ORDERINGS OF MAPPING CLASS GROUPS AFTER THURSTON

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ORDERINGS OF MAPPING CLASS GROUPS AFTER THURSTON

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ABSTRACT. We are concerned with mapping class groups of hyperbolic surfaces with nonempty boundary. We present a very natural method, due to Thurston, of finding many different left orderings of such groups. The construction uses the action of the mapping class group on the boundary of the universal cover (viewed in $\mathbf{H}^2$), including its limit points on the circle at infinity. We classify all orderings of braid groups which arise in this way. Moreover, restricting to a certain class of "nonpathological" orderings, we prove that there are only finitely many conjugacy classes of such orderings.

We shall be concerned with certain surfaces S and their mapping class groups $\mathcal{MCG}(S)$. The surfaces under consideration are compact, with a finite set of punctures and nonempty boundary, but not necessarily oriented. We recall that $\mathcal{MCG}(S)$ is the group of isotopy classes of homeomorphisms $S \to S$ which map ∂S identically and permute the punctures. It was first proved by Dehornoy [6] that braid groups (i.e. mapping class groups of punctured disks) are left orderable. A topological proof of this result was given in [9], and the extension to mapping class groups of general surfaces with boundary can be found in [22]. (Note that mapping class groups of surfaces with empty boundary have torsion, and thus cannot be left orderable.) Here we present a very natural method, due to Thurston [24], of finding many different left orderings of such groups. In brief, one equips the surface with a hyperbolic structure, lifts it to $\mathbf{H}^2$, attaches to this cover its limit points on the circle at infinity, and notices that there is a natural action of the mapping class group on the (circular) boundary of the resulting space which fixes a point, and thus an action on $\mathbf{R}$. We classify the set of orderings of braid groups which arise from Thurston's construction (not all orderings do – see the example in 2.6); more precisely, we divide these orderings into two disjoint classes, which we call orderings of finite, respectively infinite, type; the orderings inside each of the classes are classified by combinatorial means. Finite type orderings are discrete, and there exist only finitely many conjugacy classes of them. By contrast, there are uncountably many infinite type orderings, and all of them are non-discrete.

The outline of the paper is as follows. In the first section we give a short introduction to orderable groups and survey some known results about them. In the second section we present Thurston's construction. In the third section we define finite and infinite type orderings, and state our classification theorems. Sections four to six are concerned with finite type orderings: in Section 4 we describe a different method of constructing orderings, using "curve diagrams". In Section 5 we prove that the set of orderings arising from curve diagrams is very easy to understand and classify. Moreover, we prove that up to conjugacy only a finite number of orderings arise in this way. In the sixth section we prove the classification theorems for finite type orderings. The strategy is to associate to every point of $\mathbf{R}$ with orbit of finite type a curve diagram such that the orderings arising from this point and from the curve diagram agree. Thus we obtain, via curve diagram orderings, a good understanding of Thurston type orderings. In Section 7 we prove the results about the infinite type case.

1. Orderable groups

In this section we define orderable groups and survey some known results about them. The standard reference for orderings on groups is Rhemtulla and Mura's book [19].

DEFINITION 1.1. A group G is *left orderable* (respectively *right orderable*) if there is a total order $<$ on G which is invariant under left mutiplication (resp. right multiplication), that is, such that, for all $a,b \in G$, $a<b$, $a=b$ or $b<a$, and for all $g \in G$, $a<b$ implies that $ga<gb$ (resp. $ag<bg$).

A group G is *bi-orderable* or *two-sided orderable* if there is a total order on G which is respected by multiplication on the left and multiplication on the right: i.e. $a<b \Longrightarrow ga<gb$ and $ag<bg$.

Two left orderings $<$ and $\prec$ on a group G are *conjugate* if there exists a $g \in G$ such that $a \prec b$ if and only if $ag<bg$. So two left orderings are conjugate if "one is obtained from the other by right translation in the group".

REMARKS 1.2.

(1) The following observation will be extremely important in what follows. If a group G acts on the left by orientation preserving homeomorphisms on $\mathbf{R}$, then every point x in $\mathbf{R}$ with free orbit (i.e. $Stab(x) = \{1_G\}$) gives rise to a left ordering on G, by defining $g > h \;:\Longleftrightarrow\; g(x) > h(x)$. We have for every $f \in G$ that $fg(x) > fh(x) \iff g(x) > h(x)$, since the action of f preserves the orientation of $\mathbf{R}$; this implies that the ordering is indeed left invariant. Note that different points in $\mathbf{R}$ may give rise to different orderings. If a point x does not have free orbit, it still gives rise to a *partial* left invariant ordering.

(2) In fact, a countable group is left orderable if and only if it has an action by orientation preserving homeomorphisms on $\mathbf{R}$ such that only the trivial group element acts by the identity-homeomorphism, see for instance [11].

(3) A left orderable group is torsion-free: if an element x had order n, and if $1 < x$, then it would follow that $1 < x < x^2 < \cdots < x^{n-1} < x^n = 1$.

(4) The "positive cone" of the ordering, $P = \{g \in G \mid g > 1\}$ has the properties that $G = P \sqcup \{1\} \sqcup P^{-1}$, and that $PP \subset P$. Conversely, given a subset with these two properties, a left order $<$ can be defined by $a < b \;:\Longleftrightarrow\; a^{-1}b \in P$. Similarly, a right order $\prec$ is obtained from $a \prec b \;:\Longleftrightarrow\; ab^{-1} \in P$. (In particular, a group is left orderable if and only if it is right orderable.) The orders are total because of the first property, and transitive because of the second. The orders are bi-orders if and only if we have in addition that $g^{-1}Pg \subseteq P$ for all $g \in G$.

(5) The following classes of groups are bi-orderable:

(a) finitely generated torsion-free abelian groups;

(b) finitely generated free groups (this is a result of Magnus, see e.g. [13]);

(c) more generally, residually free groups, like fundamental groups of closed surfaces (this is due to Baumslag, see [26, 27, 28]).

(6) If S is a *closed* surface, then $\mathcal{MCG}(S)$ has torsion, but there exists a finite index subgroup which is torsion-free: consider the set of all elements which act as the identity on the homology $H_1(S, \mathbf{Z}_p)$, where p is a prime larger than $84(\text{genus} - 1)$. The torsion-freeness of these groups seems to be a folklore result, the analogue for the Torelli group (defined in the same way, only with $\mathbf{Z}_p$ replaced by $\mathbf{Z}$) is proved in [14]. It is an open problem whether or not these subgroups are left orderable.

We now give four examples of attractive results about orders on groups.

(1) Neville Smythe [23] used the orderability of surface groups to prove that any null-homotopic curve on a surface S is the image under projection of an embedded unknotted loop in $S \times I$.

(2) As pointed out by N. Smythe [16] in response to a question of L. Neuwirth [15, Question N], knot groups are in general not bi-orderable. For instance the trefoil knot group (which is isomorphic to the braid group on three strings B_3), is not bi-orderable. To show this, recall that B_3 contains an element Δ (the "half twist") which is not in the centre, but whose square Δ^2 is. Assume that $>$ is a bi-ordering of B_3, and let $b \in B_3$ be such that $b\Delta \neq \Delta b$, say $b\Delta > \Delta b$. Multiplying this inequality on the left by Δ and on the right by Δ^{-1} would yield $\Delta b > \Delta^2 b \Delta^{-1} = b\Delta^2\Delta^{-1} = b\Delta$, which is a contradiction.

Neuwirth reformulated the question as 'Are knot groups left orderable?'. A positive answer to this question follows from an observation by J. Howie and H. Short [12] that knot groups are locally indicable (every non-trivial finitely generated subgroup has $\mathbf{Z}$ as a homomorphic image), together with a theorem of Burns and Hale [4] that locally indicable groups are left orderable. The converse of Burns and Hale's theorem is known to be false – see [1] and [9, Theorem 5.3].

(3) We have just seen that B_3 (and hence B_n for all n) is not bi-orderable. Kim and Rolfsen [13] have recently proved that the finite index subgroup PB_n of *pure* braids *is* bi-orderable. However, no bi-ordering of PB_n extends to a left ordering of B_n [20].

(4) The Zero Divisor Conjecture, often attributed to Kaplansky, asserts that if R is a ring without zero divisors and G is a torsion-free group then the group ring RG has no zero divisors. The hypothesis that G be torsion-free is necessary, for if G contains an element x of order n then $(1-x)(1+x+\cdots+x^{n-1}) = 0$ in RG. The conjecture is known to hold for left orderable groups. In fact, it is not hard to see that left orderable groups have the "two unique product" property which implies that the conjecture holds for them (see e.g. [18], and also Delzant [7] and Bowditch [3] for some recent remarks about this property).

2. Orderings of mapping class groups using hyperbolic geometry

In this section we present the construction of orders on mapping class groups of surfaces which we learned from W. P. Thurston, and prove that they all extend the subword-ordering of Elrifai-Morton. The idea comes from the following classical situation as developed by Nielsen. As is well known, every closed surface of genus $g \geqslant 2$ can carry a hyperbolic structure; i.e. there is a homeomorphism between the universal cover $S^{\sim}$ of S and the hyperbolic plane $\mathbf{H}^2$ such that the covering transformations are isometries of $\mathbf{H}^2$. There is a natural closure $S^{\approx} \cong \overline{\mathbf{H}}^2$ of $S^{\sim} \cong \mathbf{H}^2$, defined by adding the so-called *circle at infinity* $S^1_\infty = \partial\overline{\mathbf{H}}^2$. Points of this circle can be defined as classes of geodesics, or quasi-geodesics, $\gamma\colon [0,\infty) \to \mathbf{H}^2$, staying a bounded distance apart. The covering action of $\pi_1(S)$ on $S^{\sim}$ extends to an action on $S^{\approx}$. So in particular, we have an action of $\pi_1(S)$ on the circle at infinity by homeomorphisms; this action has been much studied (for a good modern exposition of this see [10]). Even stronger, every homeomorphism of the surface lifts and extends to a homeomorphism of $S^{\approx}$; however, there is a $\pi_1(S)$-family of possible choices of lift, and therefore we get no well-defined action of $\mathcal{MCG}(S)$ on S^1_∞.

Instead of closed surfaces, Thurston considers surfaces S with nonempty boundary, a finite number of punctures, and $\chi(S) < 0$. Again, one can obtain a hyperbolic structure on S in which ∂S is a geodesic and the punctures are cusps; this time, $S^{\sim}$ is identified with a proper subset of $\mathbf{H}^2$. The boundary of this subset is just the union of the lifts of ∂S; in particular it is a union of geodesics in $\mathbf{H}^2$, and it follows that $S^{\sim}$ is convex in the hyperbolic metric. Moreover, the set of limit points of $S^{\sim}$ on the circle at infinity $\partial\overline{\mathbf{H}}^2$ is a Cantor set in $\partial\overline{\mathbf{H}}^2$. The closure $S^{\approx}$ of $S^{\sim}$ in $\overline{\mathbf{H}}^2$, i.e. $S^{\sim}$ with its limit points on the circle at infinity attached, is homeomorphic to a closed disk; $\partial S^{\approx}$ is a circle, also containing $S^{\approx} \cap \partial\overline{\mathbf{H}}^2$ as a Cantor set.

