $H(Q'[t])$ we may assume that $P(0)$ is the hyperbolic space over a free module. The class of P in $K_0(A[t])/K_0(A) = N_+(A)$ is a symmetric element. By Corollary 2.4 it can be written as $a + a^*$, hence, adding to (P, α) a suitable free hyperbolic space, we may assume that (P, α) is of the form

$$H(A^n[t]) \perp (R \oplus R^*, \beta).$$

Let R' be an $A[t]$ -module such that $R \oplus R'$ is free. Adding to (P, α) the hyperbolic space $H(R')$ we are reduced to the case in which P is free and α is an invertible ϵ -hermitian matrix with entries in $A[t]$.

LEMMA 3.2. *Let $\alpha = \epsilon\alpha^* \in M_n(A[t])$ be any ϵ -hermitian matrix. There exist an integer m and a matrix $\tau \in \text{GL}_{n+2m}(A[t])$ (actually in $E_{n+2m}(A[t])$) such that*

$$\tau^* \begin{pmatrix} \alpha & 0 \\ 0 & \chi \end{pmatrix} \tau = \alpha_0 + t\alpha_1,$$

where α_0 and α_1 are constant matrices and χ is a sum of hyperbolic blocks $\begin{pmatrix} 0 & 1 \\ \epsilon 1 & 0 \end{pmatrix}$ of various sizes.