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A FREE GROUP ACTING ON $\mathbf{Z}^2$ WITHOUT FIXED POINTS

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ABSTRACT. The group of all orientation-preserving affine transformations of the plane has a non-abelian free subgroup which stabilizes $\mathbf{Z}^2$ and which acts on $\mathbf{Z}^2$ without non-trivial fixed points.

INTRODUCTION

Let G be a group acting on a non-empty set X. The following two conditions are known to be equivalent (see [D], and Theorems 4.5 and 4.8 in [W]):

(a) *there exists a non-abelian free subgroup of* G *whose action on* X *is locally commutative;*

(b) *there exists a* G*-paradoxical decomposition of* X *using* 4 *pieces, namely a partition of* X *in parts* P_0, P_1, P_2, P_3 *and elements* α_0, α_1, α_2, α_3 *in* G *such that*

$$X = P_0 \sqcup P_1 \sqcup P_2 \sqcup P_3 = \alpha_0(P_0) \sqcup \alpha_1(P_1) = \alpha_2(P_2) \sqcup \alpha_3(P_3)\,.$$

Moreover, in the situation of (b), it can be shown that the subgroup of G generated by ${\alpha_0}^{-1}\alpha_1$ and ${\alpha_2}^{-1}\alpha_3$ is free of rank 2. (The symbol $\sqcup$ denotes disjoint union. Recall that an action of a group H on X is *locally commutative* if the stabilizer $\{h \in H \mid h(x) = x\}$ is commutative for all $x \in X$, i.e. if two elements of H which have a common fixed point commute; trivial examples of locally commutative actions are actions *without non-trivial fixed points*, for which $\{h \in H \mid h(x) = x\}$ is reduced to $\{1\}$ for all $x \in X$.)

For example, the group $SO_3(\mathbf{R})$ of rotations of the unit sphere $\mathbf{S}^2$ has such a free subgroup: this was discovered by F. Hausdorff (see, e.g., [Ś], or Theorem 2.1 in [W]). It implies the following result, for which we refer to [BT] and Theorem 3.11 in [W]; we denote by $SG_3(\mathbf{R})$ the group of all orientation-preserving isometries of $\mathbf{R}^3$.

THE BANACH-TARSKI PARADOX. *Any two bounded subsets U and V of the 3-dimensional Euclidean space $\mathbf{R}^3$ with non-empty interiors are* $SG_3(\mathbf{R})$-*equidecomposable. In other words, one can partition U into a finite number of pieces and reconstruct V from the same number of respectively* $SG_3(\mathbf{R})$-*congruent pieces.*

The Banach-Tarski paradox holds similarly for higher dimensional Euclidean spaces, but not for $\mathbf{R}$ and $\mathbf{R}^2$; the reason is that neither $SG_1(\mathbf{R})$ nor $SG_2(\mathbf{R})$, which are soluble groups, contain free subgroups of rank 2. (There are other known examples of free groups acting without non-trivial fixed points on familiar spaces. See e.g., [B], [DS], and [S2]. The proof of the Banach-Tarski paradox requires the axiom of choice, because the proof of the equivalence of conditions (a) and (b) requires it. But similar paradoxes hold for rational spheres of the form $(\sqrt{q}\,\mathbf{S}^2) \cap \mathbf{Q}^3$, as can be shown *without* the axiom of choice from the countability of rational spheres. See [S1], and [S3].) In dimension 2, von Neumann has exhibited a Banach-Tarski paradox with respect to the group $SA_2(\mathbf{R})$ of affine transformations of $\mathbf{R}^2$ that preserve area and orientation ([V], and Theorem 7.3 of [W]). The following problem was raised in [MW]; see also the discussion which follows Proposition 7.1 in [W].

PROBLEM ([MW], [W]). *Does* $SA_2(\mathbf{R})$ *contain a free subgroup of rank* 2 *whose action on* $\mathbf{R}^2$ *is locally commutative ?*

Indeed, these authors asked more specifically if the group generated by

$$\alpha\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$\beta\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

satisfies the requirements of the problem. We observe here that the answer is “no”, because both $\alpha^{-2}\beta^2$ and $\alpha^{-1}\beta^{-1}\alpha\beta$ fix the origin.

Though we cannot solve the above problem, the purpose of this note is to show that, if one replaces $\mathbf{R}^2$ by $\mathbf{Z}^2$, the new problem has a positive solution. In fact, we will prove the following result, which shows somewhat more, namely that the action on $\mathbf{Z}^2$ may be an action without non-trivial fixed points, rather than only locally commutative. We denote by $\mathrm{SA}_2(\mathbf{Z})$ the group of all transformations $\vec{x} \mapsto A\vec{x} + \vec{a}$ of $\mathbf{Z}^2$, with $A \in \mathrm{SL}_2(\mathbf{Z})$ and $\vec{a} \in \mathbf{Z}^2$.