We now fix, once and for all, a basepoint of $S^{\sim}$ anywhere on $\partial S^{\sim}$. We denote the component of $\partial S^{\sim}$ which contains the base point by Π (see Figure 1). The basepoint projects to a basepoint of S in ∂S, and Π is an infinite cyclic cover of one component of ∂S. We consider the set of geodesics in $S^{\approx}$ starting at the basepoint – they are parametrized by the interval $(0,\pi)$, according to their angle with Π. We shall denote by $\widetilde{\gamma}_\alpha$ the geodesic with angle $\alpha \in (0,\pi)$ and by γ_α its projection to S. Since $S^{\sim}$ is hyperbolically convex, each point of $\partial S^{\approx}$ can be connected to the basepoint by a unique geodesic (possibly of infinite length) in $S^{\approx}$, and for points in $S^{\approx}\backslash\Pi$ this is one of the geodesics $\widetilde{\gamma}_\alpha$ with $\alpha \in (0,\pi)$. This construction proves

LEMMA 2.1. *There is a natural homeomorphism between* $\partial S^{\approx}\backslash\Pi$ *and* $(0,\pi)$. □

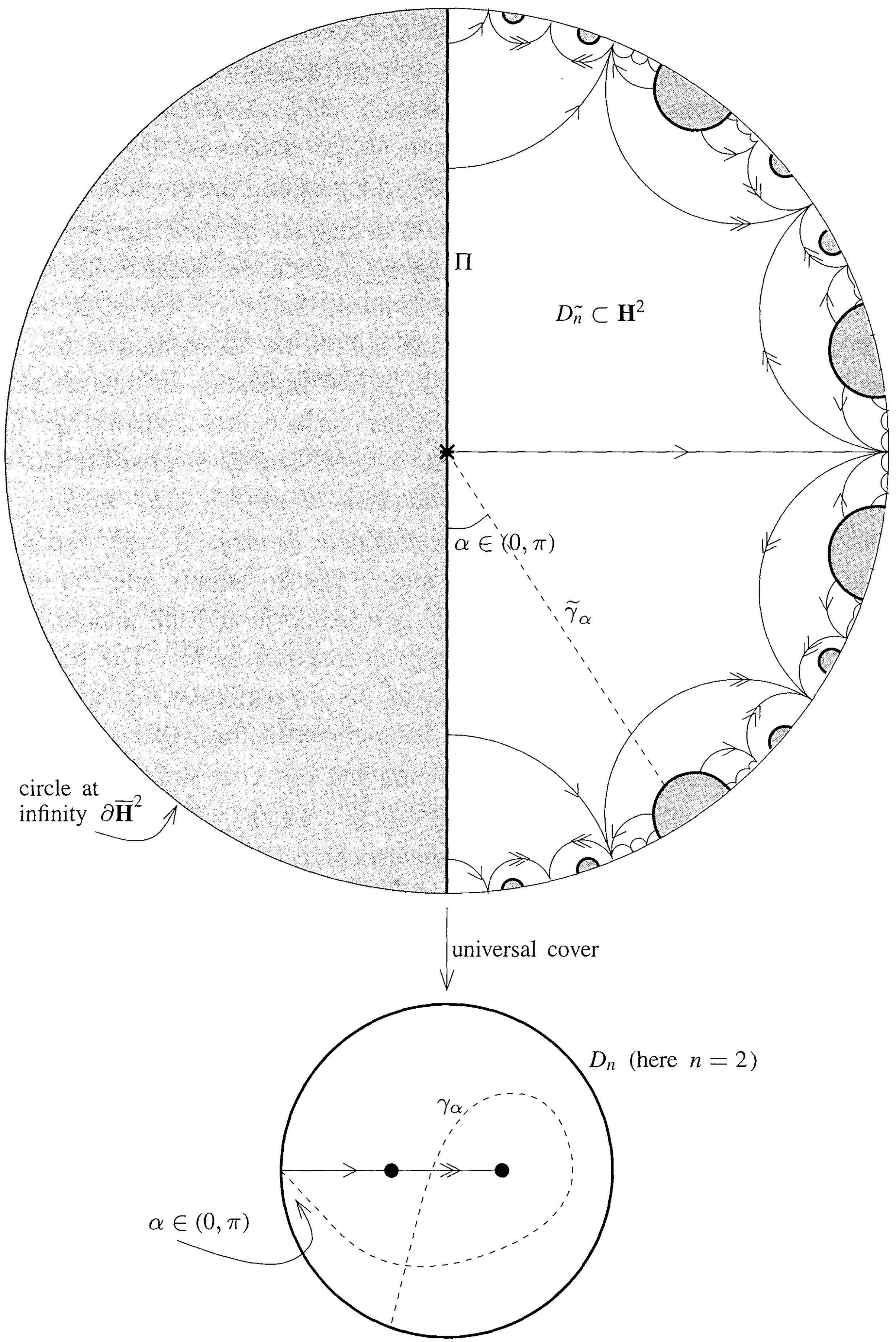

FIGURE 1

Picture of $S^\sim$ in $\mathbf{H}^2$ (here S is a twice-punctured disk)

As in the case of closed surfaces, we have an action of $\pi_1(S)$ on $S^{\bar{\bar{}}}$, which restricts to an action on $\partial S^{\bar{\bar{}}}$. However, this time we have more:

PROPOSITION 2.2. *There is a natural action by orientation preserving homeomorphisms of $\mathcal{MCG}(S)$ on $\partial S^{\bar{\bar{}}}\backslash\Pi \cong (0,\pi)$.*

Proof. Every homeomorphism $\varphi\colon S \to S$ has a canonical lift $\widetilde{\varphi}\colon S^{\sim} \to S^{\sim}$, namely the one that fixes the basepoint of $S^{\sim}$, and thus all of Π. Moreover, $\widetilde{\varphi}$ has an extension $\bar{\widetilde{\varphi}}\colon S^{\bar{\bar{}}} \to S^{\bar{\bar{}}}$. The restriction of this homeomorphism to $\partial S^{\bar{\bar{}}}$ is invariant under isotopy of φ, and fixes Π, and thus yields a well-defined orientation-preserving homeomorphism of $\partial S^{\bar{\bar{}}}\backslash\Pi$. (Note that there is no requirement for S to be orientable here.) □

COROLLARY 2.3. *$\mathcal{MCG}(S)$ is left orderable.*

Proof. No nontrivial element of $\mathcal{MCG}(S)$ acts trivially on $(0,\pi)$, because if such an element existed, it would in particular fix all liftings of the basepoint of S, and thus induce the identity-homorphism on $\pi_1(S)$; by [2, Corollary 1.8.3] it would then be isotopic to the identity, in contradiction with the hypothesis. The result now follows from Remark 1.2(2), because $(0,\pi)$ is homeomorphic to $\mathbf{R}$.

However, there is an elementary proof in our situation. We choose arbitrarily a finite generating set of $\pi_1(S)$, and denote the end points of the liftings of these elements by $s_1,\ldots,s_k \in (0,\pi)$. A left order on $\mathcal{MCG}(S)$ is now defined inductively: if $\varphi(s_1) > s_1$ then $\varphi > 1$ (and the same with $>$ replaced by $<$); if $\varphi(s_1) = s_1$, but $\varphi(s_2) > s_2$, then $\varphi > 1$ as well, and so on; this is a total order, because we have that $\varphi(s_i) = s_i$ for all i if and only if $\varphi = 1$. □

However, for the rest of the paper we shall be less interested in orderings of this type, but rather in orderings induced by the orbits of single geodesics, i.e. in orderings of the type introduced in Remark 1.2(1).

We recall the definition of a *positive Dehn twist* along a simple closed curve τ in the surface S: it can be characterised as a homeomorphism $S \to S$ which maps all but an annular neighbourhood of τ identically, and sends any arc that crosses τ to an arc that, upon entering the annular neighbourhood, turns left, spirals exactly once along τ, and then turns right to leave the annular neighbourhood through its other boundary component and continue as before. For example in the case of a punctured disk, if $\Delta \in B_n$ denotes the "half-twist braid", then Δ^2 is a Dehn twist along a curve parallel to the boundary of the disk.

PROPOSITION 2.4. *For the positive Dehn twist T along any simple closed geodesic τ in S we have $T(\alpha) \geqslant \alpha$ for any $\alpha \in (0, \pi)$. If γ_α intersects τ at least once, then the inequality is strict.*

Proof. If γ_α is disjoint from τ, then $T(\alpha) = \alpha$. If, on the contrary, γ_α intersects τ, and hence any curve isotopic to τ, any number of times (possibly infinitely often), then we denote by $T_i(\gamma_\alpha)$ $(i \in \mathbf{N})$ the curve obtained from γ_α by applying the Dehn twist to the first i intersections of γ_α with τ and ignoring all following intersections; we denote by $T_i(\alpha)$ its end point in $\partial D_n^{\bar{=}} \backslash \Pi$. We have $T(\alpha) = \lim_{i \to \infty} T_i(\alpha)$.

We now claim that $(T_i(\alpha))_{i \in \mathbf{N}}$ is a strictly increasing sequence. To simplify notation, we shall prove the special case $T_1(\alpha) > \alpha$, the proof in the general case is exactly the same. In the universal cover $D_n^{\bar{=}}$ we consider the lifting of the curve $T_1(\gamma_\alpha)$: starting at the basepoint, it sets off along $\tilde{\gamma}_\alpha$, up to the first intersection with some lifting $\tilde{\tau}$ of τ. There it turns left, walks along $\tilde{\tau}$ up to the next preimage of the intersection point, where it encounters a different lifting $\tilde{\gamma}'_\alpha$ of γ_α. There it turns right, following this lifting all the way to $\partial D_n^{\bar{=}} \backslash \Pi$. The crucial point now is that $\tilde{\gamma}_\alpha$ and $\tilde{\gamma}'_\alpha$ intersect $\tilde{\tau}$ at the same angle, because the two intersections are just different liftings of the same intersection between γ_α and τ in D_n. It follows that $\tilde{\gamma}_\alpha$ and $\tilde{\gamma}'_\alpha$ do not intersect, not even at infinity, for if they did they would determine a hyperbolic triangle in $D_n^{\bar{=}}$ two of whose interior angles already add up to 180 degrees, which is impossible. This implies the claim, and thus proves the proposition. □

COROLLARY 2.5. *All total orderings of the braid group B_n considered in this paper extend the subword-ordering of Elrifai-Morton* [8, 25]. *More precisely, if a curve τ in D_n encloses a precisely twice punctured disk and $T^{1/2}$ is the positive half-Dehn twist along τ interchanging the two punctures then $T \circ \varphi > \varphi$ for any $\varphi \in B_n$ and any ordering $>$ of Thurston-type.*

Proof. It suffices to prove that $T^{1/2}(\alpha) \geqslant \alpha$ for all $\alpha \in (0, \pi)$. If there existed an $\alpha \in (0, \pi)$ with $T^{1/2}(\alpha) < \alpha$ then it would follow that $T(\alpha) = T^{1/2} \circ T^{1/2}(\alpha) < T^{1/2}(\alpha) < \alpha$ (where the first inequality holds since $T^{1/2}$ is orientation preserving), in contradiction with the proposition. □

REMARK 2.6. Here is an example of an ordering $\prec$ of B_n that does not arise from Thurston's construction: if "$<$" is any ordering of Thurston-type, then we define an element $\varphi \in B_n$ to be in the positive cone of $\prec$ if either $ab(\varphi)$ is positive, where $ab\colon B_n \to \mathbf{Z}$ is the abelianization, or if $ab(\varphi) = 0$

and $1_{B_n} < \varphi$. In this ordering the commutator subgroup is convex [19], and we leave it to the reader to verify that no Thurston-type ordering has this property.

3. Main results

We shall mainly be interested in the case $S = D_n$ $(n \geq 2)$, where D_n is the closed unit disk in $\mathbf{C}$, with n punctures lined up in the real interval $(-1, 1)$; in this case the mapping class group is a braid group: $\mathcal{MCG}(D_n) = B_n$. We recall that for $\alpha \in (0, \pi)$ we denote by γ_α the geodesic which starts at the basepoint with angle α with ∂S, and by $\widetilde{\gamma}_\alpha$ its preimage in the universal cover starting at the basepoint of $S^{\sim}$.

DEFINITION 3.1. A geodesic γ_α, $\alpha \in (0, \pi)$, is said to be of *finite type* if it satisfies at least one of the following conditions:

(a) there exists a finite initial segment γ_α^t such that any two punctures that lie in the same path component of $S \backslash \gamma_\alpha^t$ also lie in the same path component of $S \backslash \gamma_\alpha$, or

(b) it falls into a puncture, or

(c) it spirals towards a simple closed geodesic.

