THE WITT GROUP OF LAURENT POLYNOMIALS

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ABSTRACT. We give a direct, self-contained proof of the fact that for a large class of rings A, in particular for all regular rings with involution, $W(A[t,1/t]) = W(A) \oplus W(A)$.

1. INTRODUCTION

The purpose of this note is to give a short direct proof of two fundamental theorems on the Witt group of polynomials and Laurent extensions of a ring A. These theorems were proved independently by M. Karoubi [3] and by A. Ranicki [5]. We will state them under the most general conditions on A and for their proofs we will use nothing more than a general result on the K-theory of Laurent polynomials. In the last section we will show, by two counterexamples, that the assumptions we make on A are necessary.

We begin by recalling briefly some definitions. We refer to [4] for a more detailed exposition and for the proofs of the few basic results that we will use.

Let A be an associative ring with an involution denoted by $a \mapsto a^\circ$. Except in §2 we will always assume that 2 is invertible in A. If M is a right A-module, we denote by M^* its dual $\mathrm{Hom}_A(M, A)$ endowed with the right action of A given by $fa(x) = a^\circ f(x)$ for any $f\colon M \to A$ and $a \in A$. If P is a finitely generated projective right A-module we identify it with P^{**} through the canonical isomorphism mapping $x \in P$ to $\widehat{x}\colon P^* \to A$ defined by $\widehat{x}(f) = f(x)$.

Let ϵ be 1 or -1. An *ϵ-hermitian space* over A is a pair (P, α) consisting of a finitely generated projective right A-module P and an A-isomorphism $\alpha\colon P \to P^*$ satisfying $\alpha = \epsilon\alpha^*$. For brevity ϵ-hermitian spaces will be called *spaces*. A 1-hermitian space (over a commutative ring A) is also called a *quadratic space*.

Two spaces (P,α) and (Q,β) are *isometric* if there exists an A-isomorphism $\varphi\colon P \to Q$ such that the square

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & Q \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ P^* & \xleftarrow[\varphi^*]{} & Q^* \end{array}$$

commutes. A space is *hyperbolic* if it is isometric to a space of the form

$$H(P) = \left(P \oplus P^*, \left(\begin{smallmatrix} 0 & 1 \\ \epsilon & 0 \end{smallmatrix}\right)\right) .$$

The *orthogonal sum* of two spaces (P,α) and (Q,β) is the space

$$(P,\alpha) \perp (Q,\beta) = (P \oplus Q, \alpha \oplus \beta) .$$

If (P,α) is a space and M a submodule of P we denote by $M^\perp$ the orthogonal of M, defined by the exact sequence

$$0 \longrightarrow M^\perp \longrightarrow P \xrightarrow{i^* \circ \alpha} M^*,$$

where i^* is the dual of the inclusion $i\colon M \to P$. A submodule M of P is *totally isotropic* if $M \subseteq M^\perp$. A *sublagrangian* of a space (P,α) is a totally isotropic direct factor of P. A *lagrangian* of (P,α) is a sublagrangian L such that $L = L^\perp$. For instance, P and P^* are lagrangians of $H(P)$.

The Witt group $W(A)$ of ϵ-hermitian spaces over A is the quotient of the Grothendieck group of ϵ-hermitian spaces with respect to orthogonal sums, by the subgroup generated by all hyperbolic spaces. We say that two spaces are *Witt equivalent* if they represent the same element of $W(A)$.

Consider now the rings $A[t]$ and $A[t,t^{-1}]$, endowed with the involution that fixes t and maps $a \in A$ to a°. For the ring $A[t,t^{-1}]$ we introduce a variant $W'(A[t,t^{-1}])$ of the Witt group. We first consider the Grothendieck group Q of ϵ-hermitian spaces over $A[t,t^{-1}]$ which are extended from A as $A[t,t^{-1}]$-modules, and its subgroup N generated by the hyperbolic spaces $H(P)$ where P is extended from A. We then define $W'(A[t,t^{-1}])$ as Q/N. Clearly $W'(A[t,t^{-1}])$ maps canonically to $W(A[t,t^{-1}])$. Here are our results.

A (THEOREM 3.1). *Let A be an associative ring with involution, in which 2 is invertible. The canonical homomorphism*

$$W(A) \to W(A[t])$$

is an isomorphism.

B (THEOREM 5.1). *Let A be an associative ring with involution, in which 2 is invertible. The homomorphism*

$$\psi\colon W(A) \oplus W(A) \to W'(A[t,t^{-1}])$$

mapping (ξ,η) to $\xi + t\eta$ is an isomorphism.

C (THEOREM 7.1). *Let A be an associative ring with involution, in which 2 is invertible. Let*

$$\varphi\colon W'(A[t,t^{-1}]) \to W(A[t,t^{-1}])$$

be the canonical homomorphism.

(a) *If $H^2(\mathbf{Z}/2, K_{-1}(A)) = 0$, then φ is surjective.*

(b) *If $K_0(A) = K_0(A[t]) = K_0(A[t,t^{-1}])$, then φ is an isomorphism.*

Two examples will be constructed in § 8 to show that the assumptions in (a) and in (b) cannot be omitted.

An amusing application of **B** is the following result:

D (PROPOSITION 6.8). *Let A be a commutative semilocal ring in which 2 is invertible. Let (P,α) be a quadratic space over A. If (P,α) is isometric to $(P, t\cdot\alpha)$ over $A[t,t^{-1}]$, then (P,α) is hyperbolic.*

We remark that in general, even for a commutative local ring, there is no residue map

$$Res\colon W(A[t,t^{-1}]) \to W(A)$$

satisfying the following two properties:

- For any constant space $\xi \in W(A) \subset W(A[t,t^{-1}])$, $Res(\xi) = 0$.
- For any constant space $\xi \in W(A) \subset W(A[t,t^{-1}])$, $Res(t\cdot\xi) = \xi$.

In fact, the existence of such a residue map immediately implies the injectivity of

$$\varphi \circ \psi\colon W(A) \oplus W(A) \to W(A[t,t^{-1}]),$$

which may fail, as in Example 8.1. However, there exists a residue map $Res\colon W'(A[t,t^{-1}]) \to W(A)$ (Proposition 5.2) which yields the injectivity of ψ.

We now recall three elementary, well-known facts about hermitian spaces.

PROPOSITION 1.5. *Let* (P, α) *be any space. Then:*

1. *The space* $(P, \alpha) \perp (P, -\alpha)$ *is hyperbolic.*
2. *If* L *is a lagrangian of* (P, α)*, then* (P, α) *is isometric to* $H(L)$*.*
3. *If* M *is a sublagrangian of* (P, α)*, then the map* α *induces on* $M^{\perp}/M$ *a natural structure of hermitian space that makes it Witt equivalent to* (P, α)*.*

2. K-THEORETIC PRELIMINARIES

We recall a few results proved in the twelfth chapter of Bass' book [1]. For any ring A we denote by $K_0(A)$ the Grothendieck group of finitely generated projective right A-modules and by $K_1(A)$ the abelianized general linear group of A : $K_1(A) = GL(A)/[GL(A), GL(A)]$. By Whitehead's lemma $K_1(A)$ is also the quotient of $GL(A)$ by the subgroup $E(A)$ generated by all elementary matrices over A.

For any functor F from rings to abelian groups we denote by $N_+F(A)$ the kernel of the map $F(A[t]) \to F(A)$ obtained by putting $t = 0$. Similarly, we denote by $N_-F(A)$ the kernel of $F(A[t^{-1}]) \to F(A)$ obtained by putting $t^{-1} = 0$. The inclusions of $A[t]$ and $A[t^{-1}]$ into $A[t, t^{-1}]$ define a map

$$N_+F(A) \oplus N_-F(A) \longrightarrow F(A[t, t^{-1}])$$

whose cokernel will be denoted by $LF(A)$. The functor LK_1 turns out to be naturally isomorphic to K_0, hence we will denote LK_i by K_{i-1} for $i = 1$ and also for $i = 0$.