THEOREM. *The group* $\mathrm{SA}_2(\mathbf{Z})$ *has a free subgroup* F_2 *of rank* 2 *which acts on* $\mathbf{Z}^2$ *without non-trivial fixed points, namely the subgroup generated by*

$$\zeta : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 7 & 3 \\ 9 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

and

$$\eta : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 94 & 39 \\ 147 & 61 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

The theorem implies the existence of a partition of $\mathbf{Z}^2$ into three pieces P, Q and R such that the six pieces P, Q, R, $P \sqcup Q$, $Q \sqcup R$, $R \sqcup P$ are pairwise F_2-congruent, without the axiom of choice ([S0], and Corollary 4.12 in [W]).

As observed in the discussion which follows Proposition 7.1 in [W], it is known that the above theorem does not carry over to $\mathbf{R}^2$; more precisely, it is known that a subgroup of $\mathrm{SA}_2(\mathbf{R})$ which acts on $\mathbf{R}^2$ without non-trivial fixed points is soluble, and consequently does not contain non-commutative free subgroups.

PROOF OF THE MAIN RESULT

Recall that a matrix in $\mathrm{SL}_2(\mathbf{Z})$ is *hyperbolic* if the absolute value of its trace is strictly larger than 2, or equivalently if it has an eigenvalue of absolute value strictly larger than 1.

LEMMA 0. *The subgroup of* $\mathrm{SL}_2(\mathbf{Z})$ *generated by*

$$\begin{pmatrix} 7 & 3 \\ 9 & 4 \end{pmatrix} \text{ and } \begin{pmatrix} 94 & 39 \\ 147 & 61 \end{pmatrix}$$

is free of rank 2 *and all its elements distinct from the identity are hyperbolic.*

Proof. It is well-known that the subgroup of $SL_2(\mathbf{Z})$ generated by

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

is free of rank 2, and that all its elements distinct from the identity are hyperbolic. (See Appendix B in [K], [Ma], [MW], [N], or the proof of Theorem 6.8 in [W].) The lemma follows, because

$$\begin{pmatrix} 7 & 3 \\ 9 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 94 & 39 \\ 147 & 61 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^2 \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}^2$$

(see for example Exercice 12 of Section 1.4 in [MKS]). □

The following is elementary linear algebra.

LEMMA 1. *For $A \in SL_2(\mathbf{R})$ with $\operatorname{tr} A \neq 2$ and for $\vec{a} \in \mathbf{R}^2$, the affine transformation*

$$\begin{cases} \mathbf{R}^2 & \to & \mathbf{R}^2 \\ \vec{x} & \mapsto & A\vec{x} + \vec{a} \end{cases}$$

has a unique fixed point.

Our preparations are complete.

Proof of the main theorem. The two transformations ζ and η of our main result generate a group which is free of rank 2, by Lemma 0. As both these transformations fix the point

$$\begin{pmatrix} 2/3 \\ -5/3 \end{pmatrix} \in \mathbf{R}^2,$$

each element of the group they generate fix the same point. As this point is not in $\mathbf{Z}^2$, the theorem follows by Lemma 1. □

REMARK. Let $\alpha, \beta \in SA_2(\mathbf{Z})$ be as in the introduction. Then we can prove the main theorem by using the group generated by $\alpha\beta^{-1}\alpha\beta^{-2}\alpha$ and $\beta\alpha^{-1}\beta\alpha^{-2}\beta$, because the transformations

$$\alpha\beta^{-1}\alpha\beta^{-2}\alpha : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 13 & 22 \\ 10 & 17 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -17 \\ -13 \end{pmatrix}$$

and

$$\beta\alpha^{-1}\beta\alpha^{-2}\beta : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 17 & 10 \\ 22 & 13 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -13 \\ -17 \end{pmatrix}$$

have a common fixed point $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ in $\mathbf{R}^2$ and the subgroup of $SL_2(\mathbf{Z})$ generated by

$$\begin{pmatrix} 13 & 22 \\ 10 & 17 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 17 & 10 \\ 22 & 13 \end{pmatrix}$$

is free of rank 2 and all its elements distinct from the identity are hyperbolic. See [K] and the following calculations:

$$\begin{pmatrix} 13 & 22 \\ 10 & 17 \end{pmatrix} = tu(tu^{-1})^3(tu)^2tu^{-1}tu\,,$$

$$\begin{pmatrix} 17 & 10 \\ 22 & 13 \end{pmatrix} = tu^{-1}(tu)^3(tu^{-1})^2tutu^{-1}\,,$$

where

$$t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad u = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

with $\langle t, u\rangle / \{\pm 1\} \cong \mathbf{Z}_2 * \mathbf{Z}_3$.

The referee suggested to the author that the following could be shown (without the axiom of choice).

COROLLARY. *There exists a subset E_1 of $\mathbf{Z}^2$ such that, for every finite subset F of $\mathbf{Z}^2$, the symmetric difference of E_1 and F is congruent to E_1 relative to the group* $SA_2(\mathbf{Z})$.

Proof. This is a consequence of our main result and of Theorem 2 in [My] ($S = \mathbf{Z}^2$, $G = \langle \zeta, \eta \rangle$, $M = \{\zeta\eta, \zeta^2\eta^2, \zeta^3\eta^3, \dots\}$, $\mathbf{F} = \{F \subseteq \mathbf{Z}^2 \mid F \text{ is finite}\}$). □

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