If a geodesic γ_α is not of finite type then we say it is of *infinite type*. We also define the ordering of $\mathcal{MCG}(S)$ induced by a geodesic γ_α to be of finite or infinite type if γ_α is of finite or infinite type.

An infinite type geodesic looks as follows. All its self intersections occur in some finite initial segment γ_α^t. At least one of the path components of $S \backslash \gamma_\alpha^t$ contains three or more punctures in its interior, and the closure of $\gamma_\alpha \backslash \gamma_\alpha^t$ is a geodesic lamination without closed leaves inside such a component. In particular, there is a pair of punctures which are separated by the whole geodesic, but not by any finite initial segment. (Note that the geodesic $\gamma_\alpha \backslash \gamma_\alpha^t$ is isolated from both sides – in this it is very different from leaves of geodesic laminations on surfaces without boundary.)

DEFINITION 3.2. For a geodesic γ_α of finite respectively infinite type we say that it *fills the surface in finite* respectively *infinite time* if all punctures lie in different path components of $S \backslash \gamma_\alpha$. By contrast, a geodesic γ_α *does not fill* the surface if $S \backslash \gamma_\alpha$ has a path component that contains two punctures.

The aim of the rest of the paper is to prove the following theorems. Recall that every point $\alpha \in (0,\pi)$ gives rise to a – possibly partial – ordering of $\mathcal{MCG}(S)$. The first theorem gives criteria for these orderings to be total or, equivalently, for the orbit of α to be free.

THEOREM 3.3. *Let S be any hyperbolic surface.*

(a) *If a geodesic γ_α does not fill S, then the orbit of $\alpha \in (0,\pi)$ is not free.*

(b) *If γ_α is of finite type, then the converse holds as well: if γ_α fills the surface, then α has free orbit.*

(c) *Let $\mathcal{I} := \{\alpha \mid \gamma_\alpha$ is of infinite type$\} \subset (0,\pi)$. Then $\mathcal{I}$ is uncountable, and all but countably many of its elements have free orbits. In any neighbourhood of an $\alpha \in \mathcal{I}$ there exist points of both finite and infinite type, i.e. there are $\alpha' \neq \alpha$ and $\beta \in (0,\pi)$ such that $\gamma_{\alpha'} \in \mathcal{I}$ and $\gamma_\beta \notin \mathcal{I}$.*

The next theorem gives a classification of orders of Thurston-type.

THEOREM 3.4. *If S is a punctured disk, we have:*

(a) *An ordering cannot be both of finite and infinite type.*

(b) *Given two geodesics $\gamma_\alpha, \gamma_\beta$ of finite type, one can decide whether or not they determine the same ordering.*

(c) *Given two geodesics $\gamma_\alpha, \gamma_\beta$ of infinite type, one can decide whether or not they determine the same ordering. For instance, if γ_α and γ_β are embedded, then they determine the same ordering if and only if $\beta = \Delta^{2k}(\alpha)$ for some $k \in \mathbf{Z}$ (i.e. if γ_β is obtained from γ_α by sliding the starting point $2k$ times around ∂D_n).*

(Note that part (a) is not immediately clear: it is conceivable that finite and infinite type geodesics induce the same orderings.) In fact, we shall develop machinery which gives a very good and explicit understanding of finite type orderings:

THEOREM 3.5. *There are only finitely many conjugacy classes of orderings of finite type of $\mathcal{MCG}(D_n) = B_n$. The number N_n of conjugacy classes can be calculated by the following recursive formula*

$$N_2 = 1 \quad \textit{and} \quad N_n = N_{n-1} + \sum_{k=2}^{n-2} \binom{n-2}{k-1} N_k \, N_{n-k}\,.$$

We do not know if there exists a "closed" formula for N_n. The following list gives the first few values:

n	2	3	4	5	6	7	8
N_n	1	1	3	9	39	189	1197

Theorems 3.4 and 3.5 almost certainly generalise to mapping class groups of other negatively curved surfaces, but in order to keep our machinery simple, we stick to the special case of punctured disks.

4. ORDERINGS OF MAPPING CLASS GROUPS USING CURVE DIAGRAMS

In this section we present another method for constructing left orderings on B_n, using certain diagrams on D_n, which we call *curve diagrams*. Both the definition of curve diagrams and the orderings associated to them are generalisations of similar concepts in [9].

CONVENTION. Whenever we talk about geodesics in D_n, we think of the punctures as being holes in the disk, whose neighbourhoods on the disk have the geometry of cusps. By contrast, when we talk about curve diagrams, we think of the punctures as distinguished points on, and belonging to, the disk, and we ignore the geometric structure. This changing perspective should not cause confusion.

DEFINITION 4.1. A (partial) curve diagram Γ is a diagram on D_n consisting of $j \leqslant n-1$ closed, oriented arcs which are labelled $\Gamma_1, \ldots, \Gamma_j$. Moreover, the boundary circle of D_n is labelled Γ_0, and by abuse of notation we shall refer to it as an "arc" of Γ. We require:

(1) every path component of $D_n \backslash \Gamma$ has at least one puncture in its interior,

(2) $\bigcup_{i=0}^{j} int(\Gamma_i)$ is embedded and disjoint from the punctures (where *int* denotes the interior),

(3) the starting point of the i^{th} arc lies in $\bigcup_{k=0}^{i-1} \Gamma_k$, i.e. on one of the previous arcs,

(4) the end point of the i^{th} arc lies in one of the previous arcs, or on an earlier point of the i^{th} arc, or in a puncture.

In the special case that $j = n-1$, so that in (1) every path component contains precisely one puncture, we say Γ is a total curve diagram.

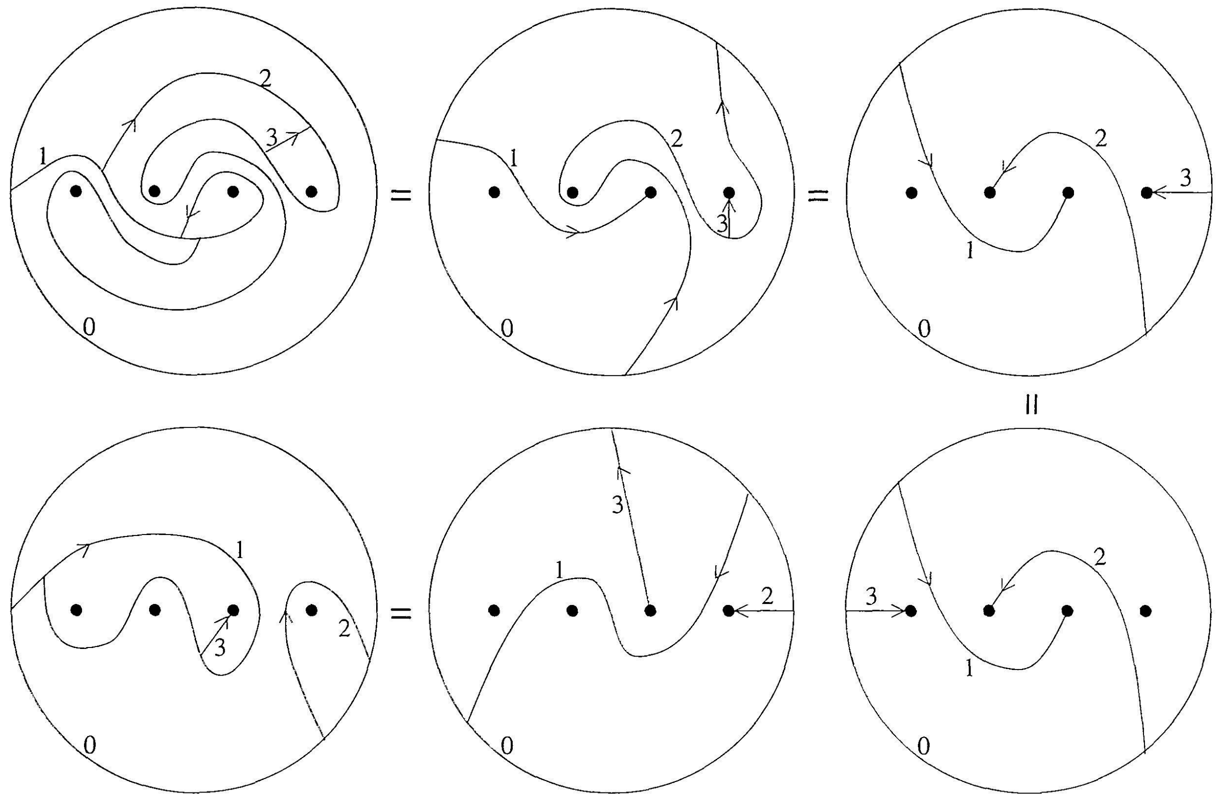

FIGURE 2

Examples of total curve diagrams on D_4
(The meaning of the equality signs will be explained in §5.)

REMARKS. For simplicity we shall sometimes label arcs $0, \ldots, j$, instead of $\Gamma_0, \ldots, \Gamma_j$. Moreover, we shall use the abbreviated notation $\Gamma_{0\cup\cdots\cup i} := \bigcup_{k=0}^{i} \Gamma_k$. Note for (1) that the number of path components of $D_n \backslash \Gamma$ equals 1 plus the number of arcs of Γ not ending in a puncture, so it can be anything between 1 and n. Note for (3) that the start point of the i^{th} arc can lie in a puncture, if this puncture was the end point of one of the previous arcs. Finally note that if $i < j$ then Γ_i is disjoint from the interior of Γ_j.

We now explain how to associate a partial left ordering of $\mathcal{MCG}(D_n) = B_n$ to a partial curve diagram (with total curve diagrams giving rise to total orderings). The essential ingredient in this definition is the well-known procedure of "pulling tight" or "reducing" two properly embedded curves in a surface. In brief, two simple closed or properly embedded curves in a surface can be isotoped into a relative position in which they have minimal possible intersection number, and this relative position is unique. Moreover, it can be found in a very naive way: whenever one sees a D-disk (or "bigon") enclosed by a pair of segments of the curves, one "squashes" it, i.e. one reduces the intersection number of the two curves, by isotoping the arcs across the disk. A systematic exposition of these ideas can for instance be found in Section 2 of [9].

Our definition of the ordering of $\mathcal{MCG}(S)$ associated to a curve diagram will be a variation of the definition in [9]. We briefly remind the reader of this comparison method. Let Γ be a partial curve diagram in which all j arcs are embedded (no curve Γ_i has end point in its own interior), and let φ and ψ be two homeomorphisms of D_n. If $\varphi(\Gamma_k) \neq \psi(\Gamma_k)$, then we will define either $\varphi < \psi$ or $\psi < \varphi$, according to the following rule. There is an $i \leq j$ such that $\varphi(\Gamma_{0\cup\cdots\cup i-1})$ and $\psi(\Gamma_{0\cup\cdots\cup i-1})$ are isotopic, whereas $\varphi(\Gamma_{0\cup\cdots\cup i})$ and $\psi(\Gamma_{0\cup\cdots\cup i})$ are not. Then we replace φ by an isotopic map, also denoted φ, such that the restrictions of φ and ψ to $\Gamma_{0\cup\cdots\cup i-1}$ are exactly the same maps. At this point, $\varphi(\Gamma_i)$ and $\psi(\Gamma_i)$ have the same starting point and lie in the same path component of $D_n \backslash \varphi(\Gamma_{0\cup\cdots\cup i-1})$. Next we "pull $\varphi(\Gamma_i)$ tight" with respect to $\psi(\Gamma_i)$, i.e. we isotope φ so as to minimise the number of intersections of $\varphi(\Gamma_i)$ and $\psi(\Gamma_i)$, as described above. This can be done by an isotopy which fixes $\varphi(\Gamma_{0\cup\cdots\cup i-1})$. Restricting finally our attention to small initial segments of $\varphi(\Gamma_i)$ and $\psi(\Gamma_i)$, we see that the two curves set off from their common starting point into the interior of a component of $D_n \backslash \varphi(\Gamma_{0\cup\cdots\cup i-1})$ in different directions, one of them "going more to the left"; if it is $\varphi(\Gamma_i)$ say, then we define $\varphi > \psi$, otherwise $\psi < \varphi$. The resulting (possibly parital) ordering is left invariant, because the *relative* position of $\chi \circ \varphi(\Gamma)$ and $\chi \circ \psi(\Gamma)$ is the same as that of $\varphi(\Gamma)$ and $\psi(\Gamma)$ for all $\chi \in \mathcal{MCG}(S)$.