THEOREM 2.1. *Let* A *be any associative ring.*
(a) *For* $i = 0$ *or* 1 *there exists a natural embedding*

$$\lambda_i \colon K_{i-1}(A) \longrightarrow K_i(A[t, t^{-1}])$$

such that the composite

$$K_{i-1}(A) \xrightarrow{\lambda_i} K_i(A[t, t^{-1}]) \to LK_i(A) = K_{i-1}(A)$$

is the identity.

(b) *The embedding* λ_i *and the canonical homomorphism*

$$N_{\pm}K_i(A) \to K_i(A[t,t^{-1}])$$

yield canonical decompositions

$$K_1(A[t,t^{-1}]) = K_1(A) \oplus N_+K_1(A) \oplus N_-K_1(A) \oplus K_0(A)$$

and

$$K_0(A[t,t^{-1}]) = K_0(A) \oplus N_+K_0(A) \oplus N_-K_0(A) \oplus K_{-1}(A).$$

Proof. See [1], Theorem 7.4 of chapter XII. □

We will also use the following well-known result.

PROPOSITION 2.2. *If* 2 *is invertible in* A, *the groups* $N_{\pm}K_1(A)$ *are uniquely divisible by* 2.

Proof. By [1], XII, 5.3, every element of $N_+K_1(A)$ can be represented by a matrix $\alpha = 1 + \nu t$, with ν a nilpotent matrix of $M_n(A)$. Let

$$P(X) = \sum_{0}^{\infty} \binom{1/2}{n} X^n \in \mathbf{Z}[1/2][X].$$

Then $P(\nu t) \in M_n(A[t])$ and $(P(\nu t))^2 = 1 + \nu t$. This shows that $N_+K_1(A)$ is divisible by 2. To show uniqueness it suffices to show that $N_+K_1(A)$ has no 2-torsion. Take $\alpha = 1 + \nu t$ as before and suppose that $\alpha^2 \in E(A[t])$. Put $s = t(2 + \nu t)$, so that $\alpha^2 = 1 + \nu s$. Since

$$t = \sum_{1}^{\infty} \binom{1/2}{n} \nu^{n-1} s^n$$

we have $M_n(A)[t] = M_n(A)[s]$. If $\alpha^2 = 1 + \nu s \in E(A[s]) = E(M_n(A)[s])$ we clearly also have $\alpha = 1 + \nu t \in E(M_n(A)[t])$. □

COROLLARY 2.3. *If* 2 *is invertible in* A, *the groups* $N_{\pm}K_0(A)$ *are uniquely divisible by* 2.

Proof. $K_0(A)$ is a direct factor of $K_1(A[X,X^{-1}])$, hence $N_{\pm}K_0(A)$ is a direct factor of $N_{\pm}K_1(A[X,X^{-1}])$. □

Assume now that A has an involution. Associating to any projective module its dual and to any matrix its conjugate transpose yields actions of $\mathbf{Z}/2$ on K_0 and K_1 which are compatible with the decompositions of Theorem 2.1. From Corollary 2.3 we immediately deduce

COROLLARY 2.4. *Suppose that A is a ring with involution, in which 2 is invertible. Then*

$$H^2(\mathbf{Z}/2, K_0(A[t,t^{-1}])/K_0(A)) = H^2(\mathbf{Z}/2, K_{-1}(A)).$$

3. THE WITT GROUP OF POLYNOMIAL RINGS

THEOREM 3.1. *Let A be an associative ring with involution, in which 2 is invertible. Let ϵ be 1 or -1 and let W be the Witt group functor of ϵ-hermitian spaces. The natural homomorphism*

$$W(A) \longrightarrow W(A[t])$$

is an isomorphism.

Proof. It suffices to show that the homomorphism $W(A[t]) \to W(A)$ given by the evaluation at $t = 0$ is an isomorphism. Surjectivity is obvious. To prove injectivity let (P, α) be a space over $A[t]$ and $(P(0), \alpha(0))$ its reduction modulo t. Suppose that $(P(0), \alpha(0))$ is isometric to some hyperbolic space $H(Q)$. Choosing a projective module Q' such that $Q \oplus Q'$ is free and adding to (P, α) the space $H(Q'[t])$ we may assume that $P(0)$ is the hyperbolic space over a free module. The class of P in $K_0(A[t])/K_0(A) = N_+(A)$ is a symmetric element. By Corollary 2.4 it can be written as $a + a^*$, hence, adding to (P, α) a suitable free hyperbolic space, we may assume that (P, α) is of the form

$$H(A^n[t]) \perp (R \oplus R^*, \beta).$$

Let R' be an $A[t]$-module such that $R \oplus R'$ is free. Adding to (P, α) the hyperbolic space $H(R')$ we are reduced to the case in which P is free and α is an invertible ϵ-hermitian matrix with entries in $A[t]$.

LEMMA 3.2. *Let $\alpha = \epsilon\alpha^* \in \mathrm{M}_n(A[t])$ be any ϵ-hermitian matrix. There exist an integer m and a matrix $\tau \in \mathrm{GL}_{n+2m}(A[t])$ (actually in $\mathrm{E}_{n+2m}(A[t])$) such that*

$$\tau^* \begin{pmatrix} \alpha & 0 \\ 0 & \chi \end{pmatrix} \tau = \alpha_0 + t\alpha_1,$$

where α_0 and α_1 are constant matrices and χ is a sum of hyperbolic blocks $\begin{pmatrix} 0 & 1 \\ \epsilon 1 & 0 \end{pmatrix}$ of various sizes.

Proof of the lemma. Write $\alpha = \gamma + \delta t^N$, where δ is constant and γ of degree less than N. Assume that N is at least 2. Since δ is ϵ-hermitian and 2 is invertible in A we can write $\delta = \sigma + \epsilon\sigma^*$. Then

$$\begin{pmatrix} 1 & t & -\sigma^* t^{N-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma + \sigma t^N + \epsilon\sigma^* t^N & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \epsilon & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ -\sigma t^{N-1} & 0 & 1 \end{pmatrix}$$

is of degree $\leq N-1$ and after $N-1$ such transformations we get a linear matrix. □

Writing $\alpha = \alpha_0 + t\alpha_1$ as $\alpha_0(1+\nu t)$ we see immediately that, α being invertible, ν is nilpotent. The formal power series

$$\tau = (1+\nu t)^{-1/2} = \sum \binom{-1/2}{k} (\nu t)^k$$

is a polynomial. From $\alpha = \epsilon\alpha^*$ we get $\alpha_0^* = \epsilon\alpha_0$ and $\nu^*\alpha_0^* = \epsilon\alpha_0\nu$. This implies that $\tau^*\alpha_0^* = \epsilon\alpha_0\tau$ and therefore

$$\tau^*\alpha\tau = \tau^*\alpha_0(1+\nu t)\tau = \alpha_0\tau(1+\nu t)\tau = \alpha_0 .$$

This proves that (P, α) is Witt equivalent to $(P(0), \alpha(0))$ and is, therefore, hyperbolic. □

4. THE WITT GROUP OF TORSION MODULES

Let M be a finitely generated right $A[t]$-module and suppose that it is a t-torsion module and that it is projective as an A-module. Obviously, it will be finitely generated over A. We denote by $M^\sharp$ the left $A[t]$-module $\mathrm{Hom}_{A[t]}(M, A[t,t^{-1}]/A[t])$ and we consider it as a right module through the involution on $A[t]$.

