2.K-theoretic PRELIMINARIES

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We now recall three elementary, well-known facts about hermitian spaces.

PROPOSITION 1.5. *Let* (P,α) *be any space. Then:*

1. *The space* $(P,\alpha) \perp (P,-\alpha)$ *is hyperbolic.*
2. *If* L *is a lagrangian of* (P,α)*, then* (P,α) *is isometric to* $H(L)$*.*
3. *If* M *is a sublagrangian of* (P,α)*, then the map* α *induces on* $M^{\perp}/M$ *a natural structure of hermitian space that makes it Witt equivalent to* (P,α)*.*

2. K-THEORETIC PRELIMINARIES

We recall a few results proved in the twelfth chapter of Bass' book [1]. For any ring A we denote by $K_0(A)$ the Grothendieck group of finitely generated projective right A-modules and by $K_1(A)$ the abelianized general linear group of A: $K_1(A) = GL(A)/[GL(A), GL(A)]$. By Whitehead's lemma $K_1(A)$ is also the quotient of $GL(A)$ by the subgroup $E(A)$ generated by all elementary matrices over A.

For any functor F from rings to abelian groups we denote by $N_+F(A)$ the kernel of the map $F(A[t]) \to F(A)$ obtained by putting $t=0$. Similarly, we denote by $N_-F(A)$ the kernel of $F(A[t^{-1}]) \to F(A)$ obtained by putting $t^{-1}=0$. The inclusions of $A[t]$ and $A[t^{-1}]$ into $A[t,t^{-1}]$ define a map

$$N_+F(A) \oplus N_-F(A) \longrightarrow F(A[t,t^{-1}])$$

whose cokernel will be denoted by $LF(A)$. The functor LK_1 turns out to be naturally isomorphic to K_0, hence we will denote LK_i by K_{i-1} for $i=1$ and also for $i=0$.

THEOREM 2.1. *Let* A *be any associative ring.*
(a) *For* $i=0$ *or* 1 *there exists a natural embedding*

$$\lambda_i \colon K_{i-1}(A) \longrightarrow K_i(A[t,t^{-1}])$$

such that the composite

$$K_{i-1}(A) \xrightarrow{\lambda_i} K_i(A[t,t^{-1}]) \to LK_i(A) = K_{i-1}(A)$$

is the identity.

(b) *The embedding* λ_i *and the canonical homomorphism*

$$N_{\pm}K_i(A) \to K_i(A[t,t^{-1}])$$

yield canonical decompositions

$$K_1(A[t,t^{-1}]) = K_1(A) \oplus N_+K_1(A) \oplus N_-K_1(A) \oplus K_0(A)$$

and

$$K_0(A[t,t^{-1}]) = K_0(A) \oplus N_+K_0(A) \oplus N_-K_0(A) \oplus K_{-1}(A).$$

Proof. See [1], Theorem 7.4 of chapter XII. □

We will also use the following well-known result.

PROPOSITION 2.2. *If* 2 *is invertible in* A, *the groups* $N_{\pm}K_1(A)$ *are uniquely divisible by* 2.

Proof. By [1], XII, 5.3, every element of $N_+K_1(A)$ can be represented by a matrix $\alpha = 1 + \nu t$, with ν a nilpotent matrix of $M_n(A)$. Let

$$P(X) = \sum_0^{\infty} \binom{1/2}{n} X^n \in \mathbf{Z}[1/2][X].$$

Then $P(\nu t) \in M_n(A[t])$ and $(P(\nu t))^2 = 1 + \nu t$. This shows that $N_+K_1(A)$ is divisible by 2. To show uniqueness it suffices to show that $N_+K_1(A)$ has no 2-torsion. Take $\alpha = 1 + \nu t$ as before and suppose that $\alpha^2 \in E(A[t])$. Put $s = t(2 + \nu t)$, so that $\alpha^2 = 1 + \nu s$. Since

$$t = \sum_1^{\infty} \binom{1/2}{n} \nu^{n-1} s^n$$

we have $M_n(A)[t] = M_n(A)[s]$. If $\alpha^2 = 1 + \nu s \in E(A[s]) = E(M_n(A)[s])$ we clearly also have $\alpha = 1 + \nu t \in E(M_n(A)[t])$. □

COROLLARY 2.3. *If* 2 *is invertible in* A, *the groups* $N_{\pm}K_0(A)$ *are uniquely divisible by* 2.

Proof. $K_0(A)$ is a direct factor of $K_1(A[X,X^{-1}])$, hence $N_{\pm}K_0(A)$ is a direct factor of $N_{\pm}K_1(A[X,X^{-1}])$. □

Assume now that A has an involution. Associating to any projective module its dual and to any matrix its conjugate transpose yields actions of $\mathbf{Z}/2$ on K_0 and K_1 which are compatible with the decompositions of Theorem 2.1. From Corollary 2.3 we immediately deduce

COROLLARY 2.4. *Suppose that A is a ring with involution, in which 2 is invertible. Then*

$$H^2(\mathbf{Z}/2, K_0(A[t,t^{-1}])/K_0(A)) = H^2(\mathbf{Z}/2, K_{-1}(A)).$$

3. THE WITT GROUP OF POLYNOMIAL RINGS

THEOREM 3.1. *Let A be an associative ring with involution, in which 2 is invertible. Let ϵ be 1 or -1 and let W be the Witt group functor of ϵ-hermitian spaces. The natural homomorphism*

$$W(A) \longrightarrow W(A[t])$$

is an isomorphism.

Proof. It suffices to show that the homomorphism $W(A[t]) \to W(A)$ given by the evaluation at $t = 0$ is an isomorphism. Surjectivity is obvious. To prove injectivity let (P, α) be a space over $A[t]$ and $(P(0), \alpha(0))$ its reduction modulo t. Suppose that $(P(0), \alpha(0))$ is isometric to some hyperbolic space $H(Q)$. Choosing a projective module Q' such that $Q \oplus Q'$ is free and adding to (P, α) the space $H(Q'[t])$ we may assume that $P(0)$ is the hyperbolic space over a free module. The class of P in $K_0(A[t])/K_0(A) = N_+(A)$ is a symmetric element. By Corollary 2.4 it can be written as $a + a^*$, hence, adding to (P, α) a suitable free hyperbolic space, we may assume that (P, α) is of the form

$$H(A^n[t]) \perp (R \oplus R^*, \beta).$$

Let R' be an $A[t]$-module such that $R \oplus R'$ is free. Adding to (P, α) the hyperbolic space $H(R')$ we are reduced to the case in which P is free and α is an invertible ϵ-hermitian matrix with entries in $A[t]$.

LEMMA 3.2. *Let $\alpha = \epsilon\alpha^* \in \mathrm{M}_n(A[t])$ be any ϵ-hermitian matrix. There exist an integer m and a matrix $\tau \in \mathrm{GL}_{n+2m}(A[t])$ (actually in $\mathrm{E}_{n+2m}(A[t])$) such that*

$$\tau^* \begin{pmatrix} \alpha & 0 \\ 0 & \chi \end{pmatrix} \tau = \alpha_0 + t\alpha_1 ,$$

where α_0 and α_1 are constant matrices and χ is a sum of hyperbolic blocks $\begin{pmatrix} 0 & 1 \\ \epsilon 1 & 0 \end{pmatrix}$ of various sizes.