We shall use the following variant of this comparison method: first we make $\varphi(\Gamma_{0\cup\cdots\cup i-1})$ and $\psi(\Gamma_{0\cup\cdots\cup i-1})$ agree for maximal possible i, as before. If the arc Γ_i is embedded, then we proceed as before to compare $\varphi(\Gamma_i)$ and $\psi(\Gamma_i)$. If the arc Γ_i has end point in the interior of Γ_i itself, then we consider the embedded arc Γ_i' which, by definition, is obtained from Γ_i by sliding the end point back along Γ_i so as to make start and end point coincide, as illustrated in Figure 3. We then ignore the original arc Γ_i, and compare $\varphi(\Gamma_i')$ and $\psi(\Gamma_i')$ as before.

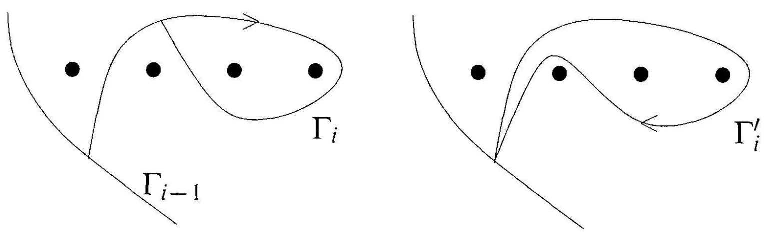

FIGURE 3

The embedded arc Γ_i' obtained from Γ_i by sliding the end point

DEFINITION 4.2. The ordering defined in this way is *the ordering associated to* the curve diagram Γ.

LEMMA 4.3. *The ordering associated to a curve diagram Γ is total if and only if Γ is a total curve diagram.*

Proof. If Γ is total, i.e. if all components of $D_n \backslash \Gamma$ are once-punctured disks, then any homeomorphism of D_n which fixes Γ is isotopic to the identity; this follows from the Alexander trick (see e.g. [21]). Conversely, if $D_n \backslash \Gamma$ has a path component which contains at least two holes, then we can push the boundary curve of this path component slightly into its interior, to make it disjoint from Γ. A Dehn twist along such a curve is a nontrivial element of B_n, and acts trivially on Γ. □

EXAMPLE. For any n, the Dehornoy ordering [6] is defined by the diagram consisting of $n-1$ horizontal line segments, connecting ∂D_n to the first (leftmost) hole, the first to the second hole, and so on. The arcs are oriented from left to right, and labelled $1, \ldots, n-1$ in this order (see [9]).

DEFINITION 4.4. A (possibly partial) order on a group G is *discrete* if the positive cone $P = \{g \in G \mid g > 1\}$ has a minimal element. (If the ordering is total then this element is necessarily unique.)

In a group with a discrete total left-invariant order every element has a unique predecessor and successor. We note that an ordering is non-discrete if and only if for all $a, c \in G$ there exists a $b \in G$ such that $a < b < c$.

LEMMA 4.5. *The total ordering associated to a total curve diagram Γ is discrete.*

Proof. The curve diagram $\Gamma_{0 \cup \cdots \cup n-2}$ (which is obtained from Γ by removing the arc of maximal index) cuts D_n into a number of once-punctured disks and one twice-punctured disk. We observe that the unique smallest element is the positive half-twist interchanging the two punctures inside this disk. □

REMARK. It is an easy exercise to prove that the partial orderings associated to partial curve diagrams are in general not discrete. However, we shall see in the proof of Theorem 3.4(a) that even such orderings have a certain discreteness property.

5. WHICH PAIRS OF CURVE DIAGRAMS DETERMINE THE SAME ORDERING ?

In this section we define an equivalence relation of curve diagrams which we call *loose isotopy*. We give a simple algorithm to decide whether or not two given curve diagrams are loosely isotopic. We prove that two curve diagrams determine the same ordering if and only if they are loosely isotopic. Moreover, the quotient of the set of loose isotopy classes of curve diagrams under the natural action of B_n is finite; we deduce that for fixed $n \geqslant 2$ there is only a finite number of conjugacy classes of orderings arising from curve diagrams.

DEFINITION 5.1. Let $\mathcal{C}$ denote the space of all curve diagrams, equipped with the natural topology (the subset topology from the space of all mappings of $n-1$ arcs into D_n). We define *loose isotopy* to be the equivalence relation on $\mathcal{C}$ generated by the following two types of equivalence:

(1) *Continuous deformation:* two curve diagrams are equivalent if they lie in the same path component of $\mathcal{C}$.

(2) *Pulling loops around punctures tight:* if some final segment of the curve Γ_i say cuts out a disk with one puncture from D_n, then this final segment can be pulled tight, so as to make Γ_i end in the puncture.

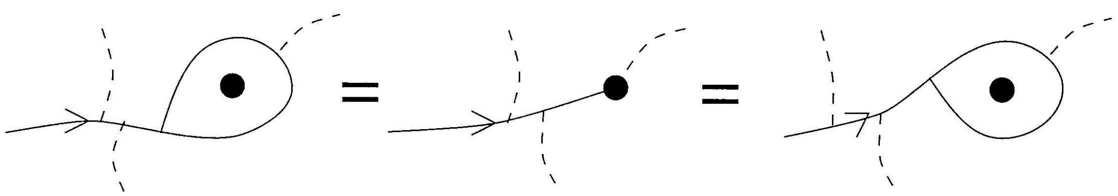

FIGURE 4
Pulling loops around punctures tight

Equivalence (2) is illustrated in Figure 4; here the dashed lines indicate any number of arcs of index greater than i which start on Γ_i. Equivalence (1) says that one is allowed to deform the diagram, to slide starting points of arcs along the union of all previous arcs, including their start and end points, and even across punctures, if they are the end points of some previous arcs. Similarly, end points of arcs are allowed to slide across the union of all "previous points of the diagram".

In order to get a feel for the meaning of this definition, the reader may want to prove that the equality signs in Figure 2 represent loose isotopies.

THEOREM 5.2. (a) *Two curve diagrams determine the same ordering of* B_n *if and only if they are loosely isotopic.*

(b) *There is an algorithm to decide whether or not two curve diagrams* Γ *and* Δ *are loosely isotopic.*

Proof. For the implication "$\Leftarrow$" of (a) we have to prove that loosely isotopic diagrams define the same ordering. The only nonobvious claim here is that the ordering is invariant under the "pulling tight" procedure.

In order to prove this, we consider a curve diagram Γ' with j arcs, the i^{th} of which is a loop (i.e. the end point equals the start point) which encloses exactly one puncture. We consider in addition the curve diagram Γ which is obtained from Γ' by squashing the curve Γ'_i to an arc from the starting point of Γ'_i to the enclosed puncture, much as in Figure 4. Let φ and ψ be two nonisotopic homeomorphisms, and more precisely assume that $\varphi >_\Gamma \psi$. Our aim is to prove that $\varphi >_{\Gamma'} \psi$. If $\varphi(\Gamma_{0\cup\cdots\cup i-1})$ and $\psi(\Gamma_{0\cup\cdots\cup i-1})$ are already nonisotopic then this is obvious since the first $i-1$ arcs of Γ and Γ' coincide. On the other hand, if $\varphi(\Gamma_{0\cup\cdots\cup i})$ and $\psi(\Gamma_{0\cup\cdots\cup i})$ are isotopic (and the difference between φ and ψ only shows up on arcs of higher index), then after an isotopy the first i arcs of $\varphi(\Gamma')$ and $\psi(\Gamma')$ coincide as well, and the result follows easily. Finally in the critical case, when the first difference occurs on the i^{th} arc of Γ, we have the two arcs $\varphi(\Gamma_i)$ and $\psi(\Gamma_i)$ which are reduced with respect to each other, with $\varphi(\Gamma_i)$ setting off more to the left. The crucial observation is now that the boundary curves of sufficiently small regular neighbourhoods of the two curves are isotopic to $\varphi(\Gamma'_i)$ respectively $\psi(\Gamma'_i)$ *and* reduced with respect to each other – see Figure 5. It is now clear that $\varphi(\Gamma'_i)$ also sets off more to the left than $\psi(\Gamma'_i)$. This completes the proof of implication "$\Leftarrow$" of (a).

We shall now explicitly describe the algorithm promised in (b), and prove the implication "$\Rightarrow$" of (a) along the way. The proof is by induction on n. For the case $n = 2$ we note that any two total curve diagrams (with one arc) are loosely isotopic. Thus there are only two loose isotopy classes of curve diagrams: the empty diagram and the one with one arc. The empty diagram induces the trivial ordering, whereas the diagram with one arc induces the ordering $\sigma_1^k > \sigma_1^l \iff k > l$. So the desired algorithm consists just of counting the number of arcs, and non loosely isotopic curve diagrams do indeed induce different orderings.

Now suppose that $n \geq 3$, that the result is true for disks with fewer than n punctures, and that we want to compare two curve diagrams $\Gamma_0, \ldots, \Gamma_j$ and $\Delta_0, \ldots, \Delta_{j'}$ in D_n, with $j, j' \leq n-1$. The arc Γ_1 ends either on ∂D_n, or in

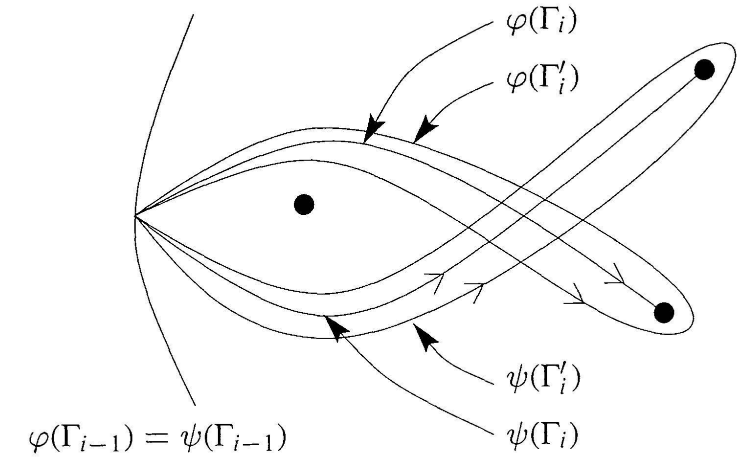

FIGURE 5

Proof that $\varphi >_\Gamma \psi \Rightarrow \varphi >_{\Gamma'} \psi$ – the critical case where the first difference between φ and ψ occurs on the arc which is being pulled tight

the interior of Γ_1 itself, or in a puncture. In the first two cases $D_n \backslash \Gamma_1$ has precisely two path components. At most one of them can contain only one puncture; if one of them does, we pull Γ_1 tight around it. If both components of $D_n \backslash \Gamma_1$ contain more than one puncture and if Γ_1 ends on itself, then we slide the end point of Γ_1 back along Γ_1, across its starting point, and into $\Gamma_0 = \partial D_n$. There are now two possibilities left: either Γ_1 is an embedded arc connecting the boundary to a puncture (Γ_1 is *nonseparating*), or it is an embedded arc connecting two boundary points, cutting D_n into two pieces, each of which has at least two punctures in its interior (Γ_1 is *separating*). We repeat this procedure for Δ_1. There are now four cases:

(1) It may be that Γ_1 is separating, while Δ_1 is not (or vice versa).

(2) It is possible that Γ_1 and Δ_1 are both nonseparating but are not isotopic with starting points sliding in ∂D_n (a criterion which is easy to check algorithmically).