Recall that, as an A-module, the quotient $A[t,t^{-1}]/A[t]$ can be written as a direct sum

$$A[t,t^{-1}]/A[t] = At^{-1} \oplus At^{-2} \oplus \cdots .$$

Thus, to any $f \in \mathrm{Hom}_{A[t]}(M, A[t,t^{-1}]/A[t])$ we can associate an A-linear map $f_{-1}\colon M \to A$, which is defined as the composite of f with the projection onto At^{-1}.

PROPOSITION 4.1. *The map*

$$\partial = \partial_M : M^\sharp = \mathrm{Hom}_{A[t]}(M, A[t,t^{-1}]/A[t]) \longrightarrow \mathrm{Hom}_A(M,A) = M^*$$

obtained by associating f_{-1} to f is a functorial A-linear isomorphism.

Proof. It is clear that ∂ is A-linear. To show that it is bijective we construct its inverse. Given any $g \in M^*$ define $\widetilde{g}$ by the (finite !) sum

$$\widetilde{g}(x) = t^{-1}g(x) + t^{-2}g(tx) + t^{-3}g(t^2x) + \cdots .$$

It is easy to check that $\widetilde{g} \in M^\sharp$, $(\widetilde{g})_{-1} = g$ and $\widetilde{f_{-1}} = f$. Functoriality is clear. □

COROLLARY 4.2. *For any finitely generated t-torsion module M which is projective as an A-module the canonical homomorphism $M \longrightarrow M^{\sharp\sharp}$ is an isomorphism.*

Proof. It suffices to remark that the diagram

$$\begin{array}{ccc} & M & \\ {\scriptstyle can}\swarrow & & \searrow{\scriptstyle can} \\ M^{\sharp\sharp} & \xrightarrow[(\partial_M^*)^{-1}\circ\partial_{M^\sharp}]{} & M^{**} \end{array}$$

commutes and that $M \xrightarrow{can} M^{**}$ is an isomorphism. □

An *ϵ-hermitian t-torsion space* (or, briefly, a *t-torsion space*) is a pair $(M, \langle,\rangle)$ consisting of a finitely generated t-torsion right $A[t]$-module M which is projective as an A-module, and a perfect ϵ-hermitian pairing $\langle\ ,\ \rangle : M \times M \to A[t,t^{-1}]/A[t]$. Giving $\langle,\rangle$ is the same, of course, as giving its adjoint $\varphi : M \to M^\sharp$ defined by $\varphi(a)(b) = \langle a, b\rangle$.

Isometries and orthogonal sums are defined in the obvious way. For any subset $X \subset M$ we define its orthogonal as

$$X^\perp = \{y \in M \mid \langle x,y\rangle = 0 \quad \forall x \in X\}.$$

A *sublagrangian* of (M,φ) is an $A[t]$-submodule L of M which satisfies the following two conditions:

(1) It is contained in its own orthogonal: $L \subseteq L^\perp$.

(2) The quotient M/L is projective over A (which is the same as saying that L, as an A-module, is a direct factor of M).

A sublagrangian L is a *lagrangian* if $L = L^{\perp}$. A t-torsion space is *metabolic* if it has a lagrangian. The Witt group of t-torsion spaces is the quotient of the Grothendieck group of t-torsion spaces with respect to orthogonal sums, modulo the subgroup generated by the metabolic spaces. We will denote it by $W_{tors}(A[t])$. Lemma 4.6 below will show that the opposite of the class of (M, φ) is the class of $(M, -\varphi)$.

LEMMA 4.3. *Let M and N be finitely generated t-torsion modules and $i\colon N \to M$ an $A[t]$-linear homomorphism. Assume that as A-modules M and N are projective. Then the map $i^{\sharp}: M^{\sharp} \to N^{\sharp}$ is surjective (respectively injective) if and only if $i^*: M^* \to N^*$ is surjective (respectively injective).*

Proof. Look:

$$\begin{array}{ccc} M^{\sharp} & \xrightarrow{i^{\sharp}} & N^{\sharp} \\ {\scriptstyle \partial_M}\downarrow & & \downarrow{\scriptstyle \partial_N} \\ M^* & \xrightarrow{i^*} & N^* \end{array}$$

□

PROPOSITION 4.4. *Let (M, φ) be a t-torsion space and L an $A[t]$-submodule of M. If M/L is projective over A, then $L = L^{\perp\perp}$ and $L^{\perp}$ is a direct factor of M as an A-module.*

Proof. First observe that as an A-module L is finitely generated and projective. Let $i\colon L \to M$ be the natural injection. By Lemma 4.3 the map $i^{\sharp} \circ \varphi$ is surjective, thus the sequence

$$0 \longrightarrow L^{\perp} \xrightarrow{j} M \xrightarrow{i^{\sharp}\circ\varphi} L^{\sharp} \longrightarrow 0$$

is exact. Hence $L^{\perp}$ is a direct factor of M as an A-module; in particular it is A-projective. Identifying L with $L^{\sharp\sharp}$ we can write the dual sequence as

$$0 \longrightarrow L \xrightarrow{i} M \xrightarrow{j^{\sharp}\circ\varphi^{\sharp}} (L^{\perp})^{\sharp} \longrightarrow 0.$$

Notice that it is exact by Lemma 4.3. Again by Lemma 4.3 the sequence

$$0 \longrightarrow L^{\perp\perp} \longrightarrow M \xrightarrow{j^{\sharp}\circ\varphi} (L^{\perp})^{\sharp} \longrightarrow 0$$

is exact because $L^{\perp}$ is a direct factor of M as an A-module. Since $\varphi^{\sharp} = \pm\varphi$, comparing the last two sequences we get the result. □

We now prove a fundamental result on the equivalence of t-torsion spaces.

THEOREM 4.5. *Let (M,φ) be an ϵ-hermitian t-torsion space and L a sublagrangian of (M,φ). The quotient $L^{\perp}/L$ carries a natural structure of t-torsion ϵ-hermitian space and its class in $W_{tors}(A[t])$ is the same as that of (M,φ).*

Proof. We first prove the following lemma.

LEMMA 4.6. *Let (M,φ) be any ϵ-hermitian t-torsion space. The space $(M,\varphi)\perp(M,-\varphi)$ is metabolic.*

Proof of Lemma 4.6. We show that the image $L=\Delta(M)$ of the diagonal map $M\xrightarrow{\Delta}M\oplus M$ is a lagrangian. The condition $L\subseteq L^{\perp}$ is immediately verified. The quotient $(M\oplus M)/L$ is isomorphic to M, hence it is projective over A. It remains to see that $L^{\perp}\subseteq L$. If $(a,b)\in L^{\perp}$ we have $0=\langle(a,b),(x,x)\rangle=\langle a-b,x\rangle$ for any $x\in M$. Since the pairing $\langle,\rangle$ is perfect, this implies $a=b$, i.e. $(a,b)\in L$. □

We now prove the theorem. By Proposition 4.4, $L^{\perp}$ is a direct factor of M as an A-module. Since $L\subseteq L^{\perp}$ is also a direct factor of M, the quotient $L^{\perp}/L$ is projective. Denoting by $\overline{a},\overline{b}$ the classes modulo L of two elements $a,b\in L$, we define the hermitian structure of $L^{\perp}/L$ by $\langle\overline{a},\overline{b}\rangle=\langle a,b\rangle$. It is clear that $\langle a,b\rangle$ only depends on $\overline{a}$ and $\overline{b}$. We first check that this pairing defines a t-torsion space. It is clearly ϵ-hermitian. The injectivity of the adjoint map $L^{\perp}/L\to(L^{\perp}/L)^{\sharp}$ follows immediately from Proposition 4.4. To show surjectivity consider any $A[t]$-linear map $f\colon L^{\perp}\to A[t,t^{-1}]/A[t]$. Since $L^{\perp}$ is a direct factor of M as an A-module, f, by Lemma 4.3, extends to an $A[t]$-linear map $\widetilde{f}\colon M\to A[t,t^{-1}]/A[t]$. Choose an $m\in M$ for which $\widetilde{f}=\langle m,\cdot\rangle$. If $\widetilde{f}$ vanishes on L, then m is in $L^{\perp}$. This proves that $L^{\perp}/L$ is a t-torsion space.