(3) It is possible that Γ_1 and Δ_1 are both separating but are not isotopic as oriented arcs, with starting and end points sliding in ∂D_n (a criterion which is equally easy to check algorithmically).

CLAIM. *In these first three cases the orderings defined by* Γ *and* Δ *do not coincide, and* Γ *and* Δ *are not loosely isotopic.*

We only need to prove the first part of the claim, the second one follows by the implication "$\Leftarrow$" of Theorem 5.2(a). We first treat the following pathological situation: if, in case (3) above, Γ_1 and Δ_1 are isotopic to each other, but with opposite orientations, then a homeomorphism of the type

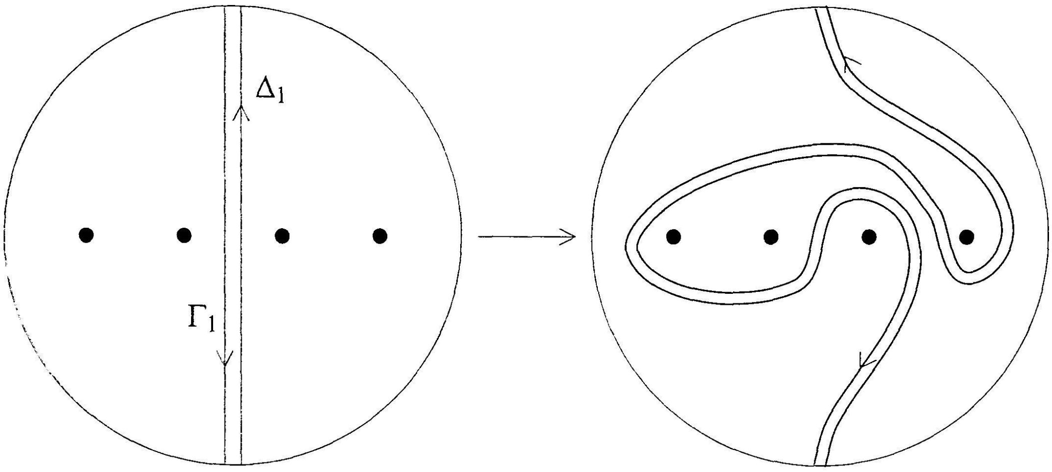

FIGURE 6
A homeomorphism which distinguishes the Γ- and Δ-orderings

indicated in Figure 6 is positive in the ordering defined by Γ, but negative in the Δ-ordering. In all other situations allowed by (1), (2) and (3), there exists a simple closed curve τ in D_n which is disjoint from Γ_1, but intersects every arc isotopic to Δ_1. (Consider, for instance, a regular neighbourhood of $\partial D_n \cup \Gamma_1$ in D_n. If Γ_1 is nonseparating then its boundary curve has this property; if Γ_1 is separating then at least one of the two boundary curves has.) We denote by $T\colon D_n \to D_n$ the positive Dehn twist along τ. The map T leaves Γ_1 invariant, while the arc $T(\Delta_1)$ is "more to the left" than Δ_1 (to see this, reduce the two arcs by making them geodesic, and apply Proposition 2.4).

Similarly, there exists a curve τ' which is disjoint from Δ_1, but not from any arc in the isotopy class of Γ_1. Then T'^{-1} sends $T(\Delta_1)$ more to the right, but not very far: $T'^{-1} \circ T(\Delta_1)$ is still to the left of the arc Δ_1, which is fixed by T'^{-1}; and T'^{-1} sends Γ_1 to the right, as well. Thus, in summary, the composition $T'^{-1} \circ T$ sends Δ_1 more to the left but Γ_1 more to the right, so that $T'^{-1} \circ T \in B_n$ is negative in the ordering determined by Γ, but positive in the Δ-ordering. This proves the claim. (One may find simpler proofs, but this one will be useful in Section 7.)

(4) The remaining possibility is that Γ_1 and Δ_1 *can* be made to coincide by isotopies which need not be fixed on ∂D_n. Such isotopies can be extended to loose isotopies of Γ or Δ.

To summarize, we can algorithmically decide whether or not there is a loose isotopy which makes Γ_1 and Δ_1 coincide. If the answer is NO (cases (1)–(3)), then Γ and Δ are not loosely isotopic, and the orderings defined by Γ and Δ do not coincide. In this case, the implication "$\Rightarrow$" of 5.2(a) is true. If the answer is YES (case (4)), then $D_n \backslash \Gamma_1 = D_n \backslash \Delta_1$ has either one or two path components, each of which is a disk with at most $n-1$ punctures. Moreover, the arcs $\Gamma_2, \ldots, \Gamma_j$ form curve diagrams in these disks (with some indices missing in each curve diagram, if the arcs are distributed among two disks), and similarly for $\Delta_2, \ldots, \Delta_{j'}$. Finally, the following conditions are equivalent:

(i) Γ and Δ are loosely isotopic,

(ii) in each path component of $D_n \backslash \Gamma_1 = D_n \backslash \Delta_1$ there is a loose isotopy between the diagrams made up of the remaining arcs of Γ respectively Δ,

(iii) the orderings of $Fix(\Gamma_1) \subseteq B_n$ induced by Γ and Δ coincide, where $Fix(\Gamma_1)$ denotes the subgroup whose elements have support disjoint from Γ_1,

(iv) the orderings of B_n defined by Γ and Δ coincide.

The equivalences between (i) and (ii), and between (iii) and (iv) are clear, whereas the equivalence of (ii) and (iii) follows from the induction hypothesis. Also by the induction hypothesis, we can decide algorithmically whether or not (ii) holds. This proves the theorem in case (4). □

We recall that for any ordering "$<$" of B_n, and every element $\rho \in B_n = \mathcal{MCG}(D_n)$, one can construct an ordering "$<_\rho$", by defining $\varphi <_\rho \psi :\Longleftrightarrow \varphi\rho < \psi\rho$, and we call $<_\rho$ "the ordering $<$ conjugated by ρ". We observe that if $<$ is induced by a curve diagram Γ, then $<_\rho$ is induced by the curve diagram $\rho(\Gamma)$. Thus two curve diagrams Γ and Δ induce conjugate orderings if and only if Γ and Δ are in the same orbit under the natural action of B_n on the set of loose isotopy classes of curve diagrams.

PROPOSITION 5.3. *Let M_n denote the number of conjugacy classes of total orderings of B_n arising from curve diagrams. Then M_n can be calculated by the following recursive formula*

$$M_2 = 1 \quad \textit{and} \quad M_n = M_{n-1} + \sum_{k=2}^{n-2} \binom{n-2}{k-1} M_k \, M_{n-k} \, .$$

REMARK. In order to avoid confusion, we recall our orientation convention: we are insisting that "more to the left" means "larger". It is for this reason that there is only one ordering of $B_2 = \mathbf{Z}$, not two, as one might expect.

Proof. We shall count the orbits of the set of loose isotopy classes of total curve diagrams under the action of B_n. The case $n = 2$ is clear, since there is only one loose isotopy class of curve diagrams. Now suppose inductively that the formula is true for up to $n-1$ strings.

For every total curve diagram in D_n there are two possibilities:

(a) the first arc of the curve diagram ends in a puncture or can be pulled tight so as to end in a puncture;

(b) the first arc cuts D_n into two disks, each of which contains at least two punctures.

For case (a) we notice that the first arc can be turned into the horizontal arc from -1 to the leftmost puncture, by an action of some appropriate element of B_n. There are now precisely M_{n-1} orbits of loose isotopy classes of curve diagrams of the remaining $n-2$ arcs in the $n-1$-punctured disk $D_n \backslash$(the first arc). So case (a) gives a contribution of M_{n-1} orbits.

The argument for case (b) is similar: the action of an appropriate element of B_n will turn the first arc of any curve diagram of type (b) into the vertical arc, oriented from bottom to top, having k punctures on its left and $n-k$ on its right, for some $k \in \{2, \dots, n-2\}$. In this case, there should be $k-1$ arcs on the left and $n-k-1$ arcs on the right of the first arc, so there are $\binom{n-2}{k-1}$ ways to distribute the remaining $n-2$ arcs over the two sides. Finally, there are M_k respectively M_{n-k} orbits of loose isotopy classes of total curve diagrams on the disk on the left respectively on the right. □

6. Replacing finite type geodesics by curve diagrams

In this section we prove the main theorems on orderings of finite type. The strategy is to associate to every geodesic of finite type a curve diagram such that the (possibly partial) orderings arising from the geodesic and the curve diagram agree. Thus we obtain, via curve diagram orderings, a good understanding of finite type orderings.

Proof of Theorem 3.3 (a). If $D_n \backslash \gamma_\alpha$ has a path component which contains at least two holes, then we can push the boundary curve of this path component slightly into its interior, to make it disjoint from γ_α. A Dehn twist along such a curve will be a nontrivial element of B_n, and act trivially on γ_α. □

We now define *the curve diagram $C(\gamma_\alpha)$ associated to a geodesic γ_α* of finite type. It is a subset of γ_α, more precisely a union of segments of γ which start and end at self-intersection points. The diagram will be disjoint from the punctures, except that the last arc may fall into a puncture. For simplicity we shall write Γ for $C(\gamma_\alpha)$ and, as before, $\Gamma_{0\cup\dots\cup i-1}$ for $\bigcup_{k=0}^{i-1} \Gamma_k$.

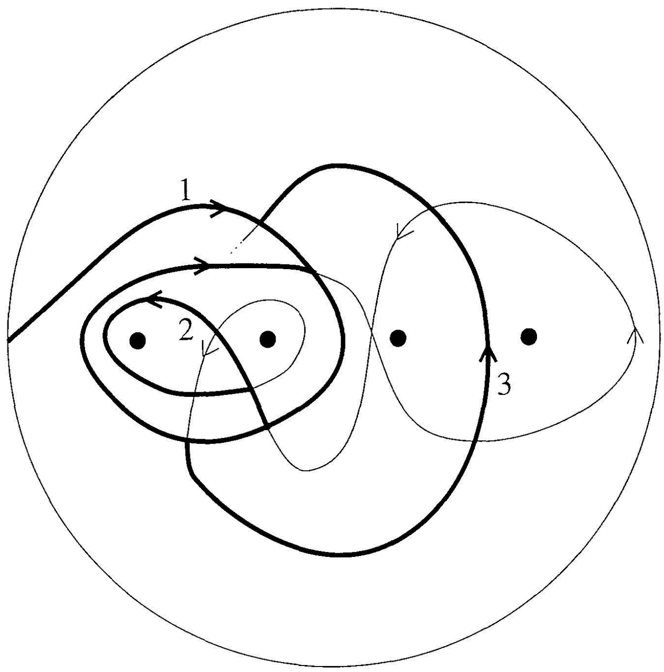

FIGURE 7
A geodesic and (in bold line) its associated curve diagram

The definition is inductive. We define $\Gamma_0 = \partial D_n$. Now suppose that we have already found $\Gamma_0, \ldots, \Gamma_{i-1}$. So every path component of $D_n \backslash \Gamma_{0 \cup \cdots \cup i-1}$ is a disk containing at least one puncture. We put down a pencil at the end point of Γ_{i-1}, start tracing out γ_α, drawing an arc Γ_i^p (with "p" standing for "potential", because Γ_i^p is potentially the new arc Γ_i). We continue drawing either up to the next intersection with $\Gamma_{0 \cup \cdots \cup i-1}$, or up to the first self intersection of Γ_i^p, or until γ_α falls into a puncture, whichever comes first. We now decide whether or not Γ_i^p has cut one of the components of $D_n \backslash \Gamma_{0 \cup \cdots \cup i-1}$ in a nontrivial way, i.e. whether it has either fallen into a puncture or cut one of the components of $D_n \backslash \Gamma_{0 \cup \cdots \cup i-1}$ into two, both of which contain at least one puncture. If yes, we let $\Gamma_i := \Gamma_i^p$, and have finished the induction step. If not, we rub out Γ_i^p, and start a new Γ_i^p at the next intersection point of γ_α with $D_n \backslash \Gamma_{0 \cup \cdots \cup i-1}$. (This intersection point is just the end point of the previous Γ_i^p, unless this endpoint is in the interior of the previous Γ_i^p . Note that in this latter case not only Γ_i^p, but the entire segment of the geodesic γ_α up to its next intersection point with $\Gamma_{0 \cup \cdots \cup i-1}$ cuts the disk in a trivial way.)