To show that $L^{\perp}/L$ is equivalent to (M,φ) we check that the image of the diagonal map $\Delta\colon L^{\perp}\to M\oplus L^{\perp}/L$ is a lagrangian of $(M,-\varphi)\perp L^{\perp}/L$ which is, therefore, metabolic. It is easy to check that $\Delta(L^{\perp})$ is contained in its own orthogonal. Conversely, if $(a,\overline{b})\in M\oplus L^{\perp}/L$ is orthogonal to every $(x,\overline{x})$, then $\langle a-b,x\rangle=0$ for every $x\in L^{\perp}$. This means that $a-b$ is in $L^{\perp\perp}$, which by Proposition 4.4 coincides with L. We thus have $(a,\overline{b})=(a,\overline{a})\in\Delta(L^{\perp})$. □

The next proposition connects the Witt group of t-torsion spaces with the Witt group of A.

PROPOSITION 4.7. *The isomorphisms*

$$\partial_M\colon \operatorname{Hom}_{A[t]}(M, A[t,t^{-1}]/A[t]) \to \operatorname{Hom}_A(M,A)$$

induce a surjective homomorphism

$$\partial^W\colon W_{tors}(A[t]) \to W(A)\,.$$

Proof. Associating to any t-torsion space (M,φ) the hermitian space $(M, \partial_M \circ \varphi)$ preserves isometries and orthogonal sums and, by Lemma 4.3, transforms metabolic t-torsion spaces into hyperbolic spaces (with the same lagrangian). Therefore it induces a homomorphism

$$\partial^W\colon W_{tors}(A[t]) \to W(A)\,.$$

To find a preimage (M,φ) of a space (M,α) over A consider M as an $A[t]$-module annihilated by t and replace $\alpha\colon M \to M^*$ by $\varphi = \partial_M^{-1} \circ \alpha$. □

5. THE WITT GROUP OF EXTENDED SPACES

Let $W'(A[t,t^{-1}])$ be the group defined in the introduction.

THEOREM 5.1. *Let A be an associative ring with involution, in which 2 is invertible. The homomorphism*

$$\psi\colon W(A) \oplus W(A) \to W'(A[t,t^{-1}])$$

mapping (ξ,η) to $\xi + t\eta$ is an isomorphism.

Proof. The injectivity of ψ is based on the following result, whose proof will be given in §6.

PROPOSITION 5.2. *There exists a homomorphism*

$$Res\colon W'(A[t,t^{-1}]) \to W(A)$$

with the following properties:

R_1 : *For any constant space $\xi \in W(A) \subset W'(A[t,t^{-1}])$, $Res(\xi) = 0$.*

R_2 : *For any constant space $\xi \in W(A) \subset W'(A[t,t^{-1}])$, $Res(t\cdot\xi) = \xi$.*

Proof. See Theorem 6.7. □

Assuming this proposition, suppose that for two elements $\xi, \eta \in W(A)$ we have $\xi + t \cdot \eta = 0$. Then $0 = Res(\xi + t \cdot \eta) = \eta$ and hence $\xi = 0$.

We now turn to the surjectivity of ψ. We have to show that every hermitian space (P, α) over $A[t, t^{-1}]$ with $P = P_0[t, t^{-1}]$ is Witt equivalent to a space of the form $(Q_0[t, t^{-1}], \alpha_0) \perp (Q_1[t, t^{-1}], t\alpha_1)$. Let P_1 be a projective A-module such that $P_0 \oplus P_1 = A^n$ for some n. Replacing (P, α) by

$$(P_0[t, t^{-1}], \alpha) \perp (P_0[t, t^{-1}], -\alpha(1)) \perp H(P_1[t, t^{-1}]),$$

we may assume that P_0 is free. Replacing α by $t^{2N}\alpha$ with a suitable N, we may also assume that α maps $P_0[t]$ into $P_0^*[t]$. By Lemma 3.2 we are reduced to the case where $\alpha = \alpha_0 + t\alpha_1$ for some ϵ-hermitian maps $\alpha_0, \alpha_1 \colon P_0 \to P_0^*$.

LEMMA 5.3. *If, for a constant matrix β,*

$$\alpha = 1 + (t-1)\beta \in \mathrm{GL}_n(A[t, t^{-1}]) \cap \mathrm{M}_n(A[t]),$$

then there exists an N such that $(1-\beta)^N \beta^N = 0$.

Proof. This is Corollary 2.4 of [2]. For the convenience of the reader we reprove it here.

Writing the inverse of α as a Laurent polynomial and equating coefficients in the identity

$$1 = \alpha\alpha^{-1} = (1 - \beta + t\beta)(\gamma_{-q}t^{-q} + \cdots + \gamma_{-1}t^{-1} + \gamma_0 + \gamma_1 t + \cdots + \gamma_p t^p)$$

we get

$$(1-\beta)\gamma_{-q} = 0,\ (1-\beta)\gamma_{-q+1} + \beta\gamma_{-q} = 0,\ \ldots,$$
$$(1-\beta)\gamma_{-1} + \beta\gamma_{-2} = 0,\ (1-\beta)\gamma_0 + \beta\gamma_{-1} = 1$$

and

$$(1-\beta)\gamma_1 + \beta\gamma_0 = 0,\ \ldots,\ (1-\beta)\gamma_p + \beta\gamma_{p-1} = 0,\ \beta\gamma_p = 0.$$

From the first line we get $(1-\beta)^q\gamma_{-1} = 0$, from the third $\beta^{p+1}\gamma_0 = 0$ and then from the middle one $\beta^{p+1}(1-\beta)^q = 0$. $\square$

We put $\beta = \alpha(1)^{-1}\alpha_1 \colon P_0 \to P_0$, so that

$$\alpha(1)^{-1}\alpha = 1 + (t-1)\beta.$$

We will repeatedly use the fact that β is adjoint with respect to $\alpha, \alpha(1), \alpha_0, \alpha_1$, by which we mean that $\alpha\beta = \beta^*\alpha$, and so on. The same clearly holds for any polynomial in β with integral coefficients.

By Lemma 5.3 we can find an integer N such that $\beta^N(1-\beta)^N = 0$. Denoting by $\mathbf{Z}[\beta]$ the subring of $\mathrm{End}_A(P_0)$ generated by β we can write $\mathbf{Z}[\beta] = \mathbf{Z}[\beta]e \times \mathbf{Z}[\beta](1-e)$, where e is an idempotent of the form $\beta+\nu$ and ν is a nilpotent matrix. Note that e and ν are polynomials in β and therefore they commute with β and with each other. If we decompose P_0 as $eP_0 + (1-e)P_0$ and represent A-linear endomorphisms of P_0 as 2×2 block matrices, we have

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1+\nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}$$

and

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \epsilon\alpha_{12}^* & \alpha_{22} \end{pmatrix} (1+(t-1)\beta).$$

Computing the product we see that the condition $\alpha^* = \epsilon\alpha$ implies that

$$\alpha_{12}(1-\nu_2) = -\nu_1^*\alpha_{12}, \qquad \alpha_{11}^* = \epsilon\alpha_{11} \quad \text{and} \quad \alpha_{22}^* = \epsilon\alpha_{22}.$$

From this we immediately deduce

$$\alpha_{12}(1-\nu_2)^k = (-\nu_1^*)^k\alpha_{12}$$

for any natural integer k. Since ν_1 and ν_2 are nilpotent, this implies that $\alpha_{12} = 0$. Thus α is of the form

$$\begin{pmatrix} \alpha_{11}t(1+\nu_1) - \alpha_{11}\nu_1 & 0 \\ 0 & \alpha_{22}(1+(t-1)\nu_2) \end{pmatrix}$$

and $(P_0[t,t^{-1}],\alpha)$ splits as a hermitian space.

Since α, α_{11} and α_{22} are symmetric, evaluating the above matrix at $t=1$ we see that

$$\alpha_{11}\nu_1 = \nu_1^*\alpha_{11} \quad \text{and} \quad \alpha_{22}\nu_1 = \nu_2^*\alpha_{22}.$$

The first block can be written as

$$\sigma_1 = \alpha_{11}t(1+\nu_1 - t^{-1}\nu_1) = \alpha_{11}t(1+(1-t^{-1})\nu_1).$$

Since $(1-t^{-1})\nu_1$ is nilpotent, the formal power series

$$\tau_1 = (1+(1-t^{-1})\nu_1)^{-1/2} = \sum \binom{-1/2}{k}((1-t^{-1})\nu_1)^k$$

is a Laurent polynomial and we can replace the first block by $\tau_1^*\sigma_1\tau_1 = \alpha_{11}t$. Similarly, the power series

$$\tau_2 = (1+(t-1)\nu_2)^{-1/2} = \sum \binom{-1/2}{k}((t-1)\nu_2)^k$$

is a Laurent polynomial and we can replace the second block by $\tau_2^*\sigma_2\tau_2 = \alpha_{22}$.

This shows that

$$(P_0[t,t^{-1}],\alpha) \simeq (P_0e[t,t^{-1}], t\alpha_{11}) \perp (P_0(1-e)[t,t^{-1}], \alpha_{22}),$$

thus proving the surjectivity of ψ. $\square$

6. The residue

In this section we construct a residue map

$$Res\colon W'(A[t,t^{-1}]) \to W(A)$$

satisfying R_1 and R_2 of §5.

The definition of *Res* will be preceded by a few preliminaries.

LEMMA 6.1. *Let P_0 be a (finitely generated) projective A-module and define $M(\alpha)$ by the exact sequence*

$$0 \longrightarrow P_0[t] \xrightarrow{\alpha} P_0^*[t] \longrightarrow M(\alpha) \longrightarrow 0,$$

where α is $A[t]$-linear. Suppose that its localization $\alpha_t\colon P_0[t,t^{-1}] \to P_0[t,t^{-1}]$ is an isomorphism. Then, as an A-module, $M(\alpha)$ is finitely generated and projective.

Proof. Decompose $P_0[t,t^{-1}]$ as a direct sum $P_0[t] \oplus t^{-1}P_0[t^{-1}]$ of A-modules. Let π be the projection onto the first summand. Then $\beta = \pi \circ \alpha_t^{-1}|_{P_0^*[t]}$ is an A-linear splitting of α. Hence $M(\alpha)$ is A-projective. It is also finitely generated as an $A[t]$-module, hence, being annihilated by a power of t, it is finitely generated as an A-module. □

Let $M = M(\alpha)$ be as in the previous lemma. Assume that α is ϵ-symmetric. We define a pairing

$$M \times M \to A[t,t^{-1}]/A[t]$$

by $\langle \overline{a}, \overline{b} \rangle = a(\alpha_t^{-1}(b))$, where a and b are representatives in $P_0^*[t]$ of $\overline{a}, \overline{b} \in M$.

LEMMA 6.2. *If α is ϵ-hermitian, then $\langle , \rangle$ is a perfect ϵ-hermitian pairing.*

Proof. Since α_t is ϵ-hermitian, denoting by $x \mapsto x^\circ$ the involution on A we have

$$\langle \overline{a}, \overline{b} \rangle = a(\alpha_t^{-1}(b)) = \epsilon(b(\alpha_t^{-1}(a)))^\circ = \epsilon \langle \overline{b}, \overline{a} \rangle^\circ .$$

This proves the first assertion.

We now check that the adjoint of $\langle , \rangle$

$$\chi\colon M \to \mathrm{Hom}_{A[t]}(M, A[t,t^{-1}]/A[t]),$$

defined as $\chi(\overline{a})(\overline{b}) = \langle \overline{a}, \overline{b} \rangle$, is an isomorphism. We first prove injectivity. Suppose that, for some a and every x in M, $\chi(\overline{a})(\overline{x}) = 0$. This means that $a(\alpha_t^{-1}(x)) \in A[t]$ for every $x \in P_0^*[t]$. We only have to show that $\alpha_t^{-1}(a) \in P_0[t]$. Consider the diagram

$$\begin{array}{ccc} P_0[t] & \xrightarrow{\sim} & \mathrm{Hom}_{A[t]}(P_0^*[t], A[t]) \\ \downarrow & & \downarrow \\ P_0[t,t^{-1}] & \xrightarrow{\sim} & \mathrm{Hom}_{A[t]}(P_0^*[t], A[t,t^{-1}]) \end{array}$$

where the horizontal arrows are the canonical ones. Since $P_0[t]$ is projective (and finitely generated!) over $A[t]$, they both are isomorphisms. Therefore an element $b \in P_0[t,t^{-1}]$ is in $P_0[t]$ if and only if, for any $x \in P_0^*[t]$, $x(b)$ is in $A[t]$. This is indeed the case for $b = \alpha_t^{-1}(a)$ because $x(\alpha_t^{-1}(a)) = \epsilon(a(\alpha_t^{-1}(x)))^\circ \in A[t]$ by the very assumption on a. Thus injectivity is proved. We now check that χ is surjective. Let $\bar{f}\colon M \to A[t,t^{-1}]/A[t]$ be an $A[t]$-linear map. Since $P_0[t]^*$ is projective, there exits an f which makes the right hand square of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_0[t] & \xrightarrow{\alpha} & P_0[t]^* & \xrightarrow{p} & M & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow f & & \downarrow \bar{f} & & \\ 0 & \longrightarrow & A[t] & \longrightarrow & A[t,t^{-1}] & \xrightarrow{q} & A[t,t^{-1}]/A[t] & \longrightarrow & 0 \end{array}$$

commute, p and q being the canonical surjections. Clearly $q \circ f \circ \alpha = 0$, hence there exists an $A[t]$-linear map $a\colon P_0[t] \to A[t]$ such $f \circ \alpha = i \circ a$, i being the inclusion $A[t] \to A[t,t^{-1}]$. We claim that $\chi(a) = \bar{f}$. For this it suffices to show that for any $b \in P_0[t]^*$ we have $a(\alpha_t^{-1}(b)) \equiv f(b)$ modulo $A[t]$. We denote by a_t the localization of a at t and by $f_t\colon P_0[t,t^{-1}]^* \to A[t,t^{-1}]$ the unique $A[t,t^{-1}]$-linear extension of f. Observing that $\alpha_t^{-1}(a) = a_t \circ \alpha_t^{-1}$ we get the following relations:

$$a(\alpha_t^{-1}(b)) = (a_t \circ \alpha_t^{-1})(b) = f_t(b) = f(b).$$

This proves that χ is surjective. □

Let now $(P_0[t,t^{-1}], \alpha)$ be an ϵ-hermitian space. For any natural integer n for which $t^{2n}\alpha(P_0[t]) \subseteq P_0[t]^*$ we define $M(\alpha, n)$ by the exact sequence

$$0 \longrightarrow P_0[t] \xrightarrow{t^{2n}\alpha} P_0^*[t] \longrightarrow M(\alpha, n) \longrightarrow 0$$

and equip it with the ϵ-hermitian structure defined above:

$$\langle \bar{a}, \bar{b} \rangle = a((t^{2n}\alpha_t)^{-1}(b)).$$

LEMMA 6.3. *Let* $\psi\colon (P_0[t,t^{-1}],\alpha) \to (Q_0[t,t^{-1}],\beta)$ *be an isometry and assume that* $\psi(P_0[t]) \subseteq Q_0[t]$, $\alpha(P_0[t]) \subseteq P_0[t]^*$ *and* $\beta(Q_0[t]) \subseteq Q_0[t]^*$. *Then* $M(\alpha)$ *and* $M(\beta)$ *are Witt equivalent* t*-torsion spaces.*