There is one special rule: if in the construction process we obtain an arc Γ_i^p which spirals *ad infinitum* towards a simple closed geodesic, then we define Γ_i to be the arc with end point in its own interior containing Γ_i^p in a regular neighbourhood, as shown in Figure 8 (this arc is unique up to loose isotopy).

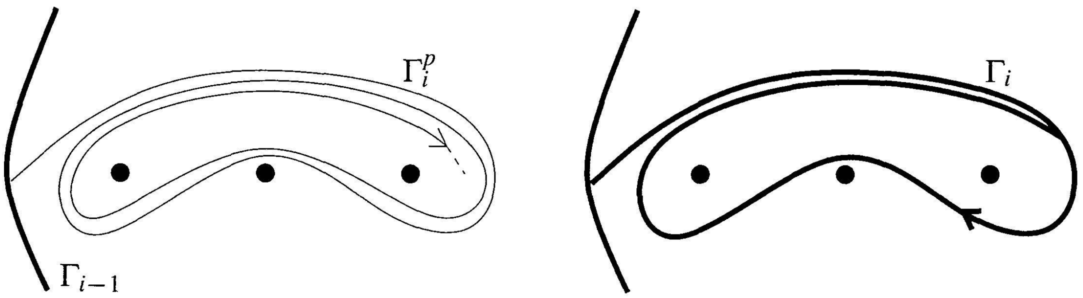

FIGURE 8
The curve diagram associated to a geodesic which spirals towards a closed geodesic

Since at most $n-1$ arcs can be constructed in this way, the process terminates after finitely many steps. We observe that the curve diagram $C(\gamma_\alpha)$ is total if and only if the geodesic γ_α fills D_n. More generally, two punctures are in the same path component of $D_n \backslash \gamma_\alpha$ if and only if they are in the same path component of $D_n \backslash C(\gamma_\alpha)$. We also note that for every geodesic γ_α and $\varphi \in B_n$ we have $C(\varphi(\gamma_\alpha)) = \varphi(C(\gamma_\alpha))$.

THEOREM 6.1. *For any $\alpha \in (0,\pi)$ and $\varphi \in B_n$ we have:*
(a) *if the curve diagrams $\varphi(C(\gamma_\alpha))$ and $C(\gamma_\alpha)$ are isotopic then $\varphi(\alpha) = \alpha$;*
(b) *if $\varphi(C(\gamma_\alpha)) > C(\gamma_\alpha)$ (in the curve diagram sense) then we have $\varphi(\alpha) > \alpha$ in $\mathbf{R}$.*

COROLLARY 6.2. *For every geodesic γ_α of finite type (where $\alpha \in (0,\pi)$), the ordering of B_n associated to α by Remark 1.2(1) coincides with the ordering associated to the curve diagram $C(\gamma_\alpha)$ by Definition 4.2.*

Proof of the theorem. We shall need a generalisation of the concept of relative "reduction" of two simple curves in D_n to the case where one of the two curves is authorised to have self-intersections, but no D-disks with itself. For instance, we shall be interested in the case where one of the two curves is a simple geodesic, and the other is a homeomorphic image of a non-simple geodesic.

Suppose that C is a disjoint collection of simple closed geodesics and properly embedded geodesic arcs connecting distinct punctures in D_n. Then we say that $\varphi(\gamma_\alpha)$ is *reducible* with respect to C if there are D-disks enclosed by $\varphi(\gamma_\alpha)$ and C, i.e. if there are finite segments of $\varphi(\gamma_\alpha)$ and of C with the same start and end points which are homotopic with fixed end points. If $\varphi(\gamma_\alpha)$ is not reducible then we say it is *reduced* with respect to C. Equivalently, any component of the preimage of $\varphi(\gamma_\alpha)$ in the universal cover $\tilde{D_n}$ intersects any component of the preimage of C at most once.

LEMMA 6.3. *One can pull $\varphi(\gamma_\alpha)$ tight with respect to C, i.e. there exists an isotopy of φ which makes $\varphi(\gamma_\alpha)$ and C reduced with respect to each other.*

Proof. The proof is an easy exercise – it is in fact similar to the proof of the "triple reduction lemma" 2.1 of [9]. □

We need some more notation. We still write Γ for $C(\gamma_\alpha)$, denote by j the number of arcs of Γ, and consider the partial curve diagrams $\Gamma_{0\cup\cdots\cup i-1}$ for $i \in \{1,\ldots,j\}$; all their arcs are geodesics. Every path component of $D_n \backslash \Gamma_{0\cup\cdots\cup i-1}$ contains at least one puncture in its interior. The boundary curve of each component with at least two punctures is isotopic to a unique simple closed geodesic, which bounds a disk (with these punctures in its interior) in D_n. Removing all these disks from D_n yields a planar surface with a number of geodesic boundary components (one of them being ∂D_n, the others corresponding to the at least twice punctured components of $D_n \backslash \Gamma_{0\cup\cdots\cup i-1}$) and a number of punctures (corresponding to once-punctured components of $D_n \backslash \Gamma_{0\cup\cdots\cup i-1}$). We denote this surface by $N\Gamma_{0\cup\cdots\cup i-1}$; it is a regular neighbourhood of $\partial D_n \cup \Gamma_{0\cup\cdots\cup i-1}$ in D_n, and contains the complete initial segment of the geodesic γ_α up to the starting point of the arc $\Gamma_i \subset \gamma_\alpha$.

We are now ready to prove the theorem. For part (a) suppose that we are given $\alpha \in (0,\pi)$, and $\varphi \in B_n$, and that the curve diagrams Γ and $\varphi(\Gamma)$ are isotopic. Then we can modify the map φ by an isotopy which fixes ∂D_n such that the restriction $\varphi|_{N\Gamma}$ becomes the identity map. But by construction of $\Gamma = C(\gamma_\alpha)$, the geodesic γ_α is entirely contained in $N\Gamma$, and is thus mapped identically. This proves part (a) of the theorem.

For part (b) suppose that we are given $\alpha \in (0,\pi)$ and $\varphi \in B_n$, and that for some $i \in \{1,\ldots,j\}$ the curve diagrams $\Gamma_{0\cup\cdots\cup i-1}$ and $\varphi(\Gamma_{0\cup\cdots\cup i-1})$ are isotopic, whereas $\varphi(\Gamma_i)$ is "more to the left" than Γ_i. Our aim is to prove that $\varphi(\alpha) > \alpha$, i.e. that the end points of the liftings of $\varphi(\gamma_\alpha)$ and γ_α on $\partial \widetilde{D}_n \backslash \Pi \cong (0,\pi)$ are different, with that of $\varphi(\gamma_\alpha)$ being "higher" in Figure 1.

Firstly, the map φ sends $\Gamma_{0\cup\cdots\cup i-1}$ to a curve diagram which is isotopic to $\Gamma_{0\cup\cdots\cup i-1}$; therefore we can assume, after an isotopy of φ which fixes ∂D_n, that the restriction $\varphi|_{N\Gamma_{0\cup\cdots\cup i-1}}$ is the identity map. Note that γ_α, being a geodesic, is already reduced with respect to the collection of geodesics $\partial N\Gamma_{0\cup\cdots\cup i-1}$, and therefore $\varphi(\gamma_\alpha)$ is also reduced with respect to $\partial N\Gamma_{0\cup\cdots\cup i-1} = \varphi(\partial N\Gamma_{0\cup\cdots\cup i-1})$.

Next, we note that the arc Γ_i will cut precisely one of the components of $D_n \backslash N\Gamma_{0\cup\cdots\cup i-1}$ in two, and leave the other components untouched. This critical component is an at least twice punctured disk, and we shall denote it by D_c. The preimage of D_c in the universal cover $\tilde{D_n}$ has many path components, but we shall be interested in one particular component $\tilde{D_c}$, namely the one which is cut in two by the segment corresponding to $\Gamma_i \subset \gamma_\alpha$ in the geodesic $\widetilde{\gamma}_\alpha$ in $\tilde{D_n}$.

We now distinguish three cases: firstly, the arc Γ_i falls into a puncture inside D_c; secondly, the arc Γ_i has its end point in $N\Gamma_{0\cup\cdots\cup i-1}$ (either on $\Gamma_{0\cup\cdots\cup i-1}$ or in the initial segment $\Gamma_i \cap N\Gamma_{0\cup\cdots\cup i-1}$ of Γ_i); thirdly, the end point of the arc Γ_i lies in the interior of D_c (and then necessarily in the interior of Γ_i).

The first case is the easiest: by an isotopy of φ which is fixed outside D_c we can pull $\varphi(\Gamma_i) \cap D_c$ tight with respect to $\Gamma_i \cap D_c$. The effect of this isotopy is to make the images of the liftings $\widetilde{\varphi}(\widetilde{\gamma}_\alpha) \cap \widetilde{D}_c$ and $\widetilde{\gamma}_\alpha \cap \widetilde{D}_c$ disjoint, except for the common starting point. Moreover, both liftings run inside $\widetilde{D}_c$ all the way to the circle at infinity. By the hypothesis that $\varphi(\Gamma) > \Gamma$, we have that an initial segment of $\widetilde{\varphi}(\widetilde{\gamma}_\alpha)$ lies to the left of the corresponding segment $\widetilde{\gamma}_\alpha$, and we conclude that its end point on the circle at infinity also lies more to the left. This proves the theorem in the first case.

LEMMA 6.4. *If γ is a (finite or infinite) geodesic starting on the boundary of the punctured disk D_c, and if φ is an automorphism of D_c which acts nontrivially on γ, then two liftings of γ and $\varphi(\gamma)$ to the universal cover $\tilde{D_c}$ of D_c with the same starting point in $\partial\tilde{D_c}$ have end points either on different components of $\partial\tilde{D_c}$ (if γ is finite) or on different points at infinity (if γ is infinite).* □

In the second case, we can pull the arc $\varphi(\Gamma_i) \cap D_c$ tight with respect to $\Gamma_i \cap D_c$ by an isotopy of φ as in the first case, thus making their liftings disjoint (except for the common starting point). We now have by hypothesis that the point of intersection of $\widetilde{\varphi}(\tilde{\Gamma_i})$ with $\partial\tilde{D_c}$ where $\widetilde{\varphi}(\tilde{\Gamma_i})$ exits $\tilde{D_c}$ lies to the left of the one of $\tilde{\Gamma_i}$. By the previous lemma, the two points will even lie on different boundary components of $\tilde{D_c}$, and therefore there is a point of $\partial\tilde{D_c}$ between these two boundary components which lies on the circle at infinity. For the liftings of our geodesic and its image this means the following: $\widetilde{\gamma}_\alpha$ and $\widetilde{\varphi}(\widetilde{\gamma}_\alpha)$ enter $\partial\tilde{D_c}$ at the same point, but exit into different components of $\tilde{D_n} \backslash \tilde{D_c}$, with $\widetilde{\varphi}(\widetilde{\gamma}_\alpha)$ choosing the one that lies more to the left. Since $\widetilde{\gamma}_\alpha$ and $\widetilde{\varphi}(\widetilde{\gamma}_\alpha)$ do not intersect $\partial\tilde{D_c}$ again, they stay inside their chosen component of

$\widetilde{D_n}\setminus\widetilde{D_c}$. Hence we have for their end points that $\varphi(\alpha) > \alpha$, and the theorem is proved in the second case.