Proof. Consider the diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \uparrow & & & & \\
 & & 0 & & K & & & & \\
 & & \downarrow & & \uparrow{\scriptstyle \hat q} & & & & \\
0 & \longrightarrow & P_0[t] & \xrightarrow{\alpha} & P_0[t]^* & \xrightarrow{q_\alpha} & M(\alpha) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\psi} & & \uparrow{\scriptstyle\psi^*} & & & & \\
0 & \longrightarrow & Q_0[t] & \xrightarrow{\beta} & Q_0[t]^* & \xrightarrow{q_\beta} & M(\beta) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle q} & & \uparrow & & & & \\
 & & L & & 0 & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}
$$

By Lemma 6.1 the module L, viewed as an A-module, is finitely generated and projective. The map ψ^* is obtained from the map ψ by dualizing over $A[t]$. We denote the cokernel of ψ^* by K and we denote the canonical map $P_0[t]^* \to K$ by $\hat{q}$. One may observe that K is isomorphic to $L^\sharp$ (see §4 for the notation) but we will not use this observation.

The $A[t]$-linear map $\theta = q_\alpha \circ \psi^*\colon Q_0[t]^* \to M(\alpha)$ induces a map $\overline{\theta}\colon M(\beta) \to \theta(Q_0[t]^*)/\theta(\beta(Q_0[t]))$. The statement will be deduced from the following claims.

(1) The map $\overline{\theta}$ is an $A[t]$-linear isomorphism.

(2) The map $\hat{q}$ induces an $A[t]$-linear isomorphism

$$\rho\colon M(\alpha)/\theta(Q_0[t]^*) \to K\,.$$

(3) $\theta(\beta(Q_0[t]))$ is a sublagrangian of $M(\alpha)$.

(4) $(\theta(\beta(Q_0[t]))^{\perp} = \theta(Q_0[t]^*)$.

(5) The map $\overline{\theta}$ is an isometry of t-torsion spaces.

In fact, by (4), (5) and Theorem 4.5, $M(\beta)$ is Witt equivalent to $M(\alpha)$.

We now prove the claims. The surjectivity of $\overline{\theta}$ is clear. To show injectivity, suppose that $x \in \ker(\theta)$. Choose a lift $\widetilde{x} \in Q_0[t]^*$ of x. There exist a $y \in Q_0[t]$ and a $z \in P_0[t]$ such that $\psi^*(\beta(y)-\widetilde{x}) = \alpha(z)$. Replacing α by $\psi^* \circ \beta \circ \psi$ we get $\psi^*(\widetilde{x}) = \psi^*(\beta(y-\psi(z)))$. Since ψ^* is injective, this shows that $\widetilde{x} \in \mathrm{Im}(\beta)$ and hence $x = 0$.

To prove (2) observe that, since $\hat{q} \circ \alpha = \hat{q} \circ \psi^* \circ \beta \circ \psi = 0$, $\hat{q}$ induces a surjective map $\rho\colon M(\alpha)/\theta(Q_0[t]^*) \to K$. Injectivity is also clear.

To prove (3) we first observe that $\theta(\beta(Q_0[t]))$ is a direct factor (as an A-module) of $M(\alpha)$. In fact, by (2), $\theta(Q_0[t]^*)$ is a direct factor (as an A-module) of $M(\alpha)$ and, by (1), $\theta(\beta(Q_0[t]))$ is a direct factor of $\theta(Q_0[t]^*)$. For any two elements $a, b \in P_0[t]^*$ let us denote by $\langle a, b\rangle_\alpha$ the element $a(\alpha_t^{-1}(b))$, and similarly for $\langle a, b\rangle_\beta$. We then have

$$\langle a, b\rangle_\beta = \langle \psi^*(a), \psi^*(b)\rangle_\alpha$$

because ψ_t is an isometry. Let now $\overline{a}, \overline{b} \in \theta(\beta(Q_0[t]))$ and $x, y \in Q_0[t]$ such that $a = \psi^*(\beta(x))$ and $b = \psi^*(\beta(y))$ are preimages of a and b. We have to check that $\langle \overline{a}, \overline{b}\rangle = 0$. This is the same as saying that $\langle a, b\rangle_\alpha$ is in $A[t]$. This is indeed the case because

$$\langle a, b\rangle_\alpha = \langle \psi^*(\beta(x)), \psi^*(\beta(y))\rangle_\alpha = \langle \beta(x), \beta(y)\rangle_\beta = \beta(x)(y) \in A[t]\,.$$

We now prove (4). For any $\overline{a} \in \theta(\beta(Q_0[t]))$ and any $\overline{b} \in M(\alpha)$ we choose preimages a and b of the form $a = \psi^*(\beta(x))$ and $b = \psi_t^*(y)$ with $x \in Q_0[t]$ and $y \in Q_0[t, t^{-1}]^*$. Then we have

$$\langle a, b\rangle_\alpha = \langle \psi^*(\beta(x)), \psi_t^*(y)\rangle_\alpha = \langle \beta(x), y\rangle_\beta = \epsilon \cdot y(x)^\circ\,,$$

which shows that, for any $y \in Q_0[t, t^{-1}]^*$, $\langle \psi^*(\beta(Q_0[t])), b\rangle_\alpha$ is in $A[t]$ if and only if $y \in Q_0[t]^*$, which is equivalent to $\overline{b} \in \theta(Q_0[t]^*)$.

We now prove (5). We already know that $\overline{\theta}$ is an $A[t]$-linear isomorphism. A computation like the one above proves that it is an isometry. □

COROLLARY 6.4. *Let $(P_0[t, t^{-1}], \alpha)$ be an ϵ-hermitian space. Let n be such that $t^{2n}\alpha(P_0[t]) \subseteq P_0[t]^*$. Then the class of $M(\alpha, n)$ in $W_{tors}(A[t])$ does not depend on the choice of n.*

COROLLARY 6.5. *Let $(P_0[t, t^{-1}], \alpha)$ and $(P_0[t, t^{-1}], \beta)$ be isometric spaces and assume that for some natural integers m and n, $t^{2m}\alpha(P_0[t]) \subseteq P_0[t]^*$ and $t^{2n}\beta(P_0[t]) \subseteq P_0[t]^*$. Then $M(\alpha, m)$ and $M(\beta, n)$ are Witt equivalent t-torsion spaces.*

Proof. Let $\psi\colon (P_0[t,t^{-1}], t^{2m}\alpha) \to (P_0[t,t^{-1}], t^{2n}\beta)$ be an isometry and let k be a natural integer such that $t^k\psi(P_0[t]) \subseteq P_0[t]^*$. Then $t^k\psi\colon (P_0[t,t^{-1}], t^{2m}\alpha) \to (P_0[t,t^{-1}], t^{2n+2k}\beta)$ is an isometry and, by Lemma 6.3, $M(\alpha, m)$ and $M(\beta, n+k)$ are Witt equivalent. Hence, by Corollary 6.4, $M(\alpha, m)$ and $M(\beta, n)$ are Witt equivalent as well. □

PROPOSITION 6.6. *Associating to any space $(P_0[t,t^{-1}], \alpha)$ the torsion space $M(\alpha, n)$ (for a suitable n) yields a homomorphism*

$$res\colon W'(A[t,t^{-1}]) \to W_{tors}(A[t]).$$

Proof. By Corollary 6.5, associating to the isometry class of a space $(P_0[t,t^{-1}], \alpha)$ the Witt class of the t-torsion space $M(\alpha, n)$ for some suitable n is a well defined map. It is obvious that the orthogonal sum of two spaces is mapped to the corresponding sum of t-torsion spaces, hence this map induces a homomorphism $\omega\colon K_H \to W_{tors}(A[t])$, where K_H is the Grothendieck group of ϵ-hermitian spaces of the form $(P_0[t,t^{-1}], \alpha)$. It is clear from the definition of $M(\alpha, n)$ that a standard hyperbolic space $H(Q_0[t,t^{-1}])$ is mapped to zero, hence ω induces a homomorphism $res\colon W'(A[t,t^{-1}]) \to W_{tors}(A[t])$. □

If we compose res with $\partial^W\colon W_{tors}(A[t]) \to W(A)$ we get a homomorphism

$$Res = \partial^W \circ res\colon W'(A[t,t^{-1}]) \to W(A)$$

which we call *residue.*

THEOREM 6.7. *The residue*

$$Res\colon W'(A[t,t^{-1}]) \to W(A)$$

satisfies the following two properties:

R_1 : *For any constant space $\xi \in W(A) \subset W(A[t,t^{-1}])$, $Res(\xi) = 0$.*

R_2 : *For any constant space $\xi \in W(A)$, $Res(t\cdot\xi) = \xi$.*

Proof. The two properties immediately follow from the construction of res. □

An amusing application of the existence of Res is the following result.

PROPOSITION 6.8. *Let A be a commutative semilocal ring in which 2 is invertible. Let (P, α) be a quadratic space over A. If (P, α) is isometric to $(P, t \cdot \alpha)$ over $A[t, t^{-1}]$, then (P, α) is hyperbolic.*

Proof. Let ξ be the class of (P, α) in $W(A)$. In $W'(A[t])$ we have $\xi = t \cdot \xi$. Applying *Res* to both sides we obtain $\xi = 0$. Since A is semilocal, by Witt's cancelletion theorem we conclude that (P, α) is hyperbolic. □

7. THE WITT GROUP OF LAURENT POLYNOMIALS

Let $W'(A[t, t^{-1}])$ be the group defined in the introduction.

THEOREM 7.1. *Let A be an associative ring with involution in which 2 is invertible. Let*

$$\varphi\colon W'(A[t, t^{-1}]) \to W(A[t, t^{-1}])$$

be the canonical homomorphism.

(a) *If $H^2(\mathbf{Z}/2, K_{-1}(A)) = 0$, then φ is surjective.*

(b) *If $K_0(A) = K_0(A[t]) = K_0(A[t, t^{-1}])$, then φ is an isomorphism.*

Proof of (a). Corollary 2.4 implies that

$$H^2(\mathbf{Z}/2, K_0(A[t, t^{-1}])/K_0(A)) = 0\,.$$

This means that every projective $A[t, t^{-1}]$-module P is in the same class as some projective module of the form

$$P_0[t, t^{-1}] \oplus Q \oplus Q^*\,,$$

where P_0 is a projective A-module. Therefore, adding to a space (P, α) a hyperbolic space $H(Q')$ with $Q \oplus Q'$ free, we may assume that P is of the form $P_0[t, t^{-1}]$. This means precisely that the class of (P, α) is in the image of $W'(A[t, t^{-1}])$. □

Proof of (b). Surjectivity is obvious, because by assumption every projective $A[t, t^{-1}]$-module is stably extended from A. Suppose that the class of a space $(P_0[t, t^{-1}], \alpha)$ vanishes in $W(A[t, t^{-1}])$. This means that, for some Q and R, there exists an isometry

$$(P_0[t, t^{-1}], \alpha) \perp H(Q) \simeq H(R)\,.$$

Adding to both sides a suitable $H(A[t, t^{-1}]^n)$ we may replace Q and R by extended modules $Q_0[t, t^{-1}]$ and $R_0[t, t^{-1}]$. Then the isometry means precisely that the class of $(P_0[t, t^{-1}], \alpha)$ vanishes in $W'(A[t, t^{-1}])$. □

We can restate assertion (b) of Theorem 7.1 as follows.

THEOREM 7.2. *Let A be an associative ring with involution, in which 2 is invertible. Assume that $K_0(A) = K_0(A[t]) = K_0(A[t,t^{-1}])$. Then there exists a natural homomorphism Res such that the sequence*

$$0 \longrightarrow W(A) \longrightarrow W(A[t,t^{-1}]) \xrightarrow{Res} W(A) \longrightarrow 0$$

is split exact. The homomorphism Res restricts to an isomorphism of $t \cdot W(A)$ onto $W(A)$.

8. TWO COUNTEREXAMPLES

In this section we show that the map $W'(A[t,t^{-1}]) \to W(A[t,t^{-1}])$, in general, is neither surjective nor injective.

EXAMPLE 8.1. We first recall the Mayer-Vietoris sequence associated to a cartesian square of commutative rings (see [1], Ch. IX, Corollary 5.12). Let

$$\begin{array}{ccc} R & \longrightarrow & S \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \bar{R} & \longrightarrow & \bar{S} \end{array}$$

be a cartesian diagram of commutative rings, with f or g surjective. Denote by $\widetilde{K_0}$ the kernel of the rank function on K_0. Then there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} K_1(\bar{R}) \times K_1(S) & \longrightarrow & K_1(\bar{S}) & \longrightarrow & \widetilde{K_0}(R) & \longrightarrow & \widetilde{K_0}(\bar{R}) \times \widetilde{K_0}(S) & \longrightarrow & \widetilde{K_0}(\bar{S}) \\ \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \wedge^{\max}} & & \downarrow{\scriptstyle \wedge^{\max}} & & \downarrow{\scriptstyle \wedge^{\max}} \\ \mathbf{G}_m(\bar{R}) \times \mathbf{G}_m(S) & \longrightarrow & \mathbf{G}_m(\bar{S}) & \longrightarrow & \mathrm{Pic}(R) & \longrightarrow & \mathrm{Pic}(\bar{R}) \times \mathrm{Pic}(S) & \longrightarrow & \mathrm{Pic}(\bar{S}) \end{array}$$

Let A be the local ring at the origin of the complex plane curve $Y^2 = X^2 - X^3$, $\widetilde{A}$ the normalisation of A and $\mathfrak{c}$ the conductor of $\widetilde{A}$ in A. Applying the big diagram above to the cartesian squares

$$\begin{array}{ccc} A & \longrightarrow & \widetilde{A} \\ \downarrow & & \downarrow \\ (A/\mathfrak{c}) & \longrightarrow & (\widetilde{A}/\mathfrak{c}) \end{array} \quad \text{and} \quad \begin{array}{ccc} A[t,t^{-1}] & \longrightarrow & \widetilde{A}[t,t^{-1}] \\ \downarrow & & \downarrow \\ (A/\mathfrak{c})[t,t^{-1}] & \longrightarrow & (\widetilde{A}/\mathfrak{c})[t,t^{-1}] \end{array}$$

it is easy to see that $\widetilde{K_0}(A[t,t^{-1}]) = \mathbf{C}^* \oplus \mathbf{Z} = \operatorname{Pic}(A[t,t^{-1}])$. This shows that a projective $A[t,t^{-1}]$-module P is stably free if and only if its maximal exterior power $\bigwedge^{\max}(P)$ is isomorphic to $A[t,t^{-1}]$.