We now turn to the third case, which includes the possibility that γ_α spirals towards a closed geodesic inside D_c. We consider the arc $\Sigma := \Gamma_i'$ as in Figure 3, and for simplicity we choose Σ to be a geodesic arc. We denote by $D_{cc} \subset D_c$ the subdisk cut off by Σ (so that $\Sigma = \partial D_{cc}$). Since Σ is geodesic, we have that $\gamma_\alpha \cap D_c$ is reduced with respect to Σ. After an isotopy of φ inside D_c we can assume by Lemma 6.3 that the first component of $\varphi(\gamma_\alpha) \cap D_c$ (the one that contains $\varphi(\Gamma_i) \cap D_c$) is also reduced with respect to Σ. By the hypothesis that $\varphi(\Gamma_i)$ sets off more to the left than Γ_i, we are now in one of the situations indicated in Figure 9.

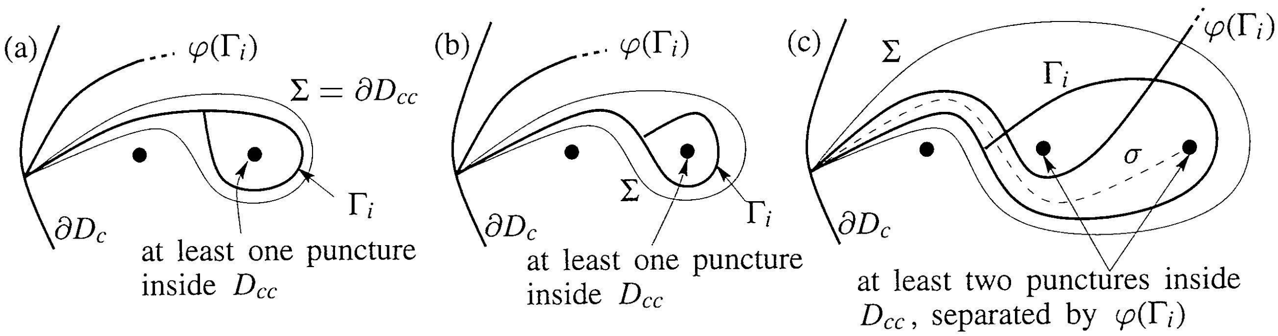

FIGURE 9
The critical disk D_c containing Γ_i and $\varphi(\Gamma_i)$

A first possiblity is that an initial segment of $\varphi(\Gamma_i) \cap D_c$ lies to the left of the tip of D_{cc} (Figures 9(a) and (b)); in the universal cover $\widetilde{D_c}$ we now have three arcs, namely $\widetilde{\varphi}(\widetilde{\gamma}_\alpha) \cap \widetilde{D_c}$, a lifting of Σ, and $\widetilde{\gamma}_\alpha \cap \widetilde{D_c}$ (and, in fact, a fourth arc, another lifting of Σ) starting at the same point of $\partial\widetilde{D_c}$, and setting off into different directions, namely in the given order from left to right. Moreover, the liftings of Σ are disjoint from the interiors of the other two arcs, by reducedness. Thus the end point of $\widetilde{\varphi}(\widetilde{\gamma}_\alpha) \cap \widetilde{D_c}$ on $\partial\widetilde{D_c}$ lies more to the left than that of $\widetilde{\gamma}_\alpha \cap \widetilde{D_c}$. Even stronger, by Lemma 6.4 they lie either on different points at infinity (in which case we are done) or they leave $\widetilde{D_c}$ through different components of $\partial\widetilde{D_c}$ (in which case we argue as above that their remainders are trapped in different components of $\widetilde{D_n}\setminus\widetilde{D_c}$, so that $\widetilde{\varphi}(\widetilde{\gamma}_\alpha)$ stays to the left of $\widetilde{\gamma}_\alpha$).

The second possibility is that some initial segment of $\varphi(\Gamma_i) \cap D_c$ lies in D_{cc} (Figure 9(c)); then D_{cc}, cut along this initial segment, has precisely two path components, each of which contains at least one puncture. Since $\varphi(\Gamma_i)$ is oriented, we can refer to them as the "left" and the "right" half of D_{cc}. We now consider a geodesic arc σ which is embedded in the right half of D_{cc}, starts at the tip of D_{cc} (i.e. at the same point as $\Gamma_i \cap D_c$ and $\varphi(\Gamma_i) \cap D_c$), and falls into one of the punctures in the right half of D_{cc}. By construction, $\gamma_\alpha \cap D_{cc}$ is reduced with respect to σ, since both are geodesics, and the first component of $\varphi(\gamma_\alpha) \cap D_{cc}$ is even disjoint from σ. In the universal cover we now have that the lifting $\widetilde{\sigma}$ of σ ends on the circle at infinity, thus separating $\widetilde{D}_{cc}$ into two components, the left one containing the lift of $\varphi(\gamma_\alpha) \cap D_{cc}$, and the right one the lift of $\gamma_\alpha \cap D_{cc}$. Thus lifts of these two curves, not being allowed to intersect any component of $\partial D_{cc}^{\sim}$ and $\partial D_c^{\sim}$ more than once, go on to hit different points of $\partial D_n^{\sim}$, with $\widetilde{\varphi}(\widetilde{\gamma}_\alpha)$ staying more to the left than $\widetilde{\gamma}_\alpha$. This completes the proof of the third case, and thus of Theorem 6.1. □

Proof of Theorem 3.3 (b). If γ_α fills D_n, then $C(\gamma_\alpha)$ is a total curve diagram, and thus induces a *total* ordering of B_n. By Corollary 6.2, the ordering of B_n associated to the point $\alpha \in (0, \pi)$ agrees with this ordering. □

Proof of Theorem 3.4 (b). For any two geodesics γ_α and γ_β of finite type one can work out their associated curve diagrams $C(\gamma_\alpha)$ and $C(\gamma_\beta)$. By Corollary 6.2 it is sufficient to decide whether or not the orderings associated to the two curve diagrams coincide, which can be done by Theorem 5.2. □

Proof of Theorem 3.5. It only remains to be proved that $N_n = M_n$ (where M_n is given in Proposition 5.3), i.e. that every curve diagram is realized up to loose isotopy as $C(\gamma_\alpha)$ for some geodesic γ_α, $\alpha \in (0, \pi)$. This is left as an exercise to the reader. □

7. ORDERINGS ASSOCIATED TO GEODESICS OF INFINITE TYPE

In this section we prove the results concerning orderings of infinite type, and explain the essential differences between finite and infinite type orderings.

We start by describing in more detail than in Section 3 the structure of geodesics of infinite type. We define the *curve diagram* $C(\gamma_\alpha)$ *associated to a geodesic of infinite type* by precisely the same inductive construction procedure as in the finite type case. Except for a finite initial segment, the last arc Γ_j will lie in some path component D_c of $D_n \backslash N\Gamma_{0 \cup \cdots \cup j-1}$, the only difference with the finite type case is that Γ_j goes on for ever, without falling into a puncture and without spiralling. The closure of Γ_j inside this critical component D_c is a geodesic lamination; the lamination has no closed leaves, for such a leaf would have to be the limit of an infinite spiral of Γ_j (see [17, Appendix]). All self-intersections of the geodesic γ_α occur inside the finite initial segment up to the entry into the punctured disk D_c ; in particular, there are only finitely many self-intersections.

Proof of Theorem 3.3 (c). We are studying the set

$$\mathcal{I} := \{\alpha \in (0, \pi) \mid \gamma_\alpha \text{ is of infinite type}\}.$$

The proof uses standard results from the theory of geodesic laminations and the Nielsen-Thurston classification of surface automorphisms [5, 17].

That $\mathcal{I}$ has uncountably many elements follows from the fact that there are uncountably many geodesic laminations of D_n, only countably many of which fall into infinite spirals. A more practical way of seeing this is to choose arbitrarily a fundamental domain of D_n by fixing n geodesic arcs, e.g. as shown in Figure 1. Thus the fundamental domain is a $2n+1$-gon with one boundary edge corresponding to ∂D_n and n pairs of boundary edges which are identified in D_n. A segment of the geodesic between any two sucessive intersections with the boundary of the fundamental domain consists of an embedded arc connecting different edges of the $2n+1$-gon. Hence constructing a geodesic of infinite type amounts to choosing an infinite "cutting sequence" of the geodesic with the boundary arcs of the fundamental domain. Often the choice will be forced upon us by the requirement that the geodesic be embedded, but there will be an infinite number of times when we have a genuine choice. Thus the set of all possible sequences of choices is uncountable.

The cutting sequence approach also makes it clear why any neighbourhood of an $\alpha \in \mathcal{I}$ in $(0, \pi)$ contains points $\alpha' \neq \alpha$ of $\mathcal{I}$ as well as $\beta \in (0, \pi)\backslash\mathcal{I}$. Given $\alpha \in (0, \pi)$ and $\epsilon > 0$, there exists an $N_\epsilon \in \mathbf{N}$ such that all geodesics γ_δ whose cutting sequences agree with the one of γ_α for at least N_ϵ terms satisfy $|\alpha - \delta| < \epsilon$. Now for any $\alpha \in \mathcal{I}$ and $\epsilon > 0$ we can find a geodesic $\gamma_{\alpha'}$ of infinite type whose cutting sequence diverges from the one of γ_α only after the $N_\epsilon{}^{\text{th}}$ term. On the other hand, we can construct a geodesic γ_β with $|\alpha - \beta| < \epsilon$ which fills D_n in finite time: just choose it to have a cutting sequence which agrees with the one of γ_α for N_ϵ terms, and to then career off along some path which decomposes D_n into disks and once-punctured disks.

Finally, the last part of Theorem 3.3(c) holds because each of the countably many elements of B_n fixes only a countable number of points $\alpha \in (0, \pi)$ with the property that γ_α fills D_n. In order to see this, we note that for *irreducible* elements of B_n Theorem 5.5 of [5] states that there is only a finite number of fixed points on the circle at infinity. If an element φ of B_n is *reducible*, then we leave it to the reader to check that the result follows from the following facts:

(1) One can find a maximal invariant system C of disjoint properly embedded arcs and circles in D_n.

(2) If φ acts nontrivially on a component of $D_n \backslash C$ which is cut in a nontrivial way by a *finite* segment of γ_α, then it acts nontrivially on γ_α (for if it didn't then the collection C would not be maximal).

(3) A geodesic γ_α that fills D_n has to enter every component of $D_n \backslash C$ at least once, and φ acts nontrivially either on the first or, failing that, on the second component of $\gamma_\alpha \cap (D_n \backslash C)$ (because it cannot act trivially on two adjacent components of $D_n \backslash C$).

(4) There is a countable infinity of isotopy classes of embedded arcs from the basepoint of D_n to C. $\square$

We recall from the beginning of the section that to every geodesic γ_α of infinite type we have associated a "critical disk" D_c which contains most of the last arc of $C(\gamma_\alpha)$. The fundamental property of geodesics of infinite type which we shall use several times is the following.

LEMMA 7.1. *For any geodesic of infinite type γ_α and for any $\epsilon > 0$ there exists a geodesic γ_{α^+} with $\alpha^+ \in (\alpha, \alpha + \epsilon)$ such that γ_{α^+} falls into a puncture and has no self-intersections inside D_c.*

Proof. It suffices to prove the lemma in the special case $D_c = D_n$, i.e. when the geodesic γ_α is embedded. We suppose, for a contradiction, that there exists an $\epsilon > 0$ such that no γ_β with $\beta \in (\alpha, \alpha + \epsilon)$ is embedded *and* falls into a puncture. Our aim is to reach the contradiction that γ_α ends in an infinite spiral.

We continue to use the notions concerning cutting sequences introduced above: we choose arbitrarily a fundamental domain, and we shall denote by γ_α^k the initial segment of γ_α up to its k^{th} intersection with the boundary of the fundamental domain. We recall that, given γ_α and $\epsilon > 0$, we can find an $N = N_\epsilon \in \mathbf{N}$ such that any geodesic γ_β with $\gamma_\beta^N = \gamma_\alpha^N$ satisfies $|\alpha - \beta| < \epsilon$. We now consider the arc γ_α^{N+1} : it ends on some boundary arc of the fundamental domain which we denote a. The orientation of γ_α gives rise to a notion of the part of a "to the left" and "to the right of" the end point of γ_α^{N+1}. The arc γ_α^{N+1} has an intersection with the interior of the "left" part of a, for if this were not the case we could obtain an embedded arc γ_β

with $\beta \in (\alpha, \alpha+\epsilon)$ by adjoining to the end point of γ_α^N an arc falling into the puncture at the left end of a; this would contradict the hypothesis. Thus it makes sense to define $\Gamma \subseteq D_n$ to be the union of γ_α^{N+1} and a segment of a from the end point of γ_α^{N+1} to the left, up to the next intersection with γ_α^{N+1} (see Figure 10).

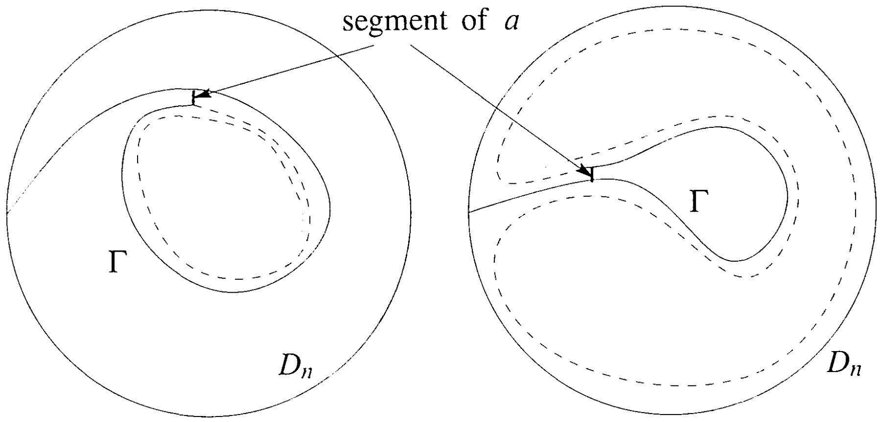

FIGURE 10
The two possible shapes of Γ, and (dashed) the resulting geodesic γ_α

We now observe that $D_n \backslash \Gamma$ has two path components, each containing at least one puncture; moreover, γ_α cannot intersect any geodesic arc connecting two punctures in the same component, because the first time it did we could drop it into the puncture at the left end of the arc and obtain a contradiction as before. It follows that γ_α has to spiral along the boundary of one of the components of $D_n \backslash \Gamma$. □

PROPOSITION 7.2. *All orderings, even partial ones, arising from geodesics* γ_α *of infinite type are non-discrete.*

Proof. We shall prove the following stronger statement: for any $\epsilon > 0$ there exists an element $\varphi \in \mathcal{MCG}(D_n) = B_n$ such that $\varphi(\alpha) \in (\alpha, \alpha+\epsilon)$.

We choose α^+ as in the previous lemma. We consider the boundary curve τ of a regular neighbourhood of $\partial D_c \cup \gamma_{\alpha^+}$ in D_c. This curve τ is disjoint from γ_{α^+}, while any curve isotopic to τ necessarily intersects γ_α. Thus for the positive Dehn twist T along τ we have that $T(\alpha) > \alpha$ (by Proposition 2.4), and that $T(\alpha^+) = \alpha^+$. It follows that $T(\alpha) \in (\alpha, \alpha^+) \subseteq (\alpha, \alpha+\epsilon)$. □

Proof of Theorem 3.4 (a). Given a geodesic γ_α of finite, and a geodesic γ_β of infinite type, our aim is to prove that γ_α and γ_β cannot induce the same orderings of B_n.

As seen in Corollary 6.2, orderings arising from geodesics which fill the surface in finite time are the same as orderings arising from total curve diagrams, which are discrete by Lemma 4.5. By contrast, we have from Proposition 7.2 that infinite type orderings are not discrete. This proves the theorem in the special case where the finite type geodesic fills the surface.

In the case where the finite type geodesic γ_α does *not* fill the surface, we consider the subsurface $D_\alpha := D_n \backslash NC(\gamma_\alpha)$, i.e. the maximal subsurface with geodesic boundary which is disjoint from γ_α. We observe that D_α is a disjoint union of disks, each containing at least two punctures. Any homeomorphism φ of D_n with support in D_α has the property that $\varphi(\alpha) = \alpha$.

If $D_\alpha \cap \gamma_\beta \neq \varnothing$ then there exists a homeomorphism φ with support in D_α such that $\varphi(\beta) \neq \beta$, and it follows that the orderings induced by α and β are different.

If, on the other hand, $D_\alpha \cap \gamma_\beta = \varnothing$, then we squash each component of D_α to a puncture; the result is a disk with say m punctures, where $m < n$, which we denote D_m. We now consider the subgroup B_m^P of $B_m = \mathcal{MCG}(D_m)$ of all mapping classes which fix those punctures of D_m that came from squashed components of D_α. This is a finite index subgroup of B_m, and the orderings of B_n determined by α and β induce quotient orderings on B_m^P. Another way to describe these quotient orderings is to repeat the Thurston-construction for the disk D_m: one can equip D_m with a hyperbolic metric, and then the geodesics γ_α and γ_β project to quasigeodesics in D_m. These quasigeodesics determine points at infinity of the universal cover of D_m, and hence give rise to orderings of B_m.

The geodesic in D_m which is homotopic to the projection of γ_α is again of finite type; the crucial observation now is that it fills D_m, so that the quotient ordering on B_m^P is discrete by Lemma 4.5. Similarly, a geodesic in D_m homotopic to the projection of γ_β is again of infinite type, hence induces, by Proposition 7.2 a non-discrete ordering on B_m, and thus also on the finite-index subgroup B_m^P. So the α- and β-orderings on B_n give rise to different quotient orderings on B_m^P, and are therefore different. □

As seen above, every geodesic of infinite type gives rise to a curve diagram "of infinite type", which is like a curve diagram of finite type, except that the arc with maximal label is, up to isotopy, an infinite geodesic which does not fall into a puncture or a spiral. All but a finite initial segment of this arc lies in the "critical disk" D_c. There is an obvious generalisation of the notion of loose isotopy:

DEFINITION 7.3. Two curve diagrams of infinite type are *loosely isotopic* if they are related by (1) continuous deformation, i.e. a path in the space of all curve diagrams of infinite type; and (2) pulling loops around punctures tight.

This is exactly the same as in the finite type case, except that no "pulling loops around punctures tight"-procedure is defined for the last arc. We are now ready to state and prove the main classification theorem for orderings of B_n of infinite type.

THEOREM 7.4. *Two geodesics γ_α and γ_β of infinite type give rise to the same (possibly partial) ordering of B_n if and only if their associated curve diagrams $C(\gamma_\alpha)$ and $C(\gamma_\beta)$ are loosely isotopic.*

Proof. By the results in the previous sections, it suffices to prove that two *embedded* geodesics γ_α and γ_β of infinite type give rise to the same ordering of B_n if and only if $\beta = \Delta^{2k}(\alpha)$ for some $k \in \mathbf{Z}$, i.e. if γ_α and γ_β are related by a slide of the starting point around ∂D_n.

The implication "$\Leftarrow$" is clear. Conversely, for the implication "$\Rightarrow$", we suppose that γ_α and γ_β are not related by a slide of the starting point, and without loss of generality we say $\alpha > \beta$. Our aim is to construct a homeomorphism which is positive in the α- and negative in the β-ordering, i.e. which sends α "more to the left" and β "more to the right". Our argument will be a refinement of the proof of the implication "$\Rightarrow$" of 5.2(a).

By Lemma 7.1 we can construct embedded geodesics $\gamma_{\alpha+}$ and $\gamma_{\beta+}$ which fall into punctures, and lie an arbitrarily small amount to the left of γ_α respectively γ_β. We define the curves $\tau_{\alpha+}$ and $\tau_{\beta+}$ to be the geodesic representatives of the boundary curves of regular neighbourhoods in D_n of $\partial D_n \cup \gamma_{\alpha+}$ and $\partial D_n \cup \gamma_{\beta+}$ respectively. We denote by $T_{\alpha+}$ respectively $T_{\beta+}$ the positive Dehn twists along these curves. Our desired homeomorphism will be of the form $T_{\alpha+}^{-k} \circ T_{\beta+}$, with carefully chosen values of α^+ and β^+, and $k \in \mathbf{N}$ very large.

We also define the two-sided infinite geodesic τ_α to be the geodesic which is disjoint from γ_α, and isotopic to the boundary of a neighbourhood of $\gamma_\alpha \cup \partial D_n$ in D_n. More formally, in the universal cover $\tilde{D}_n$ we consider two liftings of γ_α, namely $\tilde{\gamma}_\alpha$ (which starts at the basepoint of $\tilde{D}_n$), and the lifting whose starting point also lies on Π and is obtained from the basepoint of $\tilde{D}_n$ by lifting the path once around ∂D_n. The end points of these geodesics lie on the circle at infinity, and τ_α is just the projection of the geodesic connecting them.

Since γ_α and γ_β are not loosely isotopic, we have that γ_β intersects τ_α. By choosing β^+ sufficiently close to β we can now achieve that the initial segments of γ_β and γ_{β^+} up to their first point of intersection with τ_α are isotopic with end points sliding in τ_α. This gives our choice of β^+, and it remains to choose α^+ and k.

The crucial observation concerning τ_α is that it can be arbitrarily closely approximated by the curves τ_{α^+}, by choosing α^+ sufficiently close to α. More precisely, in the universal cover $\tilde{D_n}$ we consider the preimages of τ_α and of τ_{α^+}. Each of them has infinitely many path components; we choose one distinguished component for each, namely the first ones that γ_β intersects. Our observation now is that as α^+ tends to α, the end points of the distinguished component of the preimage of τ_{α^+} tend to the end points of the distinguished component of the preimage of τ_α.

We now turn to the choice of α^+. By Proposition 2.4 we have that $T_{\beta^+}(\alpha) > \alpha$. By Lemma 7.1 we can now choose α^+ close to α such that $T_{\beta^+}(\alpha) > \alpha^+ > \alpha$. By possibly pushing α^+ even closer to α, we can in addition insist (by the observation concerning τ_α above) that the initial segments of γ_β and γ_{β^+} up to their first point of intersection with τ_{α^+} are also isotopic with end points sliding in τ_{α^+}. This gives our choice of α^+.

We have arrived at the following setup: we have the three points $\beta^+ = T_{\beta^+}(\beta^+) > T_{\beta^+}(\beta) > \beta$ in $\partial\tilde{D_n}\backslash\Pi$, and they all lie between the two end points δ_l and δ_r of the distinguished lifting of τ_{α^+} (here the indices l and r stand for "left" and "right", so $\delta_l > \delta_r$). For any point δ with $\delta_l > \delta > \delta_r$ we consider the action of the positive Dehn twist T_{α^+} on the geodesic γ_δ. We observe that the limit $\lim_{k\to\infty} T_{\alpha^+}^{-k}(\delta) = \delta_r$. In particular for $\delta := \beta^+$ it follows that for sufficiently large k we have $T_{\alpha^+}^{-k}(\beta^+) < \beta$. This gives our choice of k.

To summarise, we have

$$T_{\alpha^+}^{-k} \circ T_{\beta^+}(\alpha) > T_{\alpha^+}^{-k}(\alpha^+) = \alpha^+ > \alpha$$

and

$$T_{\alpha^+}^{-k} \circ T_{\beta^+}(\beta) < T_{\alpha^+}^{-k} \circ T_{\beta^+}(\beta^+) = T_{\alpha^+}^{-k}(\beta^+) < \beta,$$

i.e. $T_{\alpha^+}^{-k} \circ T_{\beta^+}$ is positive in the α-, but negative in the β-ordering. □

Proof of Theorem 3.4 (c). This is an immediate consequence of Theorem 7.4. □

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