Let I be an ideal representing $(1,1)$ in $\mathbf{C}^* \oplus \mathbf{Z} = \operatorname{Pic}(A[t,t^{-1}])$. The module underlying the space $H(I \oplus A[t,t^{-1}] \oplus A[t,t^{-1}])$ is free. In fact it is stably free because its determinant is trivial, hence, by a well-known cancellation theorem it is free. This shows that $H(I \oplus A[t,t^{-1}] \oplus A[t,t^{-1}])$ is a quadratic space of the form $(P_0[t,t^{-1}], \alpha)$ with P_0 free of rank 6 over A. Clearly this space represents the zero element of $W(A[t,t^{-1}])$. We claim that its class in $W'(A[t,t^{-1}])$ is not trivial.

Since A is local, projective modules extended from A are free. If $H(I \oplus A[t,t^{-1}] \oplus A[t,t^{-1}])$ were hyperbolic in $W'(A[t,t^{-1}])$ it would be stably isometric to $H(A[t,t^{-1}] \oplus A[t,t^{-1}] \oplus A[t,t^{-1}])$ and hence, by the quadratic cancellation theorem (see [4], VI, 6.2.5), it would be isometric to it. Recall that, for any commutative ring R in which 2 is invertible and any finitely generated projective R-module P, the even Clifford algebra C_0 of $H(P)$ is of the form

$$C_0 = \operatorname{End}_R(\textstyle\bigwedge^{even}(P)) \times \operatorname{End}_R(\textstyle\bigwedge^{odd}(P)),$$

where $\bigwedge^{even}(P)$ (respectively $\bigwedge^{odd}(P)$) is the even (respectively odd) part of the exterior algebra of P. In the case $P = I \oplus A[t,t^{-1}] \oplus A[t,t^{-1}]$ we have

$$C_0 = \operatorname{End}_{A[t,t^{-1}]}(A[t,t^{-1}]^2 \oplus I^2) \times \operatorname{End}_{A[t,t^{-1}]}(A[t,t^{-1}]^2 \oplus I^2).$$

Suppose now that $H(I \oplus A[t,t^{-1}]^2)$ and $H(A[t,t^{-1}]^3)$ are isometric. In this case their even Clifford algebras would be isomorphic, hence the algebra $\operatorname{End}_{A[t,t^{-1}]}(A[t,t^{-1}]^2 \oplus I^2)$ would be a 4×4 matrix algebra. By Morita theory the module $A[t,t^{-1}]^2 \oplus I^2$ would be of the form J^4 for some invertible ideal J. Taking the fourth exterior power of both sides we would have $I^2 = J^4$, which is impossible because I represents $(1,1)$ in $\mathbf{C}^* \oplus \mathbf{Z}$.

This shows that, even for a one-dimensional local domain, the map $W'(A[t,t^{-1}]) \to W(A[t,t^{-1}])$ may fail to be injective.

EXAMPLE 8.2. We define a commutative ring A by the cartesian diagram of real algebras

$$\begin{array}{ccc} A & \longrightarrow & \mathbf{R}[X,Y] \\ \downarrow & & \downarrow \pi \\ \mathbf{R} & \xrightarrow{\ \iota\ } & C \end{array} \qquad (1)$$

where $C = \mathbf{R}[x,y] = \mathbf{R}[X,Y]/(X^2+Y^2-1)$, π is the canonical projection and ι the canonical injection. Then $C \oplus C$ is the direct sum of its two submodules

$$P = C\tfrac{1}{2}(y+1,-x) + C\tfrac{1}{2}(-x,1-y) \quad \text{and} \quad P' = C\tfrac{1}{2}(1-y,x) + C\tfrac{1}{2}(x,1+y)$$

and we can define an automorphism α of $C[t,t^{-1}] \oplus C[t,t^{-1}]$ as the identity on P' and multiplication by t on P. With respect to the canonical basis of $C[t,t^{-1}] \oplus C[t,t^{-1}]$,

$$\alpha = \tfrac{1}{2}\begin{pmatrix} t(1+y)+1-y & -tx+x \\ -tx+x & t(1-y)+1+y \end{pmatrix}.$$

The matrix α has determinant equal to t and thus lies in $\mathrm{GL}_2(\mathrm{C}[\mathrm{t},\mathrm{t}^{-1}])$. According to Theorem 7.4 of [1] its class in $K_1(C[t,t^{-1}])$ is the image of P by the canonical injection λ mentioned in §2. It is easy to see that P is not free over C. In fact it turns out to represent the non trivial class of $\mathrm{Pic}(C) = \mathbf{Z}/2$. Since the homomorphism ι in the cartesian square that defines A is surjective, tensoring the diagram with $\mathbf{R}[t,t^{-1}]$ yields a Milnor patching diagram

$$\begin{array}{ccc} A[t,t^{-1}] & \longrightarrow & \mathbf{R}[X,Y][t,t^{-1}] \\ \downarrow & & \downarrow \pi \\ \mathbf{R}[t,t^{-1}] & \xrightarrow{\ \iota\ } & C[t,t^{-1}] \end{array}$$

We can use this diagram and the matrix α (see for instance [1], Chapter IX, Theorem 5.1) to patch a rank 2 free module Q over $\mathbf{R}[X,Y][t,t^{-1}]$ with a rank 2 free module R over $\mathbf{R}[t,t^{-1}]$ and get a rank 2 projective module

$$M = \{(q,r) \in Q \times R \mid \alpha(\pi_*(q)) = \iota_*(r)\}$$

over $A[t,t^{-1}]$. We now equip M with a skew-symmetric structure. To do this we put on Q and on R the skew-symmetric structures defined, respectively, by the matrices

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 0 & 1/t \\ -1/t & 0 \end{pmatrix}.$$

Since $\alpha^*\tau\alpha = \sigma$, the skew-symmetric structures $\sigma\colon Q \to Q^*$ and $\tau\colon R \to R^*$ are compatible with the patching and therefore they define a skew-symmetric structure $\varphi\colon M \to M^*$ on M.

We claim that the class of this space is not in the image of $W'([t,t^{-1}])$. Extending to K_{-1} the Mayer-Vietoris sequence associated to (1) (see [1], Chapter XII, Theorem 8.3) we get an exact sequence

$$K_0(\mathbf{R}[X,Y]) \oplus K_0(\mathbf{R}) \to K_0(C) \to K_{-1}(A) \to K_{-1}(\mathbf{R}[X,Y]) \oplus K_{-1}(\mathbf{R}).$$

From the fact that regular rings have a vanishing K_{-1}, that $K_0(\mathbf{R}[X,Y]) = K_0(\mathbf{R}) = \mathbf{Z}$ and that $K_0(C) = \mathbf{Z} \oplus \mathbf{Z}/2$, where the element of order 2 is the class of P, we easily deduce that $K_{-1}(A) = \mathbf{Z}/2$, generated by the image of M. Thus, by Corollary 2.4, the class of M generates $H^2(\mathbf{Z}/2, K_0(A[t,t^{-1}])/K_0(A)) = \mathbf{Z}/2$. Consider now the homomorphism

$$\omega\colon W(A[t,t^{-1}]) \longrightarrow H^2(\mathbf{Z}/2, K_0(A[t,t^{-1}])/K_0(A))$$

obtained by associating to any space its underlying projective module. Since $\omega((M,\varphi)) \neq 0$, (M,φ) cannot be Witt equivalent to a space supported by a module extended from A. This shows that the map $W'(A[t,t^{-1}]) \to W(A[t,t^{-1}])$ is not surjective.

REMARK 8.3. We suspect that even if the assumption of (a) is satisfied the map $W'(A[t,t^{-1}]) \to W(A[t,t^{-1}])$ may not be injective, but we did not find an example to confirm our suspicion